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Associated schemes of vertex superalgebras and equivariant oriented cohomology

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ASSOCIATED SCHEMES OF VERTEX SUPERALGEBRAS
AND EQUIVARIANT ORIENTED COHOMOLOGY

by

Hao Li

A Dissertation
Submitted to the University at Albany, State University of New York
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College of Arts and Sciences
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To my parents
ABSTRACT

Part 1.

We generalize the notion of “quasi-lisse” vertex algebras to the super case. The modularity of quasi-lisse vertex superalgebra (twisted)modules is discussed. We study several families of vertex operator superalgebras from an arc (super)space point of view. We provide new examples of vertex algebras which are “chiral-quatizations” of their $C_2$-algebras $R_V$. Our examples come from certain $N = 1$ superconformal vertex algebras, Feigin-Stoyanovskiy principal subspaces, Feigin-Stoyanovskiy type subspaces, graph vertex algebras $W_\Gamma$, and extended Virasoro vertex algebras. We also give some counterexamples to the chiral-quatizations property. For principal subspaces, their characters are closely related to $q$-series identities. In particular, we obtain new fermionic character formulas for level one $A$-type principal subspaces.

Part 2.

For any Bott-Samelson resolution $q_I : \hat{X}_I \to G/B$ of the flag variety $G/B$, and any torus equivariant oriented cohomology $h_T$, we compute the restriction formula of certain basis $\eta_L$ of $h_T(\hat{X}_I)$ determined by the projective bundle formula. As an application, we show that $h_T(\hat{X}_I)$ embeds into the equivariant oriented cohomology of $T$-fixed points, and the image can be characterized by using the Goresky-Kottwitz-MacPherson (GKM) description. Furthermore, we compute the push-forward of the basis $\eta_L$ onto $h_T(G/B)$, and their restriction formula.
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CHAPTER 1
Introduction

Part 1.

Beilinson, Feigin and Mazur [19] first introduced the notions of singular support and lisse representation in order to study Virasoro (vertex) algebras. Arakawa later extended these notions to any finitely strongly generated, non-negatively graded vertex algebra $V$. More precisely, via a canonical decreasing filtration $\{F_p(V)\}$ introduced in [74], we can associate to $V$ a positively graded vertex Poisson algebra $gr^F(V)$. The spectrum of $gr^F(V)$ is called the singular support of $V$ and is denoted by $SS(V)$. With respect to this filtration, $V/F_1(V)$ is the Zhu $C_2$-algebra $R_V$. The reduced spectrum $\tilde{X}_V = \text{Spec}(R_V)$ is a Poisson variety which is called the associated variety of $V$. A large body of work has been devoted to descriptions of associated varieties for various vertex operator algebras [6, 13, 14, 11]. Certainly the most prominent examples from this point of view are well-known lisse, or $C_2$-cofinite, vertex algebras characterized by $\dim(\tilde{X}_V) = 0$. Arakawa and Kawasetsu relaxed this condition to quasi-lisse in [8] which requires that $X_V$ has finitely many symplectic leaves. Associated varieties are important in the geometry of Higgs branches in 4d/2d dualities in physics [18].

According to [6, Proposition 2.5.1], the embedding

$$R_V \hookrightarrow gr^F(V)$$

can be extended to a surjective homomorphism of vertex Poisson algebras

$$\psi : J_\infty(R_V) \twoheadrightarrow gr^F(V),$$

where $J_\infty(R_V)$ is the arc algebra of $R_V$. The map $\psi$ induces an injection from the singular support into the arc space of the associated scheme of $V$, $X_V = \text{Spec}(R_V)$,

$$\phi : SS(V) \hookrightarrow (X_V)_\infty.$$
In [12], authors showed that $\phi$ is an isomorphism as varieties if $V$ is quasi-lisse. It was shown in [104] that if the map $\psi$ is an isomorphism, then one can compute Hochschild homology of the Zhu algebra via the chiral homology of elliptic curves. Following the terminology of [104] we say that $V$ is classically free if $\psi$ is injective. Another way of expressing injectivity is by saying that $V$ is “chiral-quantization” of its $C_2$-algebra [75]. Vertex algebras with this property are common and we expect that they include many (but not all) rational vertex algebras and of course large families of irrational vertex algebras. Proving that $\psi$ is an isomorphism or finding the kernel of $\psi$ turns out to be subtle. In [10], [15] and [104], authors provided examples for which $\psi$ is not an isomorphism, including the $\mathbb{Z}_2$-orbifold of the rank one Heisenberg algebra, affine vertex algebras $L_{\text{sl}_4}(-1,0)$, most Virasoro algebras, etc. However, a full description of the kernel, if non-trivial, of the $\psi$-map is an interesting and difficult problem. Very recently, Andrews, van Ekeren and Heluani [105] found an interesting $q$-series identity that allowed them to describe the kernel of $\psi$ for the $c = \frac{1}{2}$ Ising Virasoro vertex algebra.

For a vertex algebra $V$ where $\psi$ is an isomorphism one obtains a very interesting (and important) consequence

\[ \text{ch}[V](q) = HS_q(J_\infty(R_V)) , \]

where the left-hand side is the character of $V$ and the right-hand side is the Hilbert series of the arc algebra of $R_V$. The left-hand side has often combinatorial interpretations which in turn can provide a non-trivial information about the arc space.

In Chapter 2 we introduce the basics of vertex superalgebras and (twisted) arc superspaces. In Chapter 3 we generalize the notion “quasi-lisse” to the super case. In particular, we study the modularity of quasi-lisse vertex superalgebra (twisted) modules. We show that

**Theorem 1.0.1.** Suppose $V$ is quasi-lisse, and $v \in V$ is a primary vector with $L[n]v = 0$ for $n > 0$. Let $R = R(T, T_1) = \mathbb{C}[G_2(\tau)] \otimes M(T, T_1)$. Then there exists $m \in \mathbb{N}$ and $r_i(\tau) \in R(T, T_1)$, $0 \leq i \leq m - 1$, such that

\[ (q_\tau^T \frac{d}{dq_\tau^T})^m T(v) + \sum_{i=0}^{m-1} r_i(\tau)(q_\tau^T \frac{d}{dq_\tau^T})^i T(v) = 0 , \]

where $q_\tau^T = e^{2\pi i T/T}$. 

2
In Chapter 4 we investigate the map $\psi$ in the cases of affine vertex algebras, rank one lattice vertex superalgebras including the simple $N = 2$ superconformal vertex algebra at level 1. For the later case the map $\psi$ is not an isomorphism, and we make a conjecture about its kernel. In Chapter 5 we analyze in great depth principal subspaces of lattice vertex algebras and affine vertex algebras, and show that the map $\psi$ is isomorphism for many examples. For the principal subspace $W_L$ in lattice vertex algebra $V_L$, we get:

**Theorem 1.0.2.** For a lattice $L$ of rank $n$ with a positive basis, the map $\psi$ is an isomorphism for $W_L$ if and only if its positive basis satisfies $\langle \alpha_i, \alpha_i \rangle = a$, where $a = 0$ or 1 or 2, and $\langle \alpha_i, \alpha_j \rangle = b$, where $b = 0$ or 1.

Chapter 6 was based on the joint work with Dr. Milas [76] and a sequel. In this Chapter we explore the close relations between $q$-series identities and the principal subspaces from the point view of arc algebras. In particular, we derive the new character formulas of the principal subspace of type $A$ by using quantum dilogarithm and identities from quiver representations. We also prove

**Theorem 1.0.3.** For the principal subspaces in $L_{\widetilde{sp}_4}(-1, 0)$ and $L_{\widetilde{sl}_n}(-1, 0)$, the map $\psi$ are isomorphism.

In Chapter 7 we study the classically freeness of the simple $N = 1$ vertex superalgebras associated with $N = 1$ superconformal $(p, p')$-minimal models and extended Virasoro vertex algebras. And we prove following result which is a super-analog of [104, Theorem 16.13]:

**Theorem 1.0.4.** Let $p' > p \geq 2$ satisfy that $\frac{p'-p}{2}$ and $p$ are coprime positive integers. Let $L_{p,p'}^{N=1}$ be the simple $N = 1$ vertex superalgebra associated with $N = 1$ superconformal $(p, p')$-minimal model of central charge $c_{p, p'} = \frac{3}{2}(1 - \frac{2(p'-p)^2}{pp'})$. Then the map $\psi$ is an isomorphism if and only if $(p, p') = (2, 4k), (k \in \mathbb{Z}_+)$.

**Part 2.**

Let $G/B$ be a flag variety. For each $w$ in the Weyl group $W$, and a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, one defines the variety

$$\hat{X}_{Iw} = P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_l}/B.$$
Here $P_{ij}$ is the minimal parabolic subgroup corresponding to the simple root $\alpha_{ij}$. Multiplication of all the coordinates defines a canonical map $q_{I_w} : \hat{X}_{I_w} \to G/B$, which is proper and birational over the Schubert variety $X(w)$ of $w$. This is called a Bott-Samelson resolution of $X(w)$. These resolutions play an important role in Schubert calculus and representation theory.

We are interested in $h_T(\hat{X}_{I_w})$, where $h_T$ is an (equivariant) oriented cohomology theory in the sense of Levine-Morel. Examples of $h$ include the Chow group (singular cohomology) and K-theory. For any $h$, it is proved in [34, 35, 33] that, after fixing a reduced decomposition $I_w$ for each $w \in W$, the push-forward $q_{I_w*}(1)$ in $h_T(G/B)$ of the fundamental class defines a basis of $h_T(G/B)$ over the base ring $h_T(pt)$. This enables the authors of loc.it. to construct the algebraic replacement of $h_T(G/B)$, and provides a standard setting for generalized Schubert calculus. For further study on equivariant oriented cohomology of $T$-varieties following this method, please refer to [42, 56, 71, 31, 109].

Let us consider $h_T(\hat{X}_I)$ for a general sequence $I = (i_1, ..., i_l)$. Note that the set $\hat{X}_I^T$ of $T$-fixed points of $\hat{X}_I$ is in bijection to the power set of $[l] = \{1, 2, ..., l\}$. Denote by $j : \hat{X}_I^T \to \hat{X}_I$ the canonical embedding. Our main result is the following:

**Theorem 1.0.5.** (Corollary 8.3.4) For any sequence $I$, the pull-back to $T$-fixed points $j^* : h_T(\hat{X}_I) \to h_T(\hat{X}_I^T)$ is injective.

Furthermore, we show that elements in the image of $j^*$ satisfy the Goresky-Kottwitz-MacPherson (GKM) description (see Theorem 8.3.5). Indeed, in the case where the sequence $I = (i_1, ..., i_l)$ consists of distinct $i_j$’s, we prove that the GKM description uniquely characterizes the image (Theorem 8.3.6).

Let us mention the idea of the proof briefly. Since $\hat{X}_I$ is constructed as a tower of $\mathbb{P}^1$-bundles, there are canonically defined algebra generators $\eta_j \in h_T(\hat{X}_I)$ corresponding to each parabolic subgroup $P_{ij}$ in $\hat{X}_I$. Each $\eta_j$ satisfies certain quadratic relation. Therefore, for each subset $L$ of $[l]$, denoting by $\eta_L$ the product of $\eta_j$ with $j$ in $L$, then $\{\eta_L | L \subseteq [l]\}$ forms a basis of $h_T(\hat{X}_I)$.

We compute the restriction $j^*(\eta_L)$ explicitly (Theorem 8.3.3). The computation uses the characteristic map $c : h_T(pt) \to h_T(\hat{X}_I)$ induced by the map sending a character $\lambda$ of $T$ to the first Chern class of the associated line bundle over $\hat{X}_I$. We then use the explicit
formula of $j^*(\eta_L)$ to prove Theorem 1.0.5 and use the GKM description to characterize the image of $j^*$.

As another application of the computation of $j^*(\eta_L)$, we also compute the push-forward of $\eta_L$ via the canonical map $q_I : \hat{X}_I \to G/B$. We show that the push-forward $q_{I*}(\eta_L)$ coincides with the Bott-Samelson class corresponding to the sequence $I \setminus L$.

For future applications, one would apply the restriction formula (Theorem 8.3.3) and the push-forward formula (Theorem 8.4.4) in the study of motivic Chern (mC) classes in K-theory. MC classes are certain K-theory classes associated to constructible subsets of $T$-varieties. For details, please refer [5, 96, 95]. They are closely related with the K-theoretic stable basis of Springer resolutions, defined by Maulik-Okounkov [81, 88] and studied in [100, 99]. Indeed, Mihalcea has some recent work on the relationship between push-forward of MC classes of Bott-Samelson varieties and the Kazhdan-Lusztig basis of Hecke algebra. We hope to apply the computation of the last Chapter on understanding this relationship.

The last Chapter is organized as follows: In Section 8.1 we recall necessary notions of equivariant oriented cohomology theory, formal group algebra, and the characteristic map $c$. In Section 8.2 we recall some basic facts about Bott-Samelson varieties. In Section 8.3 we compute the restriction formula (Theorem 8.3.3), which was used to prove the injectivity of the pull-back map $j^*$ and the GKM description (Theorem 8.3.5). In Section 8.4 we compute the push-forward of the basis $\{\eta_L\}$ onto $h_T(G/B)$.

The Chapter 4, 5, 7, 8 are based on papers [75], [76] and [78].
CHAPTER 2
Vertex superalgebras and arc superspaces

2.1 Vertex superalgebras

Definition 2.1.1. Let $V$ be a superspace, i.e., a $\mathbb{Z}_2$-graded vector space, $V = V_0 \oplus V_1$, where $\{0, 1\} = \mathbb{Z}_2$. If $a \in V_{p(a)}$, we say that the element $a$ has parity $p(a) \in \mathbb{Z}_2$.

A field is a formal series of the form $a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$ where $a(n) \in \text{End}(V)$ and for each $v \in V$ one has

$$a(n)v = 0$$

for $n \gg 0$.

We say that a field $a(z)$ has parity $p(a) \in \mathbb{Z}_2$ if

$$a(n)V_\alpha \in V_{\alpha + p(a)}$$

for all $\alpha \in \mathbb{Z}_2$, $n \in \mathbb{Z}$.

A vertex superalgebra contains the following data: a vector space of states $V$, the vacuum vector $1 \in V_0$, derivation $T$, and the state-field correspondence map

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1},$$

satisfying the following axioms:

- (translation coinvariance): $[T, Y(a, z)] = \partial Y(a, z)$,
- (vacuum): $Y(1, z) = Id_V$, $Y(a, z)1|_{z=0} = a$,
- (locality): $(z - w)^N Y(a, z)Y(b, w) = (-1)^{p(a)p(b)}(z-w)^NY(b, w)Y(a, z)$ for $N \gg 0$.

In particular, a vertex superalgebra $V$ is called supercommutative if $a(n) = 0$ for $n \in \mathbb{N}$. It is well-known that the category of commutative vertex superalgebras is equivalent with the
category of unital commutative associative superalgebras equipped with an even derivation.

We say that a vertex superalgebra $\mathcal{V}$ is generated by a subset $\mathcal{U} \subset \mathcal{V}$ if any element of $\mathcal{V}$ can be written as a finite linear combination of terms of the form

$$b_{(i)}^1 b_{(i_2)}^2 \cdots b_{(i_n)}^n 1$$

for $b^k \in \mathcal{U}$, $i_k \in \mathbb{Z}$, and $n \in \mathbb{N}$. If every element of $\mathcal{V}$ can be written with $i_k \in \mathbb{Z}_-$, we write $\mathcal{V} = \langle \mathcal{U} \rangle_S$ and say $\mathcal{V}$ is strongly generated by $\mathcal{U}$.

**Example 2.1.2.** Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra with a nondegenerate even supersymmetric invariant bilinear form $(\cdot, \cdot)$. We can associate the affine Lie superalgebra $\hat{\mathfrak{g}}$ to the pair $(\mathfrak{g}, (\cdot, \cdot))$.

Its universal vacuum representation of level $k$, $V_{\hat{\mathfrak{g}}}(k, 0)$, is a vertex superalgebra. In particular, when $\mathfrak{g}$ is a simple Lie algebra, $V_{\hat{\mathfrak{g}}}(k, 0)$ has an unique maximal ideal $I_{\hat{\mathfrak{g}}}(k, 0)$, and $L_{\hat{\mathfrak{g}}}(k, 0) = V_{\hat{\mathfrak{g}}}(k, 0)/I_{\hat{\mathfrak{g}}}(k, 0)$ is also a vertex superalgebra.

**Example 2.1.3.** To any $n$ dimensional superspace $A$ with a non-degenerate anti-supersymmetric bilinear form $(\cdot, \cdot)$, we can associate a Lie superalgebra $C_A$. If we fix a basis of $A$:

$$\{\phi^1, \cdots, \phi^n\},$$

the free fermionic vertex algebra $\mathcal{F}$ associated to $A$ is a vertex superalgebra strongly generated by $\phi^i_{(-\frac{1}{2})} 1$ ($i = 1, \cdots, n$), where $Y(\phi^i_{(-\frac{1}{2})} 1, z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \phi^i_{(n)} z^{-n-\frac{1}{2}}$.

**Definition 2.1.4.** A vertex superalgebra $\mathcal{V}$ is called a vertex operator superalgebra if it is $\frac{1}{2} \mathbb{Z}$-graded, $\mathcal{V} = \coprod_{n \in \frac{1}{2} \mathbb{Z}} \mathcal{V}(m)$, with a conformal vector $\omega$ such that the set of operators $\{L(n), id_{\mathcal{V}}\}_{n \in \mathbb{Z}}$ with $L(n) = \omega(n+1)$ defines a representation of Virasoro algebra on $\mathcal{V}$; that is

$$[L(n), L(m)] = (m-n)L(m+n) + \frac{m^3-m}{12} \delta_{m+n,0} c_V$$

for $m, n \in \mathbb{Z}$. We call $c_V$ the central charge of $\mathcal{V}$. We require that $L(0)$ is diagonalizable.
and it defines the $\frac{1}{2}\mathbb{Z}$ grading - its eigenvalues are called (conformal) weights. In several examples we will encounter $\frac{1}{2}\mathbb{Z}$-graded vertex superalgebras without a conformal vector. For this reason, we define the character or graded dimension as

$$\text{ch}[V](q) = \sum_{m\in\frac{1}{2}\mathbb{Z}} \dim(V(m))q^m.$$  

As we do not care about modularity here, we suppress the $q^{-\frac{c}{24}}$ factor and also view $q$ as a formal variable.

**Example 2.1.5.** [72] Let $Vir$ denote the Virasoro Lie algebra. Then the universal $Vir$-module $V_{Vir}(c, 0)$ has a natural vertex operator algebra structure with central charge $c$.

**Example 2.1.6.** [65] The universal vertex superalgebra associated with the $N = 1$ Neveu-Schwarz Lie superalgebra will be denoted by $V_{c}^{N=1}$, where $c$ is the central charge. It is a vertex operator superalgebra strongly generated by an odd vector $G_{(-\frac{3}{2})}1$ and the conformal vector $L_{(-2)}1$.

**Example 2.1.7.** [65] The universal vertex superalgebra associated with the $N = 2$ superconformal Lie algebra will be denoted by $V_{c}^{N=2}$. It is a vertex operator superalgebra strongly generated by two odd vectors $G_{(-\frac{3}{2})}1, G_{(-\frac{3}{2})}1$ and two even vectors $L_{(-2)}1, J_{(-1)}1$.

**Definition 2.1.8.** A commutative vertex superalgebra $V$ is called a vertex Poisson superalgebra if it is equipped with a linear operation,

$$V \to \text{Hom}(V, z^{-1}V[z^{-1}]),\ a \to Y_{-}(a, z) = \sum_{n\in\mathbb{N}} a_{(n)}z^{-n+1},$$

such that

- $(Ta)_n = -na_{(n-1)}$,

- $a_{(n)}b = \sum_{j\in\mathbb{N}} (-1)^{n+j+1} (\frac{(-1)^{p(a)p(b)}}{j!}) T^{j}(b_{(n+j)}a),$

- $[a_{(m)}, b_{(n)}] = \sum_{j\in\mathbb{N}} \binom{m}{j} (a_{(j)}b)_{(m+n-j)},$

- $a_{(n)}(b \cdot c) = (a_{(n)}b) \cdot c + (\frac{(-1)^{p(a)p(b)}}{b \cdot (a_{(n)}c)},$
for \( a, b, c \in V \) and \( n, m \in \mathbb{N} \).

A vertex Lie superalgebra structure on \( V \) is given by \((V,Y,T)\). So we can also say that a vertex Poisson superalgebra is a commutative vertex superalgebra equipped with a vertex Lie superalgebra structure. In fact, we can obtain a vertex Poisson superalgebra from any vertex superalgebra through standard filtration or Li’s filtration. Following [74], we can define a decreasing sequence of subspaces \( \{F_n(V)\} \) of the superalgebra \( V \), where for \( n \in \mathbb{Z} \),

\[
F_n(V) = \text{linearly spanned by the vectors } u^{(1)}_{(-1-k_1)} \cdots u^{(r)}_{(-1-k_r)} 1
\]

for \( r \in \mathbb{Z}_+, u^{(1)}, \ldots, u^{(r)} \in V, k_1, \ldots, k_r \in \mathbb{N} \) with \( k_1 + \cdots + k_r \geq n \). Then Li’s filtration of \( V \) is given by

\[
V = F_0(V) \supset F_1(V) \supset \cdots
\]
satisfying

\[
u_{(n)} v \in F_{r+s-n-1}(V) \quad \text{for} \quad u \in F_r(V), v \in F_s(V), r, s \in \mathbb{N}, n \in \mathbb{Z},
\]

\[
u_{(n)} v \in F_{r+s-n}(V) \quad \text{for} \quad u \in F_r(V), v \in F_r(V), r, s, n \in \mathbb{N}.
\]

The corresponding associated graded algebra \( gr^F(V) = \coprod_{n \in \mathbb{N}} F_n(V)/F_{n+1}(V) \) is a vertex Poisson superalgebra. Its vertex Lie superalgebra structure is given by:

\[
T(u + F_{r+1}(V)) = Tu + F_{r+2}(V),
\]

\[
Y_-(u + F_{r+1}(V), z)(v + F_{s+1}(V)) = \sum_{n \in \mathbb{N}} (u_{(n)} v + F_{r+s-n+1}(V)) z^{-n-1},
\]

for \( u \in F_r(z), v \in F_s(z) \) with \( r, s \in \mathbb{N} \). For the standard increasing filtration \( \{G_n(V)\} \), we also have the associated graded vertex superalgebra \( gr^G(V) \). In [6 Proposition 2.6.1], it was shown that

\[
gr^F(V) \cong gr^G(V)
\]
as vertex Poisson superalgebras. Thus we sometimes drop the upper index \( F \) or \( G \) for brevity.
According to [74], we know that
\[ F_n(V) = \{ u_{(-1-i)}v | u \in V, i \in \mathbb{Z}_+, v \in F_{n-i}(V) \}. \]

In particular, \( F_0(V)/F_1(V) = V/C_2(V) = R_V \) is a Poisson superalgebra according to [111]. Its Poisson structure is given by
\[ u \cdot v = u_{(-1)}v, \]
\[ \{ u, v \} = u_{(0)}v, \]
for \( u, v \in V \) where \( u = u + C_2(V) \). It was shown in [74, Corollary 4.3] that \( gr^F(V) \) is generated by \( R_V \) a differential algebra.

Next, let us compute the \( C_2 \)-algebras for some simple examples.

**Example 2.1.9.** Following notations in Example [2.1.3] let \( F \) be a free fermionic vertex superalgebra associated with an \( n \)-dimensional superspace \( A \). Clearly, the \( C_2 \)-algebra of \( F \) is
\[ R_F = \mathbb{C}[\phi_{(-\frac{1}{2})}^1, \ldots, \phi_{(-\frac{1}{2})}^n], \]
where \( \phi_{(-\frac{1}{2})}^i \) is even (resp. odd) if \( \phi^i \) is even (resp. odd) in \( A \).

**Example 2.1.10.** According to [104, 16.16], for a simple affine vertex algebras \( L_{\widehat{g}}(k,0) \), \( k \in \mathbb{N} \), where \( g \) is a simple Lie algebra, we have:
\[ R_{L_{\widehat{g}}(k,0)} = \mathbb{C}[u_{(-1)}^1, u_{(-1)}^2, \ldots, u_{(-1)}^n]/\langle U(g) \circ ((e_\theta)(-1))^{k+1} 1 \rangle, \]
where \( \{ u^1, u^2, \ldots, u^n \} \) is a basis of \( g \), \( \theta \) is the highest root of \( g \), and \( \circ \) represents the adjoint action. In particular, when \( g = sl_2 \), we have
\[ R_{L_{\widehat{sl_2}}(k,0)} \cong \mathbb{C}[e, f, h]/\langle f^i \circ e^{k+1} | i = 0, \ldots, 2k + 2 \rangle, \]
where \( e, f, h \) correspond to \( e_{(-1)}^1, f_{(-1)}^1, h_{(-1)}^1 \).

**Example 2.1.11.** For any simple Virasoro algebras \( L_{Vir}(c_{(p,p')},0) \), where \( c_{(p,p')} = 1 - \frac{6(p-p')^2}{pp'} \)
where \( p > p' \geq 2 \) and \( p, p' \) are coprime, according to [19, 104] its \( C_2 \)-algebra is isomorphic to \( \mathbb{C}[x]/\langle x^{(p-1)(p'-1)} \rangle \), where \( x \) corresponds to \( \omega = L_{(-2)}^1 \).
Example 2.1.12. The $C_2$-algebra of $V_c^{N=1}$ is $R_{V_c^{N=1}} = \mathbb{C}[x, \theta]$, where $x$ and $\theta$ correspond to even vector $L_{(-2)}1$ and odd vector $G_{(-\frac{3}{2})}1$, respectively.

Example 2.1.13. The $C_2$-algebra of $V_c^{N=2}$ is $\mathbb{C}[x, y, \theta_1, \theta_2]$ where $x, y, \theta_1, \theta_2$ correspond to $L_{(-2)}1$, $J_{(-1)}1$, $G_{(-\frac{3}{2})}^+$ and $G_{(-\frac{3}{2})}^-$, respectively. Here $\theta_1, \theta_2$ are odd variables.

### 2.2 Arc superspaces

A general introduction to the theory of supermanifolds and superschemes can be found in [79]. We are going to follow the setting in [66].

**Definition 2.2.1.** A *superspace* is a locally ringed space $(X, \mathcal{O}_X)$, where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of supercommutative rings on it such that the stalk $\mathcal{O}_x = \mathcal{O}_{X,x}$ at any point $x \in X$ is a local ring. A morphism between two superspces is a graded morphism (with respect to $\mathbb{Z}_2$ grading) of locally ringed spaces. A *superscheme* is a superspace $(X, \mathcal{O}_X)$ such that $(X, \mathcal{O}_{X,\bar{\mathbb{F}}})$ is an ordinary scheme, while $\mathcal{O}_{M,\bar{\mathbb{F}}}$ is a quasi-coherent sheaf of $\mathcal{O}_{X,0}$-module. A superscheme is *quasicompact* if underlying topological space is quasicompact. We denote the category of superspaces and superschemes by $\text{Ssp}$ and $\text{Ssch}$.

Given a superscheme $X$ we denote its underlying topological space by $\underline{X}$. The even part of $X$ is defined to be $X_{\text{rd}} = (\underline{X}, \mathcal{O}_X/(\mathcal{O}_X + (\mathcal{O}_X)^2))$ and the reduced scheme is $X_{\text{red}} = (\underline{X}, \mathcal{O}_{X,\bar{\mathbb{F}}}/\sqrt{\mathcal{O}_{X,\bar{\mathbb{F}}}})$, where $\sqrt{\mathcal{O}_{X,\bar{\mathbb{F}}}}$ is the sheaf of all nilpotent elements in structure sheaf $\mathcal{O}_{X,\bar{\mathbb{F}}}$. Note $X_{\text{rd}}$ and $X_{\text{red}}$ are both ordinary schemes.

**Example 2.2.2.** Let $R^{n|m} = \mathbb{C}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_m]$ be a polynomial superalgebra where

\[
x_1, \ldots, x_n
\]

are ordinary variables and

\[
\theta_1, \ldots, \theta_m
\]

are odd variables, then $\mathbb{A}^{n|m} = (\text{Spec}(R^{n|m}), R^{n|m})$ is a quasicompact affine superscheme of dimension $n|m$. It is clear that every superscheme of dimension $n|m$ is locally isomorphic to an affine superscheme. In particular, every $N = m$ supercurve is locally isomorphic to $\mathbb{A}^{1|m}$. Readers can refer to [59] for the general definition of supercurves.
We define a super-ind-scheme as a filtering inductive limit of quasicompact super-schemes and their closed immersions. And the category $\text{Ssch}$ is a full subcategory of super-ind-schemes $\text{Isch}$. Let $\epsilon : R^{1|m} \to \mathbb{C}$ be the augmentation morphism which maps all variables to 0. Let $(D^{1|m})^{(n)} = \text{Spec}(R^{1|m}/(\ker(\epsilon))^{n+1})$. We denote by $\widehat{R^{1|m}}$ the $\ker(\epsilon)$-completion of the polynomial ring. The truncation morphism $R^{1|m}/(\ker(\epsilon))^{n+2} \to R^{1|m}/(\ker(\epsilon))^{n+1}$ induces a closed embedding $(D^{1|m})^{(n)} \to (D^{1|m})^{(n+1)}$. Then the ind-super-scheme $D^{1|m} = \text{Spec}(\widehat{R^{1|m}}) = \lim_{n \to \infty} (D^{1|m})^{(n)}$ is called a $N = m$ superdisc.

Fix a separated superscheme of finite type over $k$. Let $\text{Ssch}/k$ be the category of superschemes over a field $k$, and $\text{Set}$ be a category of sets. Define a family of contravariant functors $F_m : \text{Ssch}/k \to \text{Set}$ by

$$F_m(Y) = \text{Hom}_k(Y \times_k (D^{1|1})^{(m)}, X).$$

The proof of the following Proposition under non-super setting is in [60], and it can be proved similarly in super case.

**Proposition 2.2.3.** The functors $F_m$ is representable by superschemes $X_m$.

Note $X_1$ is the total tangent space of $X$. We call the superschemes $X_m$ the $m$-jet superscheme. The embeddings $(D^{1|1})^{(m)} \hookrightarrow (D^{1|0})^{(m+1)}$ induces the projections $X_{m+1} \to X_m$, and we define the arc superspace of $X$ as

$$X_\infty = \lim_{m \to \infty} X_m.$$

The embedding $(D^{1|0})^{(0)} \hookrightarrow D^{1|0}$ induces the projection $\pi_\infty : X_{m+1} \to X$. We call the coordinate rings of $X_m$, $X_\infty$ and the fiber of $\pi_\infty$ over $p \in X$ the $m$-jet superalgebra, arc superalgebra and focussed arc superalgebra at $p$ of $X$, respectively.

For an affine superscheme, we can explicitly describe its jet superscheme and arc superspaces. Here we closely follow [6] and [21].

Let $f_1, f_2, \ldots, f_k$ be $\mathbb{Z}_2$-homogeneous elements in the polynomial superalgebra $R^{n|l}$. We will describe the jet superalgebra of the quotient superalgebra:

$$R = \frac{\mathbb{C}[x_1, x_2, \ldots, x_n, \theta_1, \ldots, \theta_l]}{\langle f_1, f_2, \ldots, f_k \rangle}.$$
Firstly, let us introduce new even variables \( x_j, (-\Delta_j - i) \) and odd variables \( \theta_{j', (-\Delta_{j'} - i)} \) for \( i = -1, 0, \ldots, m \), where \( \Delta_j \) and \( \Delta_{j'} \) are weights of \( x_j, (-\Delta_j) \) and \( \theta_{j', (-\Delta_{j'})} \). In most cases, we will assume that the weight of each variable is 1, although in some cases the odd weight can be shifted by \( \frac{1}{2} \). We define an even derivation \( T \) on

\[
\mathbb{C}[x_j, (-\Delta_j - i), \theta_{j', (-\Delta_{j'} - i)}] \mid -1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq j' \leq l
\]

as

\[
T(x_j, (-\Delta_j - i)) = \begin{cases} 
(-\Delta_j - i)x_j, & \text{for } i \leq m - 1 \\
0, & \text{for } i = m,
\end{cases}
\]

and

\[
T(\theta_{j', (-\Delta_{j'} - i)}) = \begin{cases} 
(-\Delta_{j'} - i)\theta_{j', (-\Delta_{j'} - i)} & \text{for } i \leq m - 1 \\
0 & \text{for } i = m.
\end{cases}
\]

Here we identify \( x_j \) and \( \theta_{j'} \) with \( x_j, (-\Delta_j + 1) \) and \( \theta_{j', (-\Delta_{j'} + 1)} \), respectively. Set

\[
J_m(R) = \frac{\mathbb{C}[x_j, (-\Delta_j - i), \theta_{j', (-\Delta_{j'} - i)}] \mid -1 \leq i \leq m, \ 1 \leq j \leq n, \ 1 \leq j' \leq l}{\langle T^j f_i \mid i = 1, \ldots, n, \ j \in \mathbb{N} \rangle}.
\]

Then the \( m \)-jet superscheme of \( X = \text{Spec}(R) \) is \( \text{Spec}(R_m) \). The arc superalgebra of \( R \) is

\[
J_\infty(R) = \lim_{m \to \infty} R_m
\]

\[
= \frac{\mathbb{C}[x_j, (-\Delta_j - i), \theta_{j', (-\Delta_{j'} - i)}] \mid -1 \leq i, \ 1 \leq j \leq n, \ 1 \leq j' \leq l}{\langle T^j f_i \mid i = 1, \ldots, k, \ j \in \mathbb{N} \rangle}.
\]

The arc superalgebra is a differential commutative superalgebra. The focused arc superalgebra at 0 is \( J_\infty^0(R) = J_\infty(R)_{x_j, (-\Delta_j + 1) = \theta_{j', (-\Delta_{j'} + 1)} = 0} \).

We denote the ideal

\[
\langle T^j f_i ; i = 1, \ldots, k, j \in \mathbb{N} \rangle
\]

by \( \langle f_1, \ldots, f_k \rangle_\theta \). Later, we sometimes write \( x(j) \) as \( x(j) \). The arc superspace of \( X \) is

\[
X_\infty = \text{Spec}(J_\infty(R)).
\]

We define the weight of each variable \( u_{(-\Delta_j)} \) to be \( \Delta + j \), where \( u = x \) or \( \theta \) if \( \Delta + j \geq 0 \) and
0 otherwise. Then $J_\infty(R) = \prod_{m \in \frac{1}{2}\mathbb{Z}} (J_\infty(R))_{(m)}$, where $(J_\infty(R))_{(m)}$ is the set of all elements in arc superalgebra with weight $m$. We define Hilbert series of $J_\infty(R)$ as:

$$HS_q(J_\infty(R)) = \sum_{m \in \frac{1}{2}\mathbb{Z}} \dim(J_\infty(R))_{(m)} q^m.$$  

When the quotient ideal of $R$ is generated by homogeneous polynomials, the arc superspace of $X$ and focussed arc superspace of $X$ over 0 are isomorphic.

**Proposition 2.2.4.** Let $I$ be a homogeneous ideal of $R$. We have the following differential algebra isomorphism:

$$J_\infty(R) \cong J^0_\infty(R).$$

**Proof.** We assume $R$ has no odd variable, and the general case can be proved similarly. Suppose $I$ is generated by homogeneous polynomials $P_1, \cdots, P_k$ of weight $d_1, \cdots, d_k$. By direct calculation, we get the following formula:

$$T^j(P_i(x_1, \cdots, x_n))|_{x_1,(-1)=\cdots=x_n,(-1)=0} = \begin{cases} 
0 & \text{if } j < d_i \\
\frac{j!}{(j-d_i+1)!} T^{j-d_i} P_i(T(x_1), \cdots, T(x_n)) & \text{if } j \geq d_i.
\end{cases} \quad (2.1)$$

Therefore, we can define a differential algebra homomorphism:

$$\phi : J_\infty(\mathbb{C}[x_1, x_2, \cdots, x_n]/I) \to J^0_\infty(\mathbb{C}[x_1, x_2, \cdots, x_n]/I)$$

by sending $x_{i,(0)}$ to $x_{i,(-1)}$. Clearly, this map is a bijection. The result follows. 

Following [6], $J_\infty(R)$ has a unique Poisson vertex superalgebra structure such that

$$u_{(n)}v = \begin{cases} 
\{u, v\}, & \text{if } n = 0 \\
0, & \text{if } n > 0
\end{cases}$$

for $u, v \in R \in J_\infty(R)$. 

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Furthermore, one can extend the embedding $R_V \hookrightarrow gr^F(V)$ to a surjective differential superalgebra homomorphism $J_\infty(R_V) \to gr^F(V)$. It is obvious that the map is a differential superalgebra homomorphism. It is surjective since $gr^F(V)$ is generated by $R_V$ as a differential algebra. Moreover, it was shown in [6] that this map is actually a Poisson vertex superalgebra epimorphism. From now on, we call this map $\psi$. The supervariety $\text{Specm}(R_V)$ and the superscheme $X_V = \text{Spec}(R_V)$ are called associated variety and associated scheme of vertex superalgebra $V$. These important objects are first studied and defined in [6]. Above argument implies that the arc superspaces of associated scheme of $V$ contains a closed subscheme $\text{Spec}(gr^F(V))$, which is called the singular support of $V$ and denoted by $SS(V)$.

The map $\psi$ is not necessarily injective, and it is an open problem to characterize rational vertex algebras for which $\psi$ is injective. Following [105], we give the following definition.

**Definition 2.2.5.** We call a vertex (super)algebra classically free if map $\psi$ is an isomorphism.

In this work, we will provide examples of classically free vertex (super)algebras from affine vertex algebras, principal subspaces of lattice vertex algebras, $N = 1$ superconformal vertex algebras, etc.

### 2.3 Twisted arc superspaces

Let $X$ be a separated superscheme of finite type over $\mathbb{C}$. Let $g : X \to X$ be a linear automorphism of order $m$. In this section we introduce the concept $g$-twisted superspace of $X$.

According to the definition of arc superspaces, we have $\text{Hom}_{\text{Sch}}(\text{Spec}(\mathbb{C}), X_\infty) = \text{Hom}_{\text{Sch}}(D^{1\dagger}, X)$. We can identify the superdisc $D^{1\dagger}$ with $\text{Spec}(\mathbb{C}[\frac{t^{1\dagger}}{\theta}])$. For $m$-th root of unity $\xi_m$ we have an automorphism $\varphi_\lambda$ of $\mathbb{C}[\frac{t^{1\dagger}}{\theta}]$ determined by $t^{1\dagger}_m \mapsto \xi_m t^{1\dagger}_m$. Since

$$\text{Hom}_{\text{Sch}}(D^{1\dagger}, X) \cong \text{Hom}_{\text{Rings}}(\Gamma(X, \mathcal{O}_X), \mathbb{C}[x][\theta]),$$

$\varphi_\lambda$ induces a natural automorphism $\phi_\lambda$ of superarcs in $X$ by composing

$$\gamma \in \text{Hom}_{\text{Rings}}(\Gamma(X, \mathcal{O}_X), \mathbb{C}[x][\theta])$$

with $\varphi_\lambda$. 

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Definition 2.3.1. The $g$-twisted arc superspaces of $X$, $X^g_{\infty}$, is the fixed sub-superscheme of $X_{\infty}$ under the automorphism $g^{-1} \circ \phi_\lambda$.

Example 2.3.2 ([101]). [Affine case] Let $R^{nl}_g$ be the quotient polynomial ring as we defined in previous section. Suppose $g$ is a linear automorphism of $R$ of order $m$. After a linear change of coordinates we may diagonalize $g$ such that its action is given by $g(x_j) = \xi_m^j x_j$ (1 $\leq$ $j$ $\leq$ $n$) and $g(\theta_{j'}) = \xi_m^{j'} \theta_{j'}$ (1 $\leq$ $j'$ $\leq$ $l$). We introduce new even variables $x_{j,(-\Delta_j + \frac{\alpha_j}{m} - i)}$ and odd variables $\theta_{j',(-\Delta_{j'} + \frac{\beta_{j'}}{m} - i)}$ for $i \geq -1$. We define an even derivation $T_g$ on

$$\mathbb{C}[x_{j,(-\Delta_j + \frac{\alpha_j}{m} - i)}, \theta_{j',(-\Delta_{j'} + \frac{\beta_{j'}}{m} - i)} \mid -1 \leq i, 1 \leq j \leq n, 1 \leq j' \leq l]$$

as

$$T_g(x_{j,(-\Delta_j + \frac{\alpha_j}{m} - i)}) = (-\Delta_j + \frac{\alpha_j}{m} - i) x_{j,(-\Delta_j + \frac{\alpha_j}{m} - i-1)} \quad \text{for} \ i \geq -1$$

and

$$T_g(\theta_{j',(-\Delta_{j'} + \frac{\beta_{j'}}{m} - i)}) = (-\Delta_{j'} + \frac{\beta_{j'}}{m} - i) \theta_{j',(-\Delta_{j'} + \frac{\beta_{j'}}{m} - i-1)} \quad \text{for} \ i \geq -1$$

Here we identify $x_j$ and $\theta_{j'}$ with $x_{j,(-\Delta_j + \frac{\alpha_j}{m} + 1)}$ and $\theta_{j',(-\Delta_{j'} + \frac{\beta_{j'}}{m} + 1)}$, respectively. The coordinate ring of $(\text{Spec}R)^g_{\infty}$ is

$$\mathbb{C}[x_{j,(-\Delta_j + \frac{\alpha_j}{m} - i)}, \theta_{j',(-\Delta_{j'} + \frac{\beta_{j'}}{m} - i)} \mid -1 \leq i, 1 \leq j \leq n, 1 \leq j' \leq l] / \langle T_g f_i \mid i = 1, \ldots, k, \ j \in \mathbb{N} \rangle$$

2.4 Complete lexicographic ordering

Following [45], we define the complete lexicographic ordering on a basis or spanning set of the arc superalgebra. Given a jet superalgebra

$$J_\infty(\mathbb{C}[y_1, y_2, \ldots, y_n]/I) = \mathbb{C}[y_{1,(-\Delta_1 - i)}, \ldots, y_{n,(-\Delta_n - i)} \mid i \in \mathbb{N}]/I_\infty,$$

where $\Delta_i$ is the weight of $y^i$, we can first define an ordering of all variables in the following way:

$$y_{1,(-\Delta_1)} < y_{2,(-\Delta_2)} < \ldots < y_{n,(-\Delta_n)} < y_{1,(-\Delta_1 - 1)} < y_{2,(-\Delta_2 - 1)} < \ldots$$

Definition 2.4.1. A monomial $u$ of $J_\infty(\mathbb{C}[y_1, y_2, \ldots, y_n]/I)$ is called an ordered monomial if

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it is of the form:

\[(y_n,(-\Delta_n-m))^{a_n+1} \cdots (y_1,(-\Delta_1-m))^{a_1+1} \cdots (y_n,(-\Delta_n))^{a_n} \cdots (y_2,(-\Delta_1))^{a_2}(y_1,(-\Delta_1))^{a_1},\]

where \(m \in \mathbb{Z}_+\) and \(a_j \in \mathbb{N}\).

It should be clear that all ordered monomials form a spanning set of the jet superalgebra. Then let us define the multiplicity of an ordered monomial as

\[\mu(u) = \sum_{i=1}^{m+1} (a_i^1 + a_i^2 + \ldots + a_i^n).\]

Given two arbitrary ordered monomials

\[u = y_n,(-\Delta_n-m))^{a_n+1} \cdots (y_1,(-\Delta_1-m))^{a_1+1} \cdots (y_n,(-\Delta_n))^{a_n} \cdots (y_2,(-\Delta_1))^{a_2}(y_1,(-\Delta_1))^{a_1}\]

and

\[v = y_n,(-\Delta_n-m))^{b_n+1} \cdots (y_1,(-\Delta_1-m))^{b_1+1} \cdots (y_n,(-\Delta_n))^{b_n} \cdots (y_2,(-\Delta_1))^{b_2}(y_1,(-\Delta_1))^{b_1},\]

we define a complete lexicographic ordering as following: If \(\mu(u) < \mu(v)\), we say that \(u < v\). If \(\mu(u) = \mu(v)\), we compare exponents of

\[y_1,(-\Delta_1), y_2,(-\Delta_1), \ldots, y_n,(-\Delta_n), \ldots, y_1,(-\Delta_1-m), y_n,(-\Delta_n-m)\]

in this order. Namely, we say \(v < u\) if \(a_1^1 < b_1^1\); if they are equal, we then compare \(a_2^1\) and \(b_2^1\), and so on. Given a polynomial \(f\), we call the greatest monomial among all its terms with respect to the complete lexicographic ordering the leading term of \(f\).
CHAPTER 3
Quasi-lisse vertex superalgebras

3.1 Quasi-lisse vertex superalgebras

Given a vertex operator superalgebra $V$, one can define its ordinary module as following.

**Definition 3.1.1.** An ordinary $V$-module is a $\lambda$-graded ($\Lambda \in \mathbb{C}$) vector space $M = \oplus_{\lambda \in \Lambda} M_{(\lambda)}$ and a linear map

$$
V \rightarrow \text{End}(M)[[z, z^{-1}]],
$$

$$
a \rightarrow Y_M(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}
$$

satisfying:

- $Y_M(1, z) = I_M$, and setting $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, we have

$$
[L(m), L(n)] = (m - n)L(m+n) + \delta_{m+n,0} \frac{m^3 - m}{12} c,
$$

$$
Y_M(L(-1)a, z) = \frac{d}{dz} Y_M(a, z), \text{ for every } a \in V.
$$

- *(locality)*: $(z - w)^N Y_M(a, z)Y_M(b, w) = (-1)^{p(a)p(b)}(z - w)^N Y_M(b, w)Y_M(a, z)$ for every $a, b \in V$.

- *(associativity)* For large enough $k$,

$$
(z_1 + z_2)^k Y_M(a, z_1 + z_2)Y_M(b, z_2)w = (z_1 + z_2)^k Y_M(Y(a, z_1)b, z_2)w, \text{ for every } a, b \in V.
$$

(3.1)

- dim$M_{(\lambda)} < \infty$,

- For any $\lambda$, $M_{(\lambda+n)} = 0$ if $n$ is large enough,
For $m \in M(\lambda)$, $L(0)(m) = \lambda m$.

Following [8], we introduce the following definition.

**Definition 3.1.2.** A finitely strongly generated vertex (super)algebra $V$ is called *lisse* if $\dim \text{Spec}(R_V) = 0$. It is called *quasi-lisse* if

- The poisson variety $(\tilde{X}_V)_{\text{red}}$ has finitely many symplectic leaves.
- The $O_{X_V,T}$ is a coherent sheaf of $O_{X_V,T}$-module.

The following result is from [43].

**Theorem 3.1.3.** Let $R$ be a finitely generated Poisson superalgebra and $R_T$ is a finitely generated $R_{\tilde{T}}$-module, and suppose that $(\text{Spec}(R))_{\text{red}}$ has finitely many symplectic leaves, then

$$\dim R/\{R, R\} < \infty.$$ 

**Theorem 3.1.4 ([12]).** For a vertex (super)algebra $V$ its singular support is isomorphic with the arc (super)space of $X_V$ as topological space if $V$ is quasi-lisse.

Given a vertex superalgebra $V = \coprod_{n \in \frac{1}{2} \mathbb{Z}} V(n)$ where $V_{\frac{1}{2}} = \coprod_{n \in \mathbb{Z}} V(n)$ and $V_T = \coprod_{n \in \mathbb{Z} + \frac{1}{2}} V(n)$, there are two binary operations defined as following: for homogeneous $a \in V$ and $b \in V$,

$$a \ast b = \begin{cases} \sum_{i \in \mathbb{N}} \binom{\text{wt}(a)}{i} a_{(i-1)} b, & \text{if } a, b \in V_{\frac{1}{2}}, \\ 0, & \text{if } a \text{ or } b \in V_T \end{cases}, \quad (3.2)$$

and

$$a \circ b = \begin{cases} \sum_{i \in \mathbb{N}} \binom{\text{wt}(a)}{i} a_{(i-2)} b, & \text{if } a \in V_{\frac{1}{2}}, \\ \sum_{i \in \mathbb{N}} \binom{\text{wt}(a)}{i} - \frac{1}{2} a_{(i-1)} b, & \text{if } a \in V_T. \end{cases} \quad (3.3)$$

Let $O(V)$ be the linear span of elements of the form $a \circ b$ in $V$. Then Zhu’s algebra $A(V)$ is defined as the quotient space $V/O(V)$ with the multiplication from $\ast$. According to [Kac],
there is a filtration \( \{ \tilde{F}_k(A(V)) \} \) on \( A(V) \) where \( \tilde{F}_k(A(V)) := \left( \bigoplus_{i \in \frac{1}{2}\mathbb{Z}, i \leq k} V(i) + O(V) \right)/O(V) \).

Its associated graded algebra,

\[
\text{gr} \tilde{F}(A(V)) = \bigoplus_{i=0}^{\infty} \frac{\tilde{F}_i(A(V))}{\tilde{F}_{i-1}(A(V))},
\]

is a commutative algebra.

**Lemma 3.1.5.** We have a surjective Poisson algebra homomorphism:

\[
\phi : (R_V)_{\tilde{\pi}} \mapsto \text{gr} \tilde{F}(A(V)), \tag{3.4}
\]

where \( \phi(a + (C_2(V))_{(p)}) = a + O(V) + \tilde{F}_{p-1}(V) \) for \( a \in ((V)_{\tilde{\pi}})_{(p)} \).

**Proof.** First, we show that the map \( \phi \) is a well-defined map. It is enough to prove

\[
(C_2(V))_{(p)} \subset O(V) + \tilde{F}_{p-1}(V).
\]

The space \((C_2(V))_{(p)}\) is spanned by the weight \( p \) elements of the form \( a_{(-2)}b \) with homogeneous elements \( a \) and \( b \). For such \( a \) and \( b \), we have \( \text{wt}(a_{(-2)}b) \leq p - 1 \) if \( i \in \mathbb{Z}_+ \). Hence

\[
a_{(-2)}b \equiv \begin{cases} 
  a \circ b \pmod{\tilde{F}_{p-1}(V)} & \text{if } a \in V_{\tilde{\pi}} \\
  0 \pmod{\tilde{F}_{p-1}(V) + O(V)} & \text{if } a \in V_{\tilde{\pi}}.
\end{cases}
\]

Therefore, \((C_2(V))_{(p)} \subset O(V) + \tilde{F}_{p-1}(V)\), and the map \( \phi \) is well-defined.

Next, let us show that \( \phi \) is a Poisson algebra homomorphism. Given two elements \( a \in (V_{\tilde{\pi}})_{(p)} \) and \( b \in (V_{\tilde{\pi}})_{(q)} \), for \( u \in a + (C_2(V))_{(p)} \) and \( v \in b + (C_2(V))_{(q)} \), we have

\[
u_{(-1)} \nu \in a_{(-1)} b + (C_2(V))_{(p+q)}
\]

and

\[
u_{(0)} \nu \in a_{(0)} b + (C_2(V))_{(p+q-1)}.
\]

Moreover, according to [111] and (3.2), (3.3), for \( u \in a + O(V) + \tilde{F}_{p-1}(V) \) and \( v \in b +
\[ O(V) + F_{q-1}(V) \text{, we have} \]
\[ u \ast v \in a_{(-1)}b + O(V) + F_{p+q-1}(V) \]
and
\[ u \ast v - v \ast u \in a_{(0)}b + O(V) + F_{p+q-2}(V). \]

So \( \phi \) is a Poisson algebra homomorphism. And the surjectivity follows from the fact that \( V_\Gamma \subset O(V) \).

\[ \textbf{Theorem 3.1.6.} \text{ Let } V \text{ be quasi-lisse vertex operator superalgebra. Then it has finitely many simple ordinary modules.} \]

\[ \textbf{Proof.} \text{ According to (3.1.5), } \text{Specm}(gr \mathfrak{A}(A(V))) \text{ is a Poisson subvariety of } \text{Specm}((R_V)_\sigma) = (X_V)_{\text{red}}, \text{ and hence has finitely many symplectic leaves. According to [43], } A(V) \text{ has finitely many simple finite-dimensional representations. The result follows from [64].} \]

3.2 Modular linear differential equation

We let \( \tau \) be an element in the upper half complex plane \( \mathbb{H} \). We first collect some facts about Eisenstein series from [111], [80] and [108].

The \textit{Eisenstein series} \( G_{2k}(\tau) \) \((k = 1, 2, 3, \cdots)\) are series
\[ G_{2k}(\tau) = -\frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n \in \mathbb{Z}_+} \sigma_{2k-1}(n)q^n, \]
where \( B_k \) is the \( k \)th Bernoulli number, \( \sigma_k(n) = \sum_{d|n} d^k \), and \( q = e^{2\pi i \tau} \in \mathbb{H} \), where \( \mathbb{H} \) is the upper half plane. For \( k \geq 2 \), \( G_{2k}(\tau) \) is a modular form of weight \( 2k \) for modular group \( SL_2(\mathbb{Z}) \). Since the character of a vertex superalgebra is a \( q \)-series, we let \( \tilde{G}_{2k}(q) \) be the \( q \)-expansion of Eisenstein series \( G_{2k}(\tau) \). We renormalize \( \tilde{G}_2(q), \tilde{G}_4(q) \text{ and } \tilde{G}_6(q) \) so that the \( q \)-expansion starts with 1, and name them \( \tilde{P}, \tilde{Q} \text{ and } \tilde{R} \), respectively. We let \( P = \tilde{P}(e^{2\pi i \tau}) \), \( Q = \tilde{Q}(e^{2\pi i \tau}) \text{ and } R = \tilde{R}(e^{2\pi i \tau}) \). Here, the \( P, Q \) and \( R \) have weight 2, 4 and 6. The polynomial algebra generated by \( P, Q \) and \( R \) is the algebra of \textit{quasi-modular forms}. All holomorphic modular forms on \( \mathbb{H} \) equal to the graded algebra \( A = \mathbb{C}[Q, R] = \oplus_k A_k \), where
$A_k$ is the subspace of $A$ spanned by all weight $k$ elements. We let $\tilde{A} = \mathbb{C}[\tilde{Q}, \tilde{R}]$. We will use the following identity later

$$\sum_{n \neq 0} \frac{n^m q^n}{1-q^n} = m! \frac{(-1)^{m+1}}{z^{m+1}} + \sum_{k \geq m+1} \binom{k-1}{m} G_k(\tau) z^{k-m-1},$$

(3.5)

where $q = e^{2\pi i \tau}$ and $q_z = e^z$.

Given an element $f \in A_k$, we define Serre derivation of weight $k$, $\vartheta_k$, on it:

$$\vartheta_k f := f' - \frac{k}{12} P(e^{2\pi i \tau}) f.$$

The Serre derivation preserves the holomorphy and modularity of $f$, i.e., $\vartheta_k f \in A_{k+2}$. The $i$th iterated Serre derivation of weight $k$ is $\vartheta_i^k = \vartheta_{k+2(i-1)} \circ \cdots \circ \vartheta_{k+2} \circ \vartheta_k$ with $\vartheta^0_k = 1$. A modular linear differential equation (MLDE) of weight $k$ is a linear differential equation

$$\vartheta^n_k f + \sum_{j=0}^{n-1} g_j \vartheta^j_k f = 0,$$

where $g_j \in A$ and the weight of $g_j$ is $2n-2j$ for each $0 \leq j \leq n-1$. The solutions of MLDE belong to $A$.

Since we will work with $q$-series later, we define formal Serre derivation $\partial_k$ and formal iterated Serre derivation $\partial^i$ of weight $k$ by

$$\partial_k f = q \frac{d}{dq} f(q) + k \tilde{G}_2(q) f(q),$$

$$\partial^i (f(q)) = \partial_{n+2i-2} (\partial^{i-1} (f(q))),$$

where $f \in A_k$ and $\partial^0 = \text{id}$.

### 3.3 n-point correlation functions on the torus

Most of materials in this Section are from [111]. Let $V$ be a vertex operator superalgebra with central charge $c$, and $M$ be an ordinary $V$-module. For an integer $n \in \mathbb{N}$, the
formal $n$-point correlation function for states $u_1, \cdots, u_n \in V$ is the formal expression

$$F_M((u_1, z_1), \cdots, (u_n, z_n), q) = tr|_M Y_M(q_1^{L(0)} u_1, q_1) \cdots Y_M(q_n^{L(0)} u_n, q_n) q^{L(0)} - \frac{c}{24},$$

where $q_i$ are formal variables.

Note 0-point correlation function equals $q^{-\frac{c}{24}} ch[M](q)$. When $n = 1$ and $u_1 \in M(k)$, we have

$$tr|_M Y(q_1^{L(0)} u_1, q_1) q^{L(0)} - \frac{c}{24} = q_1^k \sum_{m \in \mathbb{Z}} tr|_M u_1(m) q_1^{-m-1} q^{L(0)} - \frac{c}{24},$$

where the second identity follows from the axioms of ordinary module, and we denote $u_1(k)$ by $\circ(u_1)$.

For the lisse vertex operator (super)algebra $V$, one defines $n$-point correlation function

$$S_M((u_1, z_1), \cdots, (u_n, z_n), \tau) := F_M((u_1, e^{2\pi i z_1}), \cdots, (u_n, e^{2\pi i z_n}), e^{2\pi i \tau}),$$

where $z_i \in \mathbb{C}$ and $\tau \in \mathbb{H}$. Zhu showed that $S_M((u_1, z_1), \cdots, (u_n, z_n), \tau)$ converges on domain $|q| < |z_n| < \cdots < |z_1| < 1$, and is an elliptic function for each variable $z_i$ with periods 1 and $\tau$. Thus the $n$-point function is defined on a torus $\mathbb{C}/\{m \tau + n\}$ ($m, n \in \mathbb{Z}$). We can also think of formal $n$-point function $F_M((u_1, z_1), \cdots, (u_n, z_n), q)$ as a mutivariable function defined on a torus $\mathbb{C}/\{w \sim wq^n\}$ ($n \in \mathbb{Z}$). The two tori above are the same, and their local coordinates are related by $w = \phi(z) = e^{2\pi i z} - 1$. Motivated by the coordinate transformation, Zhu introduced the second vertex operator (super)algebra $(V, Y[\cdot, z], 1, \bar{\omega})$ associated to a vertex operator (super)algebra $(V, Y, 1, \omega)$, where for $v \in V$

$$Y[v, z] = Y(e^{2\pi i L(0)} v, e^{2\pi i z} - 1) = \sum_{n \in \mathbb{Z}} v[n] z^{n-1},$$

and $\bar{\omega} = 2\pi i(\omega - \frac{c}{12})$. When we study correlation functions, it is convenient to use this square bracket formalism. For instance, by using square bracket formalism we can relate 1-point
correlation functions and 2-point correlation functions as following:

\[
S_M((u_1, z_1), (u_2, z_2), \tau) = \text{tr}|_M Y_M(q_1^{L(0)}u_1, q_1)Y_M(q_2^{L(0)}u_2, q_2)q^{L(0)} - \frac{\tau}{2\pi}
\]

\[
= \text{tr}|_M Y_M(Y(q_1^{L(0)}u_1, q_1 - q_2)q_2^{L(0)}u_2, q_2)q^{L(0)} - \frac{\tau}{2\pi}
\]

\[
= \text{tr}|_M Y_M(q_2^{L(0)}Y((q_2^{-1}q_1)^{L(0)}u_1, q_2^{-1}q_1 - 1)u_2, q_2)q^{L(0)} - \frac{\tau}{2\pi}
\]

\[
= \text{tr}|_M Y_M(q_2^{L(0)}[u_1, z_1 - z_2]u_2, q_2)q^{L(0)} - \frac{\tau}{2\pi}
\]

\[
= S_M((Y[u_1, z_1 - z_2]u_2, z_2), \tau),
\]

where the second equality follows from the associativity condition (3.1), and the third equality is from the following identity by direct calculation

\[
q_i^{L(0)}Y(v, z)q_i^{-L(0)} = Y(q_i^{L(0)}v, q_i z).
\]

And we have the following Zhu recursion formula:

**Theorem 3.3.1.** [111] The following formula is true only if \( u \in V_0 \),

\[
S_M((u[-1]v, z), \tau) = \text{tr}|_M (o(u)o(v)q^{L(0)} - \frac{\tau}{2\pi})
\]

\[
+ \sum_{m \in \mathbb{Z}_+} (2\pi i)^{2m}G_{2m}(\tau)S_M((u[2m - 1]v, z), \tau),
\]

(3.7)

where \( n \in \mathbb{Z}_+ \) and \( v \in V \)

**Proof.** We assume \( q_t = e^{2\pi izt} \) (\( t = 1, 2 \)) and \( q = e^{2\pi ir} \).

For \( u \in V_k \),

\[
S_M((u, z_1), (v, z_2), \tau) = \sum_{n \in \mathbb{Z}} q_1^{-n-1+k} \text{tr}|_M (u(n)Y_M(q_2^{L(0)}v, q_2)q^{L(0)} - \frac{\tau}{2\pi}).
\]

(3.8)

When \(-n - 1 + k = 0\),

\[
\text{tr}|_M u(k-1)Y(q_2^{L(0)}v, q_2)q^{L(0)} - \frac{\tau}{2\pi} = \text{tr}|_M o(u)o(v)q^{L(0)} - \frac{\tau}{2\pi}.
\]

(3.9)
When $-n - 1 + k \neq 0$, we have

\[
tr|_M u(n)Y(q_2^{L(0)}v, q_2)q^{L(0) - \frac{c}{\pi}}
= tr|_M [u(n), Y(q_2^{L(0)}v, q_2)]q^{L(0) - \frac{c}{\pi}} + (-1)^{|u|}tr|_M (Y(q_2^{L(0)}v, q_2)u(n)q^{L(0) - \frac{c}{\pi}})
= (-1)^{|u|}q_2^{n+1-k} \sum_{m \in \mathbb{N}} (2\pi i)^{m+1} \frac{(n + 1 - k)^m}{m!} S_M((u[m]v, z_2), \tau)
+ (-1)^{|u|}(-1)^{|u|}q^{n+1-r}tr|_M Y(q_2^{L(0)}v, q_2)q^{L(0) - \frac{c}{\pi}} u(n)
= (-1)^{|u|}q_2^{n+1-k} \sum_{m \in \mathbb{N}} (2\pi i)^{m+1} \frac{(n + 1 - k)^m}{m!} S_M((u[m]v, z_2), \tau)
+ ((-1)^{|u|}q^{|u|}) (1) [-1]^{|u|}q^{n+1-r}tr|_M u(n) Y(q_2^{L(0)}v, q_2)q^{L(0) - \frac{c}{\pi}}.
\]

From this we obtain

\[
(-1)^{|u|}q_2^{n+1-k} \sum_{m \in \mathbb{N}} (2\pi i)^{m+1} \frac{(n + 1 - k)^m}{m!} S_M((u[m]v, z_2), \tau)
= (1 - (-1)^{|u|}q^{n+1-r})tr|_M u(n) Y(q_2^{L(0)}v, q_2)q^{L(0) - \frac{c}{\pi}}.
\]

Therefore, we have

\[
tr|_M u(n) Y(q_2^{L(0)}v, q_2)q^{L(0) - \frac{c}{\pi}}
= \frac{(-1)^{|u|}q_2^{n+1-k}}{(1 - (-1)^{|u|}q^{n+1-r}) \sum_{m \in \mathbb{N}} (2\pi i)^{m+1} \frac{(n + 1 - k)^m}{m!} S_M((u[m]v, z_2), \tau)}.
\]

Combining everything together, we get

\[
S_M((u, z_1), (v, z_2), \tau)
= tr|_M o(o)(u)q^{L(0) - \frac{c}{\pi}}
+ \sum_{n \neq 0} q_i^{-n-1+k} \frac{(-1)^{|u|}q_2^{n+1-k}}{(1 - (-1)^{|u|}q^{n+1-r}) \sum_{m \in \mathbb{N}} (2\pi i)^{m+1} \frac{(n + 1 - k)^m}{m!} S_M((u[m]v, z_2), \tau)}
= tr|_M o(o)(u)q^{L(0) - \frac{c}{\pi}}
+ \sum_{m \in \mathbb{Z}_+} \frac{1}{m!} S_M((u[m]v, z_2), \tau) \sum_{n+1-k \neq 0} (2\pi i)^{m+1} \frac{(n + 1 - k)^m}{m!} \frac{(-1)^{|u|}q_2^{n+1-k}}{(1 - (-1)^{|u|}q^{n+1-k})}.
\]

Now we expand the right-hand side of (3.6) and compare the constant with the right-
hand side of (3.13). We get the result.

If we replace \( u \) in above Theorem with \( L[0]u \) and we write it in terms of formal 1-point functions, we get

\[
F_M((u[-2]v, z), q) + \sum_{k=2}^{\infty} (2k - 1)G_{2k}(q)F_M((u[2k - 2]v, z), q) = 0. \tag{3.14}
\]

A vector \( v \) in \( V \) is called primary of conformal weight \( \Delta \) if \( L_n(a) = \delta_{n,0} \Delta a \) for all \( n \in \mathbb{Z}_+ \). We have an identity for \( v \in V_k \) and \( m \in \mathbb{N} \)

\[
(2\pi i)^{m+1}v[m] = m! \sum_{i \geq m} c(k, i, m)v(i), \tag{3.15}
\]

where \( c(k, i, m) \) are some constants. Hence \( v \) is primary of weight \( \Delta \) in \( (V, Y, 1, \omega) \) if and only if it is primary of weight \( \Delta \) in \( (V, Y[\cdot, z], 1, \tilde{\omega}) \). We let

\[
Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n]z^{n-1}.
\]

By using Virasoro relations, we have for \( a \in V \)

\[
L[2k - 2]L[-2]^{i - 1}a \equiv \begin{cases} 
c_k \cdot L[-2]^{i-k}a & \text{if } 2 \leq k \leq i \\
0 & \text{if } k \geq i + 1
\end{cases}. \tag{3.16}
\]

Let \( u \in V_k \),

\[
[o(u), Y(q^{L(0)}v, q)] = \sum_{i \in \mathbb{N}} \binom{k - 1}{i} Y(u_{(i)}q^{L(0)}v, q)q^{k-1-i} - Y(q^{L(0)} \sum_{i \in \mathbb{N}} \binom{k - 1}{i} u_{(i)}v, q) = (2\pi i)Y(q^{L(0)}u[0]v, q), \tag{3.17}
\]

where the last identity follows from

\[
\sum_{i \in \mathbb{N}} \binom{n}{i}u_{(i)} = \sum_{m \in \mathbb{N}} (2\pi i)^{m+1}(n + 1 - k)^m \frac{m!}{m!}u[m].
\]
By taking trace of both sides of (3.17), we get

\[ F_M((u[0]v, z), q) = 0. \] (3.18)

We will need the following Proposition.

**Proposition 3.3.2.** [39, Lemma 6.2] Let \( v \) be a primary vector of weight \( \Delta \) and \( M \) be an ordinary \( V \)-module. For each \( i \in \mathbb{Z}_+ \), there exist elements \( f_j(q) \in A \) such that

\[ S_M((L_{-2}^i v, z), \tau) = \partial^i S_M((v, z), \tau) + \sum_{j=0}^{i-1} f_j(\tau) \partial^j S_M((v, z), \tau). \] (3.19)

**Proof.** We prove this Proposition by induction on \( i \). When \( i = 1 \), the formula follows from Zhu recursion formula by letting \( u = \tilde{\omega} \). If \( i \geq 2 \), by replacing \( u, v \) with \( \tilde{\omega}, L_{-2}^{i-1} v \) in Zhu recursion formula we get the following identity

\[ S_M((L_{-2}^i v, z), \tau) = \partial S_M((L_{-2}^{i-1} v, z), \tau) + \sum_{m \in \mathbb{Z}_+} G_{2m}(\tau) S_M((L_{-2}^{2m-2} L_{-2}^{i-1} v, z), \tau). \] (3.19)

The result follows from (3.16) and the induction hypothesis. \( \Box \)

### 3.4 One-point correlation functions of quasi-lisse vertex superalgebras

We denote \( V \otimes \mathbb{C} A \) by \( V_A \). Let \([V_A, V_A]\) be the \( A \)-span of elements \( a[0]b, a,b, \) in \( V_A \). Let \( O_q(V) \) be the \( A \)-submoule of \( V_A \) generated by

\[ a[-2]b + \sum_{k=2}^\infty (2k - 1)G_{2k}(q)a[2k - 2]b, \quad (a, b \in V). \] (3.20)

**Lemma 3.4.1.** For any element \( a \) in quasi-lisse vertex operator superalgebra \( V \), there exist \( s \in \mathbb{N} \) and \( g_i(q) \in \tilde{A} \) such that

\[ L_{-2}^s a + \sum_{i=0}^{s-1} g_i(q) L_{-2}^i a \in [V_A, V_A] + O_q(V). \] (3.21)
Proof. It is enough to show that $V_A/([V_A, V_A] + O_q(V))$ is a finitely generated $A$-module. Since $V$ is quasi-lisse, $(\text{Spec}(R_V))_{\text{red}}$ has finitely many symplectic leaves and $R_V$ is a finitely generated $R_{\Gamma}$-module. According to Theorem 3.1.3 the algebra $R_V/\{R_V, R_V\}$ is finite dimensional. Following the the same argument in [8 Proposition 5.2], we get the result. \qed

**Theorem 3.4.2.** Let $V$ be a quasi-lisse vertex superalgebra. Let $a \in V_{\bar{0}}$ be primary with $L[0]a = \Delta a$. For each ordinary $V$-module $M$, the 1-point correlation function $S_M((a, z), \tau)$ is the solution of the modular linear differential equation.

**Proof.** The proof is similar to [111, Theorem 4.41]. Formulas (3.14) and (3.18) imply

$$S_M((v, z), \tau) = 0 \text{ if } v \in [V_A, V_A] + O_q(V).$$

Combining Lemma 3.4.1 we get

$$S_M((L[-2]^s a, z), \tau) + \sum_{i=0}^{s-1} g_i(\tau) S_M((L[-2]^i a, z), \tau) = 0,$$

(3.22)

where $g_i(\tau) \in A$. Applying Proposition 3.3.2 (3.22) becomes

$$\partial^s S_M((a, z), \tau) + \sum_{i=0}^{s-1} f_i(\tau) \partial^i S_M((a, z), \tau) = 0,$$

(3.23)

where $f_i(\tau) \in A$. The proof is done. \qed

### 3.5 Twisted modules

Given a vertex operator superalgebra $V$, following [111], we can define the automorphism $g$ of it and an ordinary $g$-twisted $V$-module. Let the order of $g$ be $T$.

**Definition 3.5.1.** An automorphism $g$ of $V$ is a linear automorphism of $V$ preserving $\omega$ such that the actions of $g$ and $Y(v, z)$ on $V$ are compatible in the sense that

$$gY(v, z)g^{-1} = Y(gv, z)$$

for $v \in V$. 28
We denote the automorphism group of $V$ by $\text{Aut} V$. There is a special central element of $V$, $\sigma$, which is defined as $\sigma |_V = (-1)^i id_V$.

**Definition 3.5.2.** A $g$-twisted $V$-module is a $\mathbb{C}$-linear space $M$ equipped with a linear map

$$V \to \text{End}(V)[[z^\frac{1}{T}, z^{-\frac{1}{T}}]],$$

$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v(n)z^{-n-1}$

satisfying

- For $v, w \in M$, $v(m)w = 0$ if $m$ is large enough.
- $Y_M(1, z) = id_V$.
- For $v \in V^r = \{ v \in V | gv = e^{\frac{2\pi i}{T}} v \}$, and $0 \leq r \leq T - 1$

$$Y_M(v, z) = \sum_{n \in \mathbb{Q} + Z} v(n)z^{-n-1}.$$

- (Jacobi identity) For $u \in V^r$

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y_M(v, z_1)Y_M(w, z_2) - (-1)^{|v||w|}z_0^{-1}\delta(\frac{z_0 - z_1}{z_0})Y_M(v, z_2)Y_M(u, z_1) = z_2^{-1}(\frac{z_1 - z_0}{z_2}) - \frac{T}{2}\delta(\frac{z_1 - z_0}{z_2})Y_M(Y(u, z_0)v, z_2).$$

Let $u, v \in V$ such that $g(u) = e^{\frac{2\pi i}{T}} u$ and $g(v) = e^{\frac{2\pi i}{T}} v$. From the definition we can derive the twisted Borcherds commutator \[41\] and iterated formula \[73\]:

$$[u(n), v(m)] = \sum_{i=0}^{\infty} \binom{m}{i} (u(i)v)(m+n-i), \quad \text{for } m \in \frac{r_u}{T} + \mathbb{Z}, \ n \in \frac{r_v}{T} + \mathbb{Z}. \quad (3.24)$$

$$[u(n)v] = \sum_{j=0}^{\infty} \sum_{w=0}^{\infty} (-1)^j \binom{r_u}{j} \binom{n+j}{w} \quad (3.25)$$

$$((-1)^w u(\frac{r_u+n-w}{T})v(\frac{r_v+m+w}{T}) - (-1)^{n+j-w+|u||v|} v(\frac{r_v+n+w-j-w}{T})u(\frac{r_u}{T}+j+w)) \quad (3.26)$$

for $n \in \mathbb{Z}$ and $m \in \frac{r_u+r_v}{T} + \mathbb{Z}$. 

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Example 3.5.3. (Twisted affine vertex algebra of type $A_n^{(2)}$)

If $n = 2k - 1$, the Dykin diagram of $sl_{n+1}$ is

The Dykin diagram permutations $\sigma(i) = 2k - i$ induce an automorphism $g'$ on $sl_{n+1}$.

If $n = 2k$, the Dykin diagram of $sl_{n+1}$ is

The Dykin diagram permutations $\sigma(i) = 2k + 1 - i$ induce an automorphism $g'$ on $sl_{n+1}$.

The automorphism $g'$ induces a twisted automorphism $g$ on $L_{\widehat{sl}_{n+1}}(1, 0)$ by

$$g(t^n \otimes x) = (-1)^n t^n \otimes g(x) \quad \text{for} \quad x \in sl_{n+1}.$$ 

Then $g$-fixed subalgebra, $(L_{\widehat{sl}_{n+1}}(1, 0))^g$, can be endowed with a $g$-twisted $L_{\widehat{sl}_{n+1}}(1, 0)$-module $(n = 2k + 1)$ structure:

$$Y_M((E_{i,i+1} + E_{2k-i,2k-i+1})(-1)1, z) = \sum_{n \in \mathbb{Z}} (E_{12} + E_{23}) n z^{-n-1},$$

$$Y_M((E_{i,i+1} - E_{2k-i,2k-i+1})(-1)1, z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} (E_{12} - E_{23}) n z^{-n-1}.$$ \hspace{1cm} (3.27)

(3.28)

Similary, the fixed subalgebra also has $g$ twisted module structure for $n = 2k$.

Set $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, then $Y_M(L(-1)v, z) = \frac{d}{dz} Y_M(v, z)$ for $v \in V$ and $L(n)$ satisfies the Virasoro algebra relations with central charge $c$ (cf. [37]).
Let \( M = (M, Y_M) \) be a \( g\sigma \)-twisted \( V \)-module and \( k \in \text{Aut} V \). It induces a \( k(\sigma g)k^{-1} \)-twisted \( V \)-module \( (k \circ M, Y_{k \circ M}) \), where \( k \circ M = M \) as vector spaces and \( Y_{k \circ M}(v, z) = Y_M(k^{-1}v, z) \). If \( M \) is a simple \( g\sigma \)-twisted \( V \)-module, according to [39], there exits a complex number \( \lambda \) such that

\[
M = \bigoplus_{n=0}^{\infty} M_{\lambda + \frac{n}{T}},
\]

where \( T \) is the order of \( g \). Then \( g \) induces an action on \( M \) by \( \phi(g)|_{M_{\lambda + \frac{n}{T}}} = e^{2\pi i n} \text{id}_V \).

### 3.6 Modularity of twisted modules of quasi-lisse vertex superalgebras

Let \( V \) be a vertex operator superalgebra with automorphism group \( \text{Aut} G \). Let \( g, h \in \text{Aut} V \) such that \( gh = hg \), and their orders \( o(g) = T \) and \( o(h) = T_1 \) are finite. Let \( A \) be the subgroup of \( \text{Aut} V \) generated by \( g\sigma \) and \( h\sigma \). Let \( \Gamma(T, T_1) \) be the subgroup of matrices \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) in \( SL(2, \mathbb{Z}) \) satisfying

\[
a \equiv d \equiv 1 \pmod{N}, \quad c \equiv 0 \pmod{T_1}, \quad b \equiv 0 \pmod{T}.
\]

Let \( M(T, T_1) \) be the ring of holomorphic forms on \( \Gamma(T, T_1) \) with gradation \( M(T, T_1) = \bigoplus_{k \in \mathbb{N}} M_k(T, T_1) \), where \( M_k(T, T_1) \) is the space of forms of weight \( k \). It was shown in [39] Lemma 5.1, \( M(T, T_1) \) is a Noetherian ring containing \( G_{2k}(\tau), \ (k \geq 2) \) and

\[
Q_k(\mu, \lambda, \tau) = (\mu = e^{\frac{2\pi i}{T}}, \lambda = e^{\frac{2\pi i}{T_1}}, \tau) = \frac{1}{(k-1)!} \sum_{n \in \mathbb{N}} \frac{\lambda(n + \frac{1}{T})^{k-1}e^{2\pi i(n + \frac{1}{T})}}{1 - \lambda e^{2\pi i(n - \frac{1}{T})}}, \ (k \in \mathbb{N}).
\]

The fact that \( Q_k = (\mu, \lambda, \tau) \) is a holomorphic modular form of weight \( k \) on \( (T, T_1) \) was proved in [39] Theorem 4.6.

Let \( V(T, T_1) = M(T, T_1) \otimes_{\mathbb{C}} V \), \( C_2(V)_{M(T, T_1)} = C_2(V) \otimes_{\mathbb{C}} M(T, T_1) \), and \( R_{M(T, T_1)} = V(T, T_1)/C_2(V)_{M(T, T_1)} \). For \( v \in V \) with \( gv = \mu^{-1}v, hv = \lambda^{-1}v \), one can introduce a \( M(T, T_1) \)-submodule of \( V(T, T_1) \), say \( O(g, h) \), which consists of the following elements:

- \( v[0]w, \ w \in V, \ (\mu, \lambda) = (1, 1) \).
- \( v[-2]w + \sum_{k=2}^{\infty} (2k - 1)G_{2k}(\tau) \otimes v[2k - 2]w, \ (\mu, \lambda) = (1, 1) \).
• \( v, \quad ([v], \mu, \lambda) \neq (1, 1, 1). \)

• \( \sum_{k=0}^{\infty} Q_k(\mu, \lambda, \tau) \otimes v[k - 1]w, \quad (\mu, \lambda) \neq (1, 1). \)

Here \( V(T, T_1), R_{M(T, T_1)}, O(g, h) \) plays the same role as \( V_A, R_A, O(V) \) do in non-twisted setting.

**Lemma 3.6.1.** For quasi-lisse vertex superalgebra \( V, \) \( V(T, T_1)/O(g, h) \) is a finitely generated \( M(T, T_1) \)-module.

**Proof.** The proof is similar to the one of [8, Lemmas 5.2]. Define an increasing filtration \( G \) on \( V(T, T_1) \) by

\[
G_p V(T, T_1) = \oplus_{\Delta \leq q} V_{\Delta} \otimes M(T, T_1).
\]

It also induces filtration on \( O(g, h) \) and \( V(T, T_1)/O(g, h) \):

\[
G_p O(g, h) = G_p V(T, T_1) \cap O(g, h) \quad \text{and} \quad G_p (V(T, T_1)/O(g, h)) = G_p V(T, T_1)/G_p O(g, h).
\]

Note \( gr^G V(T, T_1) = V(T, T_1). \) Thus,

\[
gr^G (V(T, T_1)/O(g, h)) = V(T, T_1)/gr^G O(g, h).
\]

Next, we show

\[
\{ R_{M(T, T_1)}, R_{M(T, T_1)} \} + C_2(V_{M(T, T_1)}) \subset gr^G O(g, h). \tag{3.29}
\]

Note \( \{ R_{M(T, T_1)}, R_{M(T, T_1)} \} \) is \( M(T, T_1) \)-span of \( \{ v_0 w, v_{(-2)} w \} \) \( v, w \in V \). When \( (\mu, \lambda) = (1, 1) \), by using (3.15) we get

\[
v[-2]w + \sum_{k=2}^{\infty} (2k - 1) G_{2k}(\tau) \otimes v[2k - 2]w = v_{(-2)}w + c_1 v_{(-1)}w + c_0 v_0 w + \sum_{k \in \mathbb{N}} f_k(\tau) v_k w,
\]

where \( f_k(\tau) \in M(T, T_1) \). The left-hand side of above identity belongs to \( O(g, h) \). Without loss of generality, suppose \( v \) and \( w \) are homogeneous elements of weights \( m \) and \( n \), the weight of \( v_i w \) equals \( m + n - i - 1 \). Hence, \( v_0 w, v_{(-2)} w \in gr^G O(g, h) \) which implies (3.29) in this case.
Now suppose $\mu, \lambda \neq (1, 1)$. In this case, $O(g, h)$ contains elements of form

$$-v[-1]w + \sum_{k \in \mathbb{Z}_+} Q_k(\mu, \lambda, T) \otimes v[k - 1]w.$$  \hfill (3.30)

By using similar arguments we used in the above paragraph, we have $v_{(0)}w \in \text{gr}^G O(g, h)$. In (3.30) we replace $v$ with $L[-1]v$. Since $(L[-1]v)[n] = -nv[n - 1]$, (3.30) becomes

$$v[-2]w + \sum_{k \in \mathbb{Z}_+} (k - 1)Q_k(\mu, \lambda, T) \otimes v[k - 2]w.$$  

Repeating the same argument again, we get $v_{(-2)}w \in \text{gr}^G O(g, h)$. Therefore (3.29) is also true in this case. We are done. \hfill \Box

Under twisted setting, one can also define the 1-point function (cf. [40]).

**Definition 3.6.2.** The space of $(g, h)$ 1-point functions $C(g, h)$ is a $\mathbb{C}$-linear space consisting of functions

$$S : V(T, T_1) \times \mathbb{H} \to \mathbb{C}$$

satisfying the following conditions:

- $S(v, \tau)$ is holomorphic in $\tau$ for $v \in V(T, T_1)$.
- $S(v, \tau)$ is $\mathbb{C}$-linear in $\tau$ and for $f \in M(T, T_1), v \in V$,

$$S(f \otimes v, \tau) = f(\tau)S(v, \tau).$$

- $S(v, \tau) = 0$ if $v \in O(g, h)$.
- If $v \in V$ with $\sigma v = gv = hv = v$, then

$$S(L[-2]v, \tau) = \partial S(v, \tau) + \sum_{l=2}^{\infty} G_{2l}(\tau)S(L[2l - 2]v, \tau).$$  \hfill (3.31)

**Example 3.6.3.** Let $M$ be a simple $g\sigma$-twisted $V$-module. One can define a twisted trace function $T$ which is linear in $v \in V$, and defined for homogeneous $v \in V$ as following (cf. [40]):

$$T(v) = T_M(v, (g, h), \tau) = \text{tr}|_MY_M(q^{L(0)}v, z)\phi(\sigma h)q^{L(0) - \frac{n}{2}}.$$
In [40] Theorem 6.2, authors showed that if \( h \circ M \), \( \sigma \circ M \) and \( M \) are isomorphic then \( T(v) \in \mathcal{C}(g,h) \). (In the paper, authors put the \( C_2 \)-cofinite condition in the Theorem. But the whole argument does not depend on that condition).

All arguments in [39] Section 6 are based on Lemma 3.6.1 and (3.31). By using their machinery, under the same setting as previous example we get

**Theorem 3.6.4.** Suppose \( V \) is quasi-lisse, and \( v \in V \) is a primary vector with \( L[n]v = 0 \) for \( n > 0 \). Let \( R = R(T, T_1) = \mathbb{C}[G_2(\tau)] \otimes M(T, T_1) \). Then there exists \( m \in \mathbb{N} \) and \( r_i(\tau) \in R(T, T_1), 0 \leq i \leq m - 1 \), such that

\[
(q_{\frac{\tau}{T}}^{n} \frac{d}{dq_{\frac{\tau}{T}}})^m T(v) + \sum_{i=0}^{m-1} r_i(\tau)(q_{\frac{\tau}{T}}^{n} \frac{d}{dq_{\frac{\tau}{T}}})^i T(v) = 0,
\]

where \( q_{\frac{\tau}{T}} = e^{2\pi i \tau / T} \).
4.1 Affine and lattice vertex algebras

In this section we analyze the Poisson (super)algebra $R_V$ and the injectivity of the $\psi$ map for some familiar examples of affine and lattice vertex algebras.

Example 4.1.1. It was shown in [6, Proposition 2.7.1] that for any simple Lie algebra $g$, we have $J_\infty(R_{\hat{g}}(k,0)) \cong gr^F(V_{\hat{g}}(k,0))$.

Proposition 4.1.2. [7, Example 4.10] For the free fermionic vertex superalgebra $\mathcal{F}$ (see 2.1.3), $J_\infty(R_\mathcal{F}) \cong gr^F(\mathcal{F})$ as vertex Poisson superalgebras.

Proof. We use Arakawa’s argument in [6, Proposition 2.7.1]. We include the proof for completeness. Here we still follow the notations from Example 2.1.3. According to [65, Section 3.6], we can choose a conformal vector such that $\mathcal{F}$ is $\frac{1}{2}\mathbb{N}$-graded. We consider the standard filtration $G$ on $\mathcal{F}$. Firstly, we have $\mathcal{F} \cong U(A[t^{-1}]t^{-1})$ as vector superspaces. Moreover,

$$G^m(\mathcal{F}) = \left\{ u^1_{-(k_1)} \cdots u^r_{-(k_r)} \mathbf{1} \mid k_i \in \frac{1}{2} + \mathbb{N}, r \in \mathbb{N}, r \leq 2m \right\},$$

where $m \in \frac{1}{2}\mathbb{N}$ and $u^i \in \{\phi^1, \ldots, \phi^n\}$. So $gr^G(\mathcal{F}) \cong S(A[t^{-1}]t^{-1}) \cong J_\infty(R_\mathcal{F})$ as Poisson vertex superalgebra. Therefore, $gr^G(\mathcal{F}) \cong gr^F(\mathcal{F}) \cong J_\infty(R_\mathcal{F})$. \hfill \endproof

Similarly, we can show that $\psi$ is an isomorphism for vertex superalgebra $V_{\hat{g}}(k,0)$, where $g$ is a Lie superalgebra satisfying conditions in Example 2.1.2, and for superconformal vertex algebras $V_c^{N=1}$ and $V_c^{N=2}$.

Let

$$V_{\sqrt{p}\mathbb{Z}} = M(1) \otimes \mathbb{C}[\sqrt{p}\mathbb{Z}],$$

be a rank one lattice vertex algebra (resp. superalgebra) constructed from an integral lattice $L = \mathbb{Z}\alpha \cong \sqrt{p}\mathbb{Z}$, where $\langle \alpha, \alpha \rangle = p$ is even (resp. odd). It has a conformal vector $\omega = \frac{1}{2p} \alpha^2(1) \mathbf{1}$. As usual, we denote the extremal lattice vectors by $e^{n\alpha}$, $n \in \mathbb{Z}$.
Proposition 4.1.3. For the lattice vertex algebra $V_{\sqrt{p}Z}$ we have

$$R_{V_{\sqrt{p}Z}} \cong \mathbb{C}[x, y, z]/\langle x^2, y^2, xy = z^p, xz, yz \rangle.$$ 

When $p$ is odd, $x$ and $y$ are odd vectors.

Proof. According to the following calculations:

$$(e^\alpha)(-2)(e^{-\alpha}) - \frac{\alpha(-1)^{p+1}1}{(p+1)!} \in C_2(V_{\sqrt{p}Z}),$$

$$(e^\alpha)(-2)(1) - \alpha(-1)e^\alpha \in C_2(V_{\sqrt{p}Z}),$$

$$(e^\alpha)(-p-1)(e^\alpha) - e^{2\alpha} \in C_2(V_{\sqrt{p}Z}),$$

$$(e^{-\alpha})(-2)(1) - \alpha(-1)e^{-\alpha} \in C_2(V_{\sqrt{p}Z}),$$

$$(e^{-\alpha})(-p-1)(e^{-\alpha}) - 2e^{-2\alpha} \in C_2(V_{\sqrt{p}Z}),$$

we know that all vectors except for $\alpha(-1)1, \ldots, \alpha^p(-1)1, e^\alpha, e^{-\alpha}$ and $1$ are zero in $R_{V_{\sqrt{p}Z}}$.

Then we will show that all those vectors are indeed nonzero in $R_{V_{\sqrt{p}Z}}$. Suppose there exist $u, v \in V_{\sqrt{p}Z}$ such that $u_{(-2)}v = e^\alpha$. Then $wt(a_{(-2)}b) = wt(a) + wt(b) + 1 = \frac{p}{2}$, which implies that $u, v \in \pi_0$, where $\pi_0$ is the Heisenberg subalgebra $\mathbb{C}[\alpha(n)]_{n \in \mathbb{Z}} \cdot 1$. This is a contradiction. So the equivalent class $\overline{e^\alpha}$ is nonzero in $R_{V_{\sqrt{p}Z}}$. By using similar weight argument, we can show that equivalence classes

$$\overline{e^{-\alpha}}, \overline{1}, \overline{\alpha(-1)1}, \ldots, \overline{\alpha^p(-1)1}$$

are all nonzero in $R_{V_{\sqrt{p}Z}}$. Moreover, we have

$$(e^\alpha)(-1)(e^{-\alpha}) - \frac{\alpha(-1)^p1}{p!} \in C_2(V_{\sqrt{p}Z}).$$

Then the map $\phi : R_{V_{\sqrt{p}Z}} \to \mathbb{C}[x, y, z]/\langle x^2, y^2, xy = z^p, xz, yz \rangle$ by sending $\overline{e^\alpha}$ to $x$, $\overline{e^{-\alpha}}$ to $y$, $\overline{1}$ to $1$ and $\sqrt{\frac{1}{p!} \alpha(-1)\overline{1}}$ to $z$ is an isomorphism.

\[\square\]

Remark 4.1.4. According to the Frenkel-Kac construction, we know that $V_{\sqrt{2}Z} \cong L_{sl_2}^-(1, 0)$. 

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Following Proposition 4.1.3, we have $R_{\hat{sl}_2}^{-1}(1,0) \cong \mathbb{C}[e, f, h]/\langle e^2, f^2, ef = h^2, eh, fh \rangle$. According to [104] 16.16, one can also compute $R_{\hat{sl}_2}^{-1}(1,0)$ directly.

**Corollary 4.1.5.** Let $p$ be a positive odd integer, then the even part of $R_{V_{\sqrt{p}Z}}$, i.e., $(R_{V_{\sqrt{p}Z}})_{\overline{\Pi}}$, is isomorphic to the associated graded algebra $gr^F(A_{V_{\sqrt{p}Z}})$.

**Proof.** According to [87] Theorem 3.3, we know that $A_{V_{\sqrt{p}Z}} \cong \mathbb{C}[x]/(F_p(x))$ where $F_p(x) = x(x + 1)(x - 1) \cdots (x + \frac{(p-1)}{2})(x - \frac{(p-1)}{2})$ in which $x$ corresponds to $[\alpha_{(-1)}1]$ in $A_{V_{\sqrt{p}Z}}$. Then according to Proposition 4.1.3 we have $(R_{V_{\sqrt{p}Z}})_{\overline{\Pi}} \cong gr^F(A_{V_{\sqrt{p}Z}}) \cong \mathbb{C}[x]/\langle x^p \rangle$ via $\phi$.

**Remark 4.1.6.** If $L = \sqrt{2kZ}$ ($k \in \mathbb{Z}_+$) is an even lattice, above result is only true for $k = 1$. Indeed, according to [38] $z^{p-1}$ is a nontrivial element in the kernel of $\phi$ in (3.4).

In [104], authors proved the map $\psi$ is an isomorphism for $L_{\hat{sl}_2}(k,0)$ by using a PBW-type basis of $L_{\hat{sl}_2}(k,0)$ from [83] and Gröbner bases. In [50], E. Feigin essentially proved the same result by using PBW filtration and a technique called “degeneration procedure”. His method is very useful for proving the injectivity of map $\psi$. In Chapter 6 we would use his idea of filtration to study the classically freeness of Feigin-Stoyanovsky principal subspaces.

In the following, we briefly explain how his results imply classically freeness for $L_{\hat{sl}_2}(k,0)$.

**Proposition 4.1.7.** The map $\psi : J_\infty(R_{L_{\hat{sl}_2}(k,0)}) \cong gr^F(L_{\hat{sl}_2}(k,0))$ is an isomorphism of vertex Poisson algebras.

**Proof.** According to Example 2.1.10, the $C_2$-algebra $R_{L_{\hat{sl}_2}(k,0)}$ is isomorphic to

$$\mathbb{C}[e, f, h]/\langle f^i \circ e^{k+1} | 0 \leq i \leq 2k + 2 \rangle = \mathbb{C}[e, f, h]/\langle e^{k+1}, e^kh, e^{k-1}h^2 - 2e^kf, \cdots, f^{k+1} \rangle.$$

It is clear that $\psi(u_{(-i)}) = \overline{u_{(-i)}1}$ for $u \in \{e, f, h\}$ and $i \in \mathbb{Z}_+$. Let $u(z) = \sum_{n \leq -1} u(n)z^{-n-1}$ where $u \in \{e, f, h\}$. Now we consider $e(z)^{k+1}$. The coefficient of $z^n$ equals $T_n(e_{(-1)}^{k+1})$ up to a
scalar multiple for \( n \in \mathbb{N} \). And we have similar results for \( e^k h, e^{k-1} h^2 - 2e^k f, \ldots, f^{k+1} \). Thus

\[
\mathcal{J}_\infty(R_{\mathfrak{sl}_2}(k,0)) \cong \frac{\mathbb{C}[e_{(-1-i)}, f_{(-1-i)}, h_{(-1-i)} \mid i \in \mathbb{N}]}{(e(z)^{k+1}, e(z)^k h(z), e(z)^k h^2 - 2e(z)^k f(z), \ldots, f(z)^{k+1})},
\]

where

\[
\langle e(z)^{k+1}, e(z)^k h(z), e(z)^k h^2 - 2e(z)^k f(z), \ldots, f(z)^{k+1} \rangle
\]

means the ideal generated by the Fourier coefficients of

\[
e(z)^{k+1}, e(z)^k h(z), e(z)^k h^2 - 2e(z)^k f(z), \ldots, f(z)^{k+1}.
\]

It is sufficient to show

\[
HS_q(\mathcal{J}_\infty(R_{\mathfrak{sl}_2}(k,0))) = \text{ch}[L_{\mathfrak{sl}_2}(k,0)](q).
\] (4.1)

To that end, we use the results from [50]. E. Feigin constructed three quotient polynomial algebras \( B_k, C_k \) and \( D_k \) as follows.

- The quotient of the algebra \( B_k \) in variables \( e_{-1-i}, h_{-1-i}, f_{-1-i}, i \in \mathbb{N} \) is generated by Fourier coefficients of series:

\[
e(z)^i h(z)^{k+1-i} \quad (i = 1, \ldots, k+1)
\]

and

\[
h(z)^i f(z)^{k+1-i} \quad (i = 0, \ldots, k+1),
\]

- Let \( u^l(z) = \sum_{i \in \mathbb{Z}} z^{-l-i} u_i^l \), where \( u = e, h, f \) and \( u_i^l = 0 \) for \( i > -l \). Then the quotient polynomial algebra \( C_k \) in variables \( u_i^l, l = 1, \ldots, k \), is generated by Fourier coefficients of series:

\[
u_i^l(z)^{\alpha} u_i^m(z)^{\beta} \quad \text{for } u = e, f, h, \text{ and } \alpha + \beta < \min(l, m),
\]

\[
e_i^l(z)^{\alpha} h_i^m(z)^{\beta} \quad \text{for } \alpha + \beta < \max(0, l + m - k),
\]

\[
h_i^l(z)^{\alpha} f_i^m(z)^{\beta} \quad \text{for } \alpha + \beta < \max(0, l + m - k).
\]
• Define a lattice $Q$ generated by vectors $p_i, q_i, r_i \in \mathbb{R}^N, i = 1, \ldots, k$, with scalar products:

$$
\langle p_i, p_j \rangle = \langle q_i, q_j \rangle = \langle r_i, r_j \rangle = 2\delta_{i,j}, \langle p_i, q_j \rangle = \langle q_i, r_j \rangle = \delta_{i,k+1-j}, \langle p_i, r_j \rangle = 0.
$$

The algebra $D_k$ is generated from the highest weight vector with the Fourier coefficients of

$$
\sum_{i=1}^{k} Y(e^{p_i}, z), \sum_{i=1}^{k} Y(e^{q_i}, z), \sum_{i=1}^{k} Y(e^{r_i}, z).
$$

By using certain filtrations [50, Lemma 3.2, Lemma 3.4], one gets

$$
HS_q(J_\infty(R_{\widehat{sl}_2}(k,0))) \leq HS_q(B_k) \leq HS_q(C_k). \quad (4.2)
$$

The “degenerate procedures” [50, Lemma 3.5, Proposition 3.1] give us:

$$
HS_q(D_k) \geq HS_q(C_k) \quad \text{and} \quad \text{ch}[\widehat{L}_{\widehat{sl}_2}(k,0)](q) \geq HS_q(D_k). \quad (4.3)
$$

Combining (4.2), (4.3) and the fact that $\psi$ is surjective, we get (4.1).

Before we prove next result, let us fix some notation first. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of type $C_n, n \geq 2$. Here we assume that $\mathfrak{g}$ has a basis

$$
\{x^i | 1 \leq i \leq (2n+1)n\}.
$$

Let $\theta$ be the maximal root of $\mathfrak{g}$, and $x_\theta$ the corresponding maximal root vector. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g}$ and its universal vacuum representation is $V_{\widehat{\mathfrak{g}}}(1,0)$ for $k \in \mathbb{Z}_+$. Set

$$
R = U(\mathfrak{g}) \circ (x_\theta)^2(-1)1, \quad \overline{R} = \text{Span}_C \{r_{(n)} | r \in R, n \in \mathbb{Z}\},
$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, and $\circ$ is the adjoint action. Then $\widehat{\mathfrak{g}}$-module $V_{\widehat{\mathfrak{g}}}(1,0)$ has a maximal submodule $I_{\widehat{\mathfrak{g}}}(1,0)$ generated by $\overline{R} \cdot 1$. Let $L_{\widehat{\mathfrak{g}}}(1,0)$ denote the simple quotient $V_{\widehat{\mathfrak{g}}}(1,0)/I_{\widehat{\mathfrak{g}}}(1,0)$. Now we are ready to prove:

**Theorem 4.1.8.** The map $\psi$ is an isomorphism for the affine vertex algebra $L_{\widehat{\mathfrak{g}}}(1,0)$. 

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Proof. It is clear that the $C_2$-algebra of $L_\mathfrak{g}(1,0)$ is $R_{L_\mathfrak{g}(1,0)} = S(\mathfrak{g})/\langle U(\mathfrak{g}) \circ e^2_\theta \rangle$, where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. We denote the algebra

$$\mathbb{C}[x^i_{(-j)} | j \in \mathbb{Z}_+] / \langle U(\mathfrak{g}) \circ e^2_\theta(z) \rangle$$

by $Q$, where $e_\theta(z) = \sum_{n \in \mathbb{Z}_-} (e_\theta)_{(n)} z^{-n-1}$. Following the similar argument in Proposition 4.1.7, we see that $J_\infty(R_{L_\mathfrak{g}(1,0)}) \cong Q$. In order to show that $\psi$ is an isomorphism, it is enough to prove that $gr^F(L_\mathfrak{g}(1,0))$ and $Q$ have the same basis. Notice that

$$I = \overline{R} \cap \mathbb{C}[x^i_{(-j)} | j \in \mathbb{Z}_+] = \langle U(\mathfrak{g}) \circ e^2_\theta(z) \rangle.$$

We can define an order on all monomials of $\mathbb{C}[x^i_{(-j)} | j \in \mathbb{Z}_+]$ in the sense of [94, Section 8]. From the same paper, we know that every nonzero homogeneous polynomial $\mathbb{C}[x^i_{(-j)} | j \in \mathbb{Z}_+]$ has an unique largest monomial. For an arbitrary nonzero polynomial $u$, we define the leading term $lt(u)$ as the largest monomial of the nonzero homogeneous component of the smallest weight, which is unique. We denote all monomials in $\mathbb{C}[x^i_{(-j)} | j \in \mathbb{Z}_+]$ by $\mathcal{P}$. We clearly have $\mathcal{P}$ as a spanning set of $Q$. Since $u = 0$ in $Q$ if $u \in I$, the leading term $lt(u)$ equals the linear combination of other terms. Therefore, $\mathcal{P} \setminus \langle lt(U) \rangle$ is a smaller spanning set of $Q$. And we denote it by $\mathcal{RR}$. Meanwhile according to [94, Theorem 11.3], we know that $\psi(\mathcal{RR})$ is a basis of $gr(L_\mathfrak{g}(1,0))$. Together with the surjectivity of $\psi$, we have that $\mathcal{RR}$ is a basis of $Q$. So $\psi$ is an isomorphism.

Remark 4.1.9. In [51], E. Feigin showed that the map $\psi$ is an isomorphism for $L_\mathfrak{g}(1,0)$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra. The proof was based on study of “PBW filtrations” and some Demazure modules in $L_\mathfrak{g}(1,0)$.

4.2 $N = 2$ vertex superalgebra at level $c = 1$

In this section we study the simple $N = 2$ superconformal vertex algebra of central charge $c = 1$, denoted by $L_1^{N=2}$. The odd lattice vertex algebra $V_{\sqrt{3}2}$ is known to be isomorphic to $L_1^{N=2}$. Here we identify $\frac{1}{3} \alpha_{(-1)} 1$ with $J_{(-1)} 1$, $\frac{1}{\sqrt{3}} e^{\pm \alpha}$ with $G_{(\pm \frac{3}{2})} 1$ and $\frac{1}{6} (\alpha_{(-1)} \alpha_{(-1)} 1)_{(-1)} 1$ with $L_{(-2)} 1$. 
According to [1, 3], the maximal submodule of $V_1^{N=2}$ is generated by

$$G^+(-\frac{3}{2}) G^+(-\frac{1}{2}) \mathbf{1} \quad \text{and} \quad G^+(-\frac{1}{2}) G^+(-\frac{3}{2}) \mathbf{1}.$$  

By identifying $G^+$ with $G^+(-\frac{3}{2}) \mathbf{1}$, $G^-$ with $G^+(-\frac{1}{2}) \mathbf{1}$ and $h$ with $J_{(-1)} \mathbf{1}$, we have

$$R_{L_1^{N=2}} \cong \mathbb{C}[G^+, G^-, h]/\langle (G^+)^2, (G^-)^2, G^+G^-=h^3, G^+h, G^-h \rangle.$$  

For $J_\infty(R_{L_1^{N=2}})$ we identify $G^+, G^-, h$ with $G^+(-\frac{3}{2}), G^-(-\frac{3}{2}), h(-1)$. We have

$$J_\infty(R_{L_1^{N=2}}) \cong \frac{\mathbb{C}[G^+(-\frac{3}{2} - i), G^-(-\frac{3}{2} - i), h(1-i)|i \in \mathbb{N}]}{\langle (G^+(z))^2, (G^-(z))^2, G^+(z)G^-(z) = h(z)^3, G^+(z)h(z), G^-(-z)h(z) \rangle},$$

where $G^\pm(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}_n} G^\pm(n)z^{-\frac{3}{2}}, h(z) = \sum_{n \in \mathbb{Z}_n} h(n)z^{-n-1}$. The map $\psi$ is not an isomorphism in this case because the images of nonzero elements

$$G^+(-\frac{5}{2}) G^+(-\frac{3}{2}) \quad \text{and} \quad G^+(-\frac{3}{2}) G^-(-\frac{3}{2})$$

in the arc superalgebra under $\psi$, i.e., $G^+(-\frac{5}{2}) G^+(-\frac{3}{2}) \mathbf{1}$ and $G^-(-\frac{3}{2}) G^-(-\frac{3}{2}) \mathbf{1}$ are null vectors. Thus

$$\langle a, b \rangle_\theta = \langle T^i(G^+(-\frac{5}{2}) G^+(-\frac{3}{2})), T^i(G^+(-\frac{5}{2}) G^+(-\frac{3}{2})) | i \in \mathbb{N} \rangle \subset \ker(\psi),$$

where $a = G^+(-\frac{5}{2}) G^+(-\frac{3}{2})$ and $b = G^-(-\frac{3}{2}) G^-(-\frac{3}{2})$.

Let us consider

$$J_\infty(R_{L_1^{N=2}})/\langle a, b \rangle_\theta.$$  

We will write down a spanning set of $J_\infty(R_{L_1^{N=2}})/\langle a, b \rangle_\theta$. We let the ordered monomial be a monomial of the form

$$G^-(-n - \frac{1}{2})^a_1 h(-n)^b_1 G^+(-n - \frac{1}{2})^c_1 \cdots G^+(-\frac{5}{2})^a_2 h(-2)^b_2 G^+(-\frac{5}{2})^c_2 \cdots$$

$$G^-(-\frac{3}{2})^a_1 h(-1)^b_1 G^+(-\frac{3}{2})^c_1.$$  

Then we have a complete lexicographic ordering on the set of ordered monomials in the sense
Now let us find the leading terms of the Fourier coefficients of

\[ G^+(z)G^-(z) = h^3(z), \quad G^+(z)h(z), \quad G^-(z)h(z), \quad T^i(a), \quad T^i(b). \]

(a) Leading term of \( G^+(z)h(z) \):

- \( n \) is even, the leading term of the coefficient of \( z^n \) is

\[ h(\frac{-2 - n}{2})G^+(\frac{-3 + n}{2}). \]

- \( n \) is odd, the leading term of the coefficient of \( z^n \) is

\[ G^+(\frac{-4 - n}{2})h(\frac{-1 - n}{2}). \]

(b) Leading term of \( G^-(z)h(z) \):

- \( n \) is even, the leading term of the \( z^n \)-th coefficient is

\[ G^-(\frac{-3 + n}{2})h(\frac{-2 - n}{2}). \]

- \( n \) is odd, the leading term of the \( z^n \)-th coefficient is

\[ h(\frac{-3 - n}{2})G^-\left(\frac{-2 - n}{2}\right). \]

(c) Leading term of \( G^+(z)G^-(z) = h^3(z) \):

- The leading term of the constant term is

\[ h(-1)h(-1)h(-1). \]

- \( n \) is even and not equal to 0, the leading term of the coefficient of \( z^n \) is

\[ G^-\left(\frac{-3 + n}{2}\right)G^+\left(\frac{-3 + n}{2}\right). \]
- $n$ is odd, the leading term of the coefficient of $z^n$ is

$$G^+ \left( -\frac{n-4}{2} \right) G^- \left( -\frac{n-2}{2} \right).$$

(d) Leading term of $T^n(a)$ or $T^n(b)$:

- $n$ is even, the leading term is

$$G^\pm \left( -\frac{n-5}{2} \right) G^\pm \left( -\frac{n-3}{2} \right).$$

- $n$ is odd, the leading term is

$$G^\pm \left( -\frac{n-6}{2} \right) G^\pm \left( -\frac{n-2}{2} \right).$$

Clearly all ordered monomials constitute a spanning set of $J_\infty (R_{L_1^{N+2}})/\langle a, b \rangle$. Since all polynomials we considered above equal zero in $J_\infty (R_{L_1^{N+2}})/\langle a, b \rangle$, the leading term of each can be written as a linear combination of all other terms. Thus if we want to get a “smaller” spanning set, all above leading terms can not appear as segments of an ordered monomial. Therefore, we can impose some difference conditions on ordered monomials by using these leading terms to get a new spanning set.

**Definition 4.2.1.** We call an ordered monomial a $Gh$-monomial, if it satisfy the following conditions:

(i) either $b_i$ or $c_i$ is 0 and either $b_i$ or $c_{i+1}$ is 0,

(ii) either $a_i$ or $b_i$ is 0 and either $a_i$ or $b_{i+1}$ is 0,

(iii) $b_1 \leq 2$, $a_1 + c_2 \leq 1$, and $a_i + c_i + c_{i+1} \leq 1$ for $i \geq 2$.

(iv) $c_i + c_{i+1} + c_{i+2} \leq 1$ and $a_i + a_{i+1} + a_{i+2} \leq 1$.

Here constraints (i)-(iv) are coming from leading terms in (a)-(d), respectively. Then we have the following:

**Proposition 4.2.2.** $Gh$-monomials form a spanning set of

$$A = J_\infty (R_{L_1^{N+2}})/\langle a, b \rangle.$$
Let us write down the first few terms of the Hilbert series of $A$.

**Example 4.2.3.** For $i \leq 5$, $Gh$-monomials give us basis of $A_i$:

\begin{align*}
A_1 : & h(-1) \\
A_2 : & G^+(-\frac{3}{2}), G^-(-\frac{3}{2}) \\
A_3 : & h(-1)^2, h(-2) \\
A_4 : & G^+(-\frac{5}{2}), G^-(-\frac{5}{2}) \\
A_5 : & G^-(-\frac{3}{2})G^-(-\frac{3}{2}), h(-1)h(-2), h(-3) \\
A_6 : & G^-(-\frac{5}{2})h(-1), G^+(-\frac{7}{2}), G^-(-\frac{7}{2}), G^+(-\frac{3}{2})h(-2) \\
A_7 : & G^-(-\frac{3}{2})G^-(-\frac{5}{2}), h(-2)^2, h(-1)^2h(-2), h(-3)h(-1), h(-4) \\
A_8 : & G^-(-\frac{5}{2})h(-1)^2, G^+(-\frac{9}{2}), G^-(-\frac{9}{2}), \\
& G^-(-\frac{7}{2})h(-1), G^+(-\frac{7}{2})h(-1), h(-3)G^-(-\frac{3}{2}), h(-3)G^+(-\frac{3}{2}) \\
A_9 : & G^-(-\frac{3}{2})G^+(-\frac{7}{2}), G^-(-\frac{3}{2})G^-(-\frac{7}{2}), h(-1)h(-4), \\
& h(-1)h(-2)^2, h(-1)^2h(-3), h(-2)h(-3), h(-5).
\end{align*}

We have $HS_q(A) = 1 + q + 2q^\frac{3}{2} + 2q^2 + 2q^\frac{5}{2} + 3q^3 + 4q^\frac{7}{2} + 5q^4 + 7q^\frac{9}{2} + 7q^5 + O(q^{\frac{11}{2}})$. Meanwhile

\[
\text{ch}[L_1^{N=2}](q) = \text{ch}[V_{\sqrt{3}Z}](q) = \frac{\sum_{n \in \mathbb{Z}} q^{\frac{3}{2}n^2}}{\prod_{n \in \mathbb{Z}^+} (1 - q^n)} = 1 + q + 2q^\frac{3}{2} + 2q^2 + 2q^\frac{5}{2} + 3q^3 + 4q^\frac{7}{2} + 5q^4 + 6q^\frac{9}{2} + 7q^5 + O(q^{\frac{11}{2}}).
\]

Since in weight $\frac{9}{2}$ dimension of $A$ is bigger than the dimension of $V_{\sqrt{3}Z}$ by 1, the induced map

\[\overline{\psi} : J_\infty(R_{L_1^{N=2}})/\langle a, b \rangle_0 \to gr(L_1^{N=2})\]

is not injective. It is not hard to see that the one dimensional kernel of $\overline{\psi}$ in weight $\frac{9}{2}$ is spanned by

\[c = G^-(-\frac{9}{2}) - \frac{1}{3} h(-3)G^-(-\frac{3}{2}) - G^-(-\frac{7}{2})h(-1) + \frac{1}{3} G^-(-\frac{5}{2})h(-1)^2.\]
We make the following conjecture:

**Conjecture 4.2.4.** The induced map \( \hat{\psi} : J_\infty(R_{L_1^{N=2}}) / \langle a, b, c \rangle \otimes \partial \to gr(L_1^{N=2}) \) is an isomorphism.
5.1 Basics of principal subspaces

Principal subspaces of affine vertex algebras (at least in a special case) were introduced by Feigin and Stoyanovsky [49] and further studied by several people; see [22, 25, 27, 36, 46, 46] and references therein. In [91, 92], M. Primc studied Feigin-Stoyanovsky type subspaces which are analogs of principal subspaces but easier to analyze. They are further investigated for many integral levels and types [17, 63, 102, 103]. Here we follow notation from [85], where principal subspaces are defined for general integral lattices (not necessarily positive definite). As in [85], we let \( V_L = M(1) \otimes C[L] \) denote a lattice vertex algebra. We fix a \( \mathbb{Z} \)-basis \( B = \{ \alpha_1, \ldots, \alpha_n \} \) of \( L \). Let \( e^{\alpha_i} \) be an element in the group algebra \( C[L] \). Then the principal subspace associated to \( B \) and \( L \), is defined as

\[
W_L(B) := \langle e^{\alpha_1}, \ldots, e^{\alpha_n} \rangle,
\]

that is the smallest vertex algebra that contains extremal vectors \( e^{\alpha_i} \). Once \( B \) is fixed, we shall drop \( B \) in the parenthesis and write \( W_L \) for convenience.

Let \( \mathfrak{g} \) be a simple finite dimensional complex Lie algebra of type \( A, D \) or \( E \), and \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \). We choose simple roots \( \{ \alpha_1, \ldots, \alpha_n \} \) of \( (\mathfrak{g}, \mathfrak{h}) \), and let \( \Delta^+ \) denote the set of positive roots. Let \( (\cdot, \cdot) \) be a rescaled Killing form on \( \mathfrak{g} \) such that \( (\alpha_i, \alpha_i) = 2 \) for \( i = 1, \ldots, n \) (as usual we identify \( \mathfrak{h} \) and \( \mathfrak{h}^* \) via the Killing form). Fundamental weights of \( \mathfrak{g} \), \( \{ \omega_1, \ldots, \omega_n \} \subset \mathfrak{h}^* \), are defined by \( (\omega_i, \alpha_j) = \delta_{i,j} \) (1 \( \leq i, j \leq n \)).

Let \( \mathfrak{n}_+ = \prod_{\alpha \in \Delta^+} Cx_\alpha \), where \( x_\alpha \) is a corresponding root vector, and \( \widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes C[t, t^{-1}] \) is its affinization. For an affine vertex algebra \( L_{\mathfrak{g}}(k, 0) \), \( k \neq -h^\vee \), which is isomorphic to \( L(k\Lambda_0) \) as \( \mathfrak{g} \)-module, we define the (FS)-principal subspace of the simple \( \mathfrak{g} \)-module \( L_{\mathfrak{g}}(k, 0) \) as

\[
W_{\Lambda_k,0} := U(\widehat{\mathfrak{n}}_+) \cdot 1,
\]
where $\mathbb{1}$ is the vacuum vector. It is easy to see that this is a vertex algebra (without conformal vector). For $k = 1$, we have $W_L \cong W_{\Lambda_1,0}$, where $L$ is the root lattice spanned by simple roots.

We fix a fundamental weight $\omega = \omega_m$ and set

$$\Gamma = \{ \alpha \in \Delta | (\omega, \alpha) = 1 \},$$

where $\Delta$ is the root system of $\mathfrak{g}$ and $\mathfrak{g}_1 := \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the $\alpha$-root space. This Lie algebra is commutative. We let $\mathfrak{g}_1 \otimes \mathbb{C}[t, t^{-1}]$ be $\hat{\mathfrak{g}}_1$. Then we can define the so-called Feigin-Stoyanovsky type subspace of $L_{\hat{\mathfrak{g}}}(k, 0)$ as

$$W'_{\Lambda_k,0} := U(\hat{\mathfrak{g}}_1) \cdot v_{k\Lambda_0}.$$ 

Unlike the FS subspace, this vertex subalgebra is commutative. We denote

$$\tilde{\Gamma}^- = \{ x_\gamma(-r) | \gamma \in \Gamma, \ r \in \mathbb{Z}_+ \}.$$

$$\tilde{\Gamma} = \{ x_\gamma(-r) | \gamma \in \Gamma, \ r \in \mathbb{Z} \}.$$

Notice that $U(\hat{\mathfrak{g}}_1) \cong \mathbb{C}[\tilde{\Gamma}]$. Therefore, we can identify the elements in $W'_{\Lambda_k,0}$ with the elements in $\mathbb{C}[\tilde{\Gamma}^-]$. For any element in $W'_{\Lambda_k,0}$,

$$v = x_{\beta_1}(m_1) \cdots x_{\beta_l}(m_l), \ \beta_i \in \Gamma,$$

we define colored weight as

$$cwt(v) = \sum_{i=1}^{l} \beta_i$$

for later use.

### 5.2 Root lattices of type $A$

Following the notations in [27], we can prove the following result.

**Proposition 5.2.1.** For $\mathfrak{g} = sl_2$, we have $W_{\Lambda_k,0} \cong gr(W_{\Lambda_k,0}) \cong J_\infty(\mathbb{C}[x]/(x^{k+1}))$ for $k \in \mathbb{Z}_+$. 

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Proof. It is clear that $R_{W_{\Lambda_{k,0}}} = \mathbb{C}[x]/(x^{k+1})$. The result follows from Theorem 3.1 in [27]. □

**Remark 5.2.2.** When $k = 1$, $W_{\Lambda_{1,0}}$ of type $A$ is isomorphic to $J_\infty(\mathbb{C}[x]/(x^2))$. By using different methods to calculate the Hilbert-Poincare series, see [21] and [16], one can derive the famous Rogers-Ramanujan identities.

For the rest of this subsection, we let $L$ be the $A_{n-1}$ root lattice with the rescaled Killing form $(\cdot,\cdot)$ such that $(\alpha, \alpha) = 2$ for any root and the standard $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_{n-1}$ of simple roots. We are going to prove that $\psi$ is an isomorphism for the principal subspace $W_L$ corresponding to this basis. In the following, we will identify $W_L$ and $W_{\Lambda_{1,0}}$. Firstly we prove the following proposition:

**Proposition 5.2.3.** Given elements $\alpha$, $\beta$, $\gamma$ and $\tau$ in lattice $L$, we have

\[(e^\alpha)(-1)e^\beta = 0, \quad \text{if } (\alpha, \beta) \in \mathbb{Z}_+.\] (5.1)

\[(e^\alpha)(-1)e^\beta = \frac{\epsilon(\alpha, \beta)}{\epsilon(\gamma, \tau)}(e^\gamma)(-1)e^\tau, \quad \text{if } (\alpha, \beta) = (\gamma, \tau) \quad \text{and} \quad \alpha + \beta = \gamma + \tau.\] (5.2)

Proof. From the definition of vertex operators from [65], we have

\[Y(e^\alpha, z)e^\beta = \epsilon(\alpha, \beta)z^{(\alpha, \beta)}\text{Exp} \left( \sum_{n \in \mathbb{Z}_-} \frac{-\alpha(n)}{n} z^{-n} \right) e^{\alpha+\beta},\]

where $\epsilon(\alpha, \beta)$ is a 2-cocycle constant. We have

\[(e^\alpha)(-1)e^\beta = \text{Coeff}_0(Y(e^\alpha, z)e^\beta) = 0\]

since the minimal power of $z$ above is greater than 0. The coefficients of $z^0$ of $Y(e^\alpha, z)e^\beta$ and $Y(e^\gamma, z)e^\tau$ are $\epsilon(\alpha, \beta)e^{\alpha+\beta}$ and $\epsilon(\gamma, \tau)e^{\gamma+\tau}$. The identity (2) follows from this fact and the given condition. □

It is clear that all quotient relations in $R_{L_{\Lambda_{1,0}(1,0)}}$ come from (1) and (2). Thus

\[R_{L_{\Lambda_{1,0}(1,0)}} \cap \mathbb{C}[E_{i,j} | 1 \leq i < j \leq n] = R_{W_L}.\]
We let $E_{i,j}$ be the $(i,j)$-th elementary matrix. Therefore, \{E_{i,j}\}_{1 \leq i < j \leq n}$ is the set of all positive root vectors. It is not hard to see that the $C_2$-algebra $R_{WL}$ equals $\mathbb{C}[E_{i,j}|1 \leq i < j \leq n]/I$, where we denote the equivalence class of $(E_{i,j})(-1)^1$ by $E_{i,j}$. In [47, Corollary 2.7] (see also [49] for $\mathfrak{g} = sl_3$,) authors have written down the graded decomposition of $R_{L_{sl_n}(1,0)}$. By restricting it to its principal subspace, we have:

**Proposition 5.2.4.** The $C_2$-algebra of $W_L$ equals

$$\mathbb{C}[E_{i,j}|1 \leq i < j \leq n]/\langle \sum_{\sigma \in S_2} E_{i_1,j_1} E_{i_2,j_2} | j_1 > i_2 \rangle$$

where $1 \leq i_1 \leq i_2 \leq n$ and $1 \leq j_1 \leq j_2 \leq n$.

Moreover, we have the following combinatorial $q$-identity which will be proven in next Chapter, where we also prove more general identities [76].

**Theorem 5.2.5.** Let $A$ be the Cartan matrix $((\alpha_i, \alpha_j))_{1 \leq i,j \leq n-1}$ of type $A_{n-1}$, $n \geq 2$, and

$$n = (n_1,2, \ldots, n_{n-1},n) = (n_{i,j})_{1 \leq i < j \leq n}.$$

Then we have

$$\sum_{n \in \mathbb{N}^{(n-1)/2}} q^{B(n)} \prod_{1 \leq i < j \leq n} (q)_{n_{i,j}} = \sum_{k = (k_1, \ldots, k_{n-1}) \in \mathbb{N}^{n-1}} q^{kA(k)} \prod_{1 \leq i < j \leq n} (q)_{k_1} (q)_{k_2} \cdots (q)_{k_{n-1}}, \quad (5.3)$$

where

$$B(n) = \sum_{1 \leq i_1 < j_1 \leq n \atop 1 \leq i_2 < j_2 \leq n \atop 1 \leq i_1 < i_2 \leq n \atop j_1 \leq j_2 \leq n \atop j_1 > j_2} n_{i_1,j_1} n_{i_2,j_2}.$$

**Example 5.2.6.** For $sl_4$, we have the following $q$-series identity:

$$\sum_{n \in \mathbb{N}^6} \frac{q^{n_1^2+n_2^2+n_3^2+n_4^2+n_5^2+n_6^2+n_1n_4+n_1n_6+n_2n_4+n_2n_5+n_3n_5+n_3n_6+n_4n_6+n_5n_6+n_4a_5}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}}$$

$$= \sum_{k \in \mathbb{N}^3} \frac{q^{k_1^2-k_1k_2+k_2^2-k_2k_3+k_3^2}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}},$$

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where we use multiindices \( n = (n_1, n_2, \cdots, n_6) \) and \( k = (k_1, k_2, k_3) \). By doing following replacement:

\[
\begin{align*}
n_{1,2} &\leftrightarrow n_1, \quad n_{2,3} \leftrightarrow n_2, \quad n_{3,4} \leftrightarrow n_3, \\
n_{1,3} &\leftrightarrow n_4, \quad n_{2,4} \leftrightarrow n_5, \quad n_{4,5} \leftrightarrow n_6,
\end{align*}
\]

we recover the formula in Theorem 5.2.5.

Now we are ready to prove:

**Theorem 5.2.7.** The map \( \psi \) is an isomorphism between \( J_{\infty}(R_{WL}) \) and \( gr(W_L) \).

**Proof.** From Proposition 5.2.4 we know that \( J_{\infty}(R_{WL}) \) is isomorphic to

\[
\mathbb{C}[E_{i,j}(n)|n \leq -1, 1 \leq i < j \leq n]/\langle \sum_{\sigma \in S_2} E_{i_1,j_{\sigma_1}}(z)E_{i_2,j_{\sigma_2}}(z)|j_1 > i_2 \rangle,
\]

where \( E_{i,j}(z) = \sum_{n \leq -1} E_{i,j}(n)z^{-n-1} \) and \( 1 \leq i_1 \leq i_2 \leq n, 1 \leq j_1 \leq j_2 \leq n \). In order to simplify notation, we first order \( \{E_{i,j}\}_{1 \leq i < j \leq n} \) as

\[
E_{1,2}, E_{1,3}, \cdots, E_{1,n}, E_{2,3}, \cdots, E_{2,n}, \cdots, E_{n-1,n},
\]

and we denote this sequence by \( \{E_m\}_{1 \leq m \leq n(n-1)/2} \) (i.e., \( E_1 = E_{1,2}, E_2 = E_{1,3} \) etc.). We then have a spanning set of arc algebra with each element of the form:

\[
E_1(-n_1^1) \cdots E_1(-n_1^{k_1})E_2(-n_2^1) \cdots E_2(-n_2^{k_2}) \cdots,
\]

where \( 1 \leq n_m^k \leq \cdots \leq n_1 \) for \( 1 \leq m \leq n(n-1)/2 \). Here \( k_s = 0 \) when we don’t have terms involving \( E_s \). Now we can reduce this spanning set by using quotient relations as following:

- **difference two condition at distance 1:** If we have \( E_m(z)^2 = 0 \) in the quotient of the arc algebra, then we can impose a condition: \( n_m^p \geq n_m^{p+1} + 2 \) \( (1 \leq p \leq k_m - 1) \) on above spanning set.

- **boundary condition:** If we have \( E_s(z)E_t(z) + \cdots = 0 \) \( (s < t) \), we can impose a condition: \( n_s^{k_t} \geq k_t + 1 \).
Therefore, we have a reduced spanning set which implies

\[ HS_q(J_\infty(R_{W_L})) \leq \sum_{n \in \mathbb{N}^{(n-1)/2}} q^{B(n)} \prod_{1 \leq i < j \leq n} (q)_{n_{i,j}}. \]

And it is well-known that

\[ \text{ch}[\text{gr}(W_L)](q) = \sum_{k = (k_1, \ldots, k_n) \in \mathbb{N}^n} q^{k_{1}k_{1}^T} \frac{q^{k_{1}k_{1}'}}{(q)_{k_1}(q)_{k_2} \cdots (q)_{k_n}}. \]

Surjectivity of \( \psi \), and identity (2) together imply that \( \psi \) is an isomorphism and the image of above spanning set under \( \psi \) is a basis of \( W_L \).

\[ \text{Remark 5.2.8.} \] Using result in [85], we can write down a basis of \( W_L \) by using \((e^{\alpha})_{(j)}\), where \( \alpha_i \) is a simple root of \( sl_n \) and \( j \) can be greater than or equal to 0. If we want the subscript \( j \) to be always less than 0, we have to include \((e^{\beta})_{(j)}\), where \( \beta \) is a positive root.

It is clear \( E_m = E_{i_{m,j_{m}}} \) is a root vector of a positive root

\[ \beta_m := \alpha_{i_m} + \alpha_{i_m+1} + \cdots + \alpha_{j_m-1}. \]

Above proposition gives us a new basis of \( W_L \):

\[ (e^{\beta_1}_{(-n_1^1)} \cdots (e^{\beta_1}_{(-n_1^1)})(e^{\beta_2}_{(-n_2^2)} \cdots (e^{\beta_2}_{(-n_2^2)})(e^{\beta_3}_{(-n_3^3)} \cdots (e^{\beta_3}_{(-n_3^3)})(e^{\beta_M}_{(-n_M^M)})^1), \]

where \( M = \frac{n(n-1)}{2} \), \( n_{M}^{k_{M}} \in \mathbb{Z}_+, \) \( n_{n_m}^{p+1} \geq n_{m}^{p+1} + 2 \) \( (1 \leq p \leq k_m - 1) \) and \( n_{s}^{k_s} \geq k_t + 1 \) if \( 1 \leq s < t \leq M, \) \( i_t < j_s \leq j_t \).

Following the notations in Example 3.5.3, we have the following presentation about the principal subspace of type \( A_n^2 \), which is \( (W_L)^g \) where \( L \) is the root lattice of type \( A_n \).

\[ \text{Corollary 5.2.9. We have} \]

\[ \text{gr}((W_L)^g) = J_{\infty}^g(C[E_{i,j}|1 \leq i < j \leq n + 1]/(\sum_{\sigma \in S_2} E_{i_1,j_{\sigma_1}} E_{i_2,j_{\sigma_2}}|j_1 > i_2)), \]

where \( 1 \leq i_1 \leq i_2 \leq n + 1 \) and \( 1 \leq j_1 \leq j_2 \leq n + 1 \).

\[ \text{Remark 5.2.10.} \] By using tools of vertex algebras, the structure of the principal subspaces
of type $A_n^2$ was studied in depth in [29], [30], etc.. Here, Corollary 5.2.9 provides us another possible approach to study the twisted principal subspaces from the point view of commutative algebras.

5.3 Feigin-Stoyanovsky type subspaces

In this section, we consider Feigin-Stoyanovsky type subspaces of affine vertex algebras of type $A_n$ at level 1. We first consider the special case when $\omega = \omega_1$. For any element of the $A_n$ root lattice, $\alpha = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n$, we define a subspace of $W_{A_1,0}'$ as $(W_{A_1,0}')^\alpha := \{ v \in W_{A_1,0}' | cwt(v) = \alpha \}$. It is not hard to see that $(W_{A_1,0}')^\alpha$ is nontrivial if and only if $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. According to [103, (3.8)], we have

$$\text{ch}[(W_{A_1,0}')^\alpha](q) = \frac{q^{\sum_{i=1}^{n} m_i^2 - \sum_{i=1}^{n-1} m_i m_{i+1}}}{(q)_{m_n} (q)_{m_{n-1}-m_n} \cdots (q)_{m_1-m_2}}.$$ 

Then

$$\text{ch}[W_{A_1,0}'](q) = \sum_{0 \leq m_n \leq \cdots \leq m_1} \frac{q^{\sum_{i=1}^{n} m_i^2 - \sum_{i=1}^{n-1} m_i m_{i+1}}}{(q)_{m_n} (q)_{m_{n-1}-m_n} \cdots (q)_{m_1-m_2}}$$

$$= \sum_{(l_1, \ldots, l_n) \in \mathbb{N}^n} \frac{q^{\sum_{i=1}^{n} l_i^2 + \sum_{1 \leq i < j \leq n} l_i l_j}}{(q)_{l_1} (q)_{l_2} \cdots (q)_{l_n}}.$$ 

Moreover, in this case,

$$\Gamma = \{ \beta_1 := \alpha_1, \beta_2 := \alpha_1 + \alpha_2, \cdots, \beta_n := \alpha_1 + \cdots + \alpha_n \}.$$ 

Note that

$$L = \mathbb{Z} \beta_1 \oplus \cdots \oplus \mathbb{Z} \beta_n$$

is a lattice with basis $\{ \beta_1, \cdots, \beta_n \}$. Then we have

$$W_L \cong W_{A_1,0}'.$$
It is not hard to see that
\[ \langle \beta_i, \beta_i \rangle = 2, \quad \text{if } i = 1, \cdots, n \]
\[ \langle \beta_i, \beta_j \rangle = 1, \quad \text{if } 1 \leq i \neq j \leq n. \]

According to Proposition 5.2.3, we have that the $C_2$-algebra of $W_L$ is
\[ \mathbb{C}[x_1, \cdots, x_n]/\langle x_ix_j | 1 \leq i \leq j \leq n \rangle. \]

By similar argument in previous section, we get
\[ HS_q(J_\infty(\mathbb{C}[x_1, \cdots, x_n]/\langle x_ix_j | 1 \leq i \leq j \leq n \rangle)) = \text{ch}[W_L](q), \]
which implies isomorphism between $J_\infty(\mathbb{R}^\Gamma_{1,0})$ and $gr(W^\prime_{\Lambda,0})$. Similarly we can also prove the isomorphism in cases where $\omega = \omega_i$, $2 \leq i \leq n$ by making use of [103, (3.21)].

### 5.4 Principal subspaces and arc (super)algebras from graphs

In this part we study principal subspaces and arc algebras coming from graphs. We begin from any graph $G$ with $k$ vertices and possibly with loops (and for simplicity we assume no double edges). We denote the vertices of $G$ by \{v_1, v_2, \cdots, v_k\}. We denote by $\Gamma := \Gamma(G)$ the (symmetric) incidence matrix of $G$ and by $(L(\Gamma), \langle , \rangle)$ rank $k$ lattice with basis $\alpha_1, \ldots, \alpha_k$, such that $\langle \alpha_i, \alpha_j \rangle = (\Gamma)_{i,j}$. The incidence matrix of the graph induces a quadratic form
\[ \Gamma \rightarrow \frac{1}{2} Q(x_1, \ldots, x_k), \]
where
\[ Q(x_1, \ldots, x_k) = \sum_{i,j=1}^{k} x_ix_j, \]
where we sum over all edges $E(G)$. Out of monomials appearing in the sum we form the arc algebra $J_\infty(\mathbb{R}^\Gamma)$, where
\[ R^\Gamma = \mathbb{C}[x_1, \cdots, x_k]/\langle \cup_{v_i,v_j \in E(G)} x_ix_j \rangle. \]
We let \( W_{L(\Gamma)} \subset V_{L(\Gamma)} \) be the principal subspace corresponding to \( \{e^{\alpha_i}\}_{i=1}^k \) inside the lattice vertex algebra \( V_{L(\Gamma)} \). For simplicity we write \( W_\Gamma \) for \( W_{L(\Gamma)} \).

**Example 5.4.1.** Consider the graph \( \circ - \circ - \circ \). Then \( \Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \), and \( W_\Gamma = \langle e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3} \rangle \) where \( L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \) with \( \langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_3 \rangle = 1 \) (zero otherwise), \( R_\Gamma = \mathbb{C}[x_1, x_2, x_3]/(x_1x_2, x_2x_3) \), and \( Q(x_1, x_2, x_3) = x_1x_2 + x_2x_3 \).

**Theorem 5.4.2.** If the bilinear form associated with \( \Gamma \) is non-degenerate, that is \( \Gamma \) is invertible, then there exists a unique conformal vector in lattice vertex algebra such that eigenvalue of \( L_{(0)} \) defines grading such that:

\[
\text{wt}(e^{\alpha_i}) = \frac{3}{2} \quad \text{if} \quad \langle \alpha_i, \alpha_i \rangle = 1,
\]

\[
\text{wt}(e^{\alpha_i}) = 1 \quad \text{if} \quad \langle \alpha_i, \alpha_i \rangle = 0.
\]

Moreover, the graded dimension is given by:

\[
\text{ch}[W_\Gamma](q) = \sum_{n_1, \cdots, n_k \in \mathbb{N}} q^{n_1 + n_2 + \cdots + n_k + \frac{1}{2} Q(n_1, \cdots, n_k)} (q)_{n_1} \cdots (q)_{n_k}.
\]

**Proof.** Clearly, we have the standard conformal vector in the lattice vertex algebra given by \( \omega_{st} = \frac{1}{2} \sum_{i=1}^n u_{(-1)}^{(i)} u_{(-2)}^{(i)} \), where \( \{u^{(1)}, \cdots, u^{(n)}\} \) is an orthonormal basis with respect to the bilinear form associated with \( \Gamma \). We know that

\[
L_{st(0)}(e^{\alpha_i}) = \frac{\langle \alpha_i, \alpha_i \rangle}{2}.
\]

It is clear that by adding a linear combination of \( \{(\alpha_i)_{(-2)}1\}_{i=1}^n \), we will still get a conformal vector. Now assume that \( \omega_{st} + \sum_{i=1}^n a_i(\alpha_i)_{(-2)}1 \), where \( a_i \in \mathbb{C} \) would give us expected weights. Then we have a system of linear equations. The non-degeneracy of the bilinear form implies that there is an unique solutions set. Thus we always have a conformal vector with the grading:

\[
\text{wt}(e^{\alpha_i}) = \frac{3}{2} \quad \text{if} \quad \langle \alpha_i, \alpha_i \rangle = 1,
\]

\[
\text{wt}(e^{\alpha_i}) = 1 \quad \text{if} \quad \langle \alpha_i, \alpha_i \rangle = 0.
\]
By applying [85, Corollary 4.14], we can write a combinatorial basis of $W_{\Gamma}$. Now let us use this basis to write down the character. Firstly, the generating function of colored partition into $(n_1, n_2, \cdots, n_k)$ parts is $\frac{1}{(q)_{n_1} \cdots (q)_{n_k}}$. It is clear that

$$\text{ch}[W_{\Gamma}](q) = \sum_{k_1, \cdots, k_k \in \mathbb{N}} q^{\text{wt}(f_{(n_1, \cdots, n_k)})} / (q)_{n_1} \cdots (q)_{n_k},$$

where $f_{(n_1, \cdots, n_k)}$ is the vector in $W_{\Gamma}$ of charge $(n_1, \cdots, n_k)$ with the minimal weight. For the $n_i$ part, there is an unique element $u_{n_i}$ of the minimal weight which is

$$e^{\alpha_i}(-1 - \sum_{j=1}^{i-1} \langle \alpha_i, \alpha_j \rangle n_j - (n_i - 1) \langle \alpha_i, \alpha_i \rangle) \cdots e^{\alpha_i}(-1 - \sum_{j=1}^{i-1} \langle \alpha_i, \alpha_j \rangle n_j) 1.$$

The weight of $u_{n_i}$ is

$$\frac{n_i}{2} (2 \sum_{j=1}^{i-1} \langle \alpha_i, \alpha_j \rangle n_j + \text{wt}((e^{\alpha_i})_{(-1)} 1) + (n_i - 1) \langle \alpha_i, \alpha_i \rangle)$$

$$= \sum_{j=1}^{i-1} \langle \alpha_i, \alpha_j \rangle n_i n_j + \frac{n_i^2}{2} \langle \alpha_i, \alpha_i \rangle + (-\langle \alpha_i, \alpha_i \rangle) 2 + \text{wt}((e^{\alpha_i})_{(-1)} 1)) n_i.$$

Therefore,

$$\text{wt}(f_{(n_1, \cdots, n_k)}) = \sum_{i=1}^{k} \text{wt}(u_{n_i})$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{i-1} \langle \alpha_i, \alpha_j \rangle n_i n_j + \frac{n_i^2}{2} \langle \alpha_i, \alpha_i \rangle + (-\langle \alpha_i, \alpha_i \rangle) 2 + \text{wt}((e^{\alpha_i})_{(-1)} 1)) n_i$$

$$= n_1 + n_2 + \cdots + n_k + \frac{1}{2} Q(n_1, \cdots, n_k).$$

Thus we proved the claimed identity.

Remark 5.4.3. If the lattice $L$ is degenerate, then $V_L$ has no conformal vector which can give us expected weights. But we can still view $W_L$ as a graded vertex algebra, if we define the weight of $e^{\alpha_i}$ as above. Then the character formula is still valid for singular $\Gamma$.

Before we prove next result, let us generalize [89, Theorem 4.3.1].
Proposition 5.4.4. We have an isomorphism

$$gr(W_\Gamma) \cong \mathbb{C}[x_i(p)|i = 1, \ldots, k, p \in \mathbb{Z}_-]$$

$$\left(\sum_{m=0}^{l-1} \frac{m + \langle \alpha_i, \alpha_j \rangle - 1)!}{m!}\langle \alpha_i, \alpha_j \rangle x_i(-\langle \alpha_i, \alpha_j \rangle - m)x_j(l + m)|1 \leq i, j \leq k, l \leq -1\right).$$

Proof. First, we define a map \( \pi \) from \( \mathbb{C}[x_i(p)|i = 1, \ldots, k, p \in \mathbb{Z}_-] \) to \( gr(W_\Gamma) \) by sending \( x_i(p) \) to \( e^{\alpha_i}_{(p)}1 \). We denote the ideal

$$\langle \sum_{m=0}^{l-1} \frac{m + \langle \alpha_i, \alpha_j \rangle - 1)!}{m!}\langle \alpha_i, \alpha_j \rangle x_i(-\langle \alpha_i, \alpha_j \rangle - m)x_j(l + m)|1 \leq i, j \leq k, l \in \mathbb{Z}_-\rangle$$

by \( I_\Gamma \). We can use the same argument from [89] to show that \( I_\Gamma \subset ker(\pi) \).

We prove \( ker(\pi) \subset I_\Gamma \) by contradiction. Suppose there exists an element \( a \in \mathbb{C}[x_i(p)|i = 1, \ldots, k, p \in \mathbb{Z}_-] \) such that \( a \in ker(\pi) \) and \( a \notin I_\Gamma \). Suppose \( a \) is homogeneous with respect to weight and charge. Choose \( r \) such that \( a \) contains some element \( x_r(p) \) as a factor. We assume that \( a \) has the minimum weight among all elements that satisfy above conditions. Again from the same argument from [89], this \( a \) can be written as \( bx_r(-1) \), where \( b \in \mathbb{C}[x_i(p)|i = 1, \ldots, k, p \in \mathbb{Z}_-] \). We prove the case when \( \langle \alpha_r, \alpha_r \rangle = 0 \). For other cases, it is proved in [89]. Firstly we define a map \( e^{\alpha}_r : W_\Gamma \to W_\Gamma \) as

$$e^{\alpha}_r((e^{\alpha_j}_{(m)}1) = (e^{\alpha_j}_{(m)})(e^{\alpha_r}_{(-1)})1.$$}

Then we lift this map to

$$x_r : \mathbb{C}[x_i(p)|i = 1, \ldots, k, p \in \mathbb{Z}_-] \to \mathbb{C}[x_i(p)|i = 1, \ldots, k, p \in \mathbb{Z}_-],$$

which is defined as

$$x_r(x_i(j)) = x_i(j)x_r(-1).$$
Since $a \in \ker(\pi)$, $\pi(a) = \pi(bx_r(-1)) = 0$. Then

$$e^{-\alpha_r(\pi(bx_r(-1)))} = \pi(b) = 0,$$

which implies that $b \in \ker(\pi)$. If $b \in I_I$, then $a = x_r(b) \in x_rI_I \subset I_I$ which contradicts with our assumption. If $b \notin I_I$, then $b$ is an element such that $b \in \ker(\pi)$ and $b \notin I_I$ but with the weight strictly less than the weight of $a$. This also contradicts our assumption. Thus we proved the claim.

**Theorem 5.4.5.** We have that

$$gr(W_I) \cong J_{\infty}(C[y_1, y_2, \ldots, y_k]/\langle\langle\alpha_i, \alpha_j\rangle y_i y_j | 1 \leq i, j \leq k\rangle).$$

**Proof.** From the definition of arc superalgebra, we know that

$$T^{(-l-1)}(\langle\alpha_i, \alpha_j\rangle y_i y_j) = \sum_{m=0}^{-l-1} c^l_m \langle\alpha_i, \alpha_j\rangle y_i(-\langle\alpha_i, \alpha_j\rangle - m)y_j(l + m),$$

where $c^l_m$ is a constant coefficient. Therefore,

$$J_{\infty}(C[y_1, y_2, \ldots, y_k]/\langle\langle\alpha_i, \alpha_j\rangle y_i y_j | 1 \leq i, j \leq k\rangle)$$

has quotient relation

$$\left(\sum_{m=0}^{-l-1} c^l_m \langle\alpha_i, \alpha_j\rangle y_i(-\langle\alpha_i, \alpha_j\rangle - m)y_j(l + m) | 1 \leq i, j \leq k, l \in \mathbb{Z}_-\right).$$

Together with Proposition 5.4.4, we get an isomorphism of differential algebras induced from the map $\psi : x_i(-1) \rightarrow y_i(-1)$.

When $\langle\alpha_i, \alpha_i\rangle = 1$, we increase the weight of $y_i(-1)$ by $\frac{1}{2}$. Then clearly we have

$$HS_q(J_{\infty}(R_I)) = ch[W_I](q).$$

### 5.5 Positive lattices

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Given a lattice $L$ of rank $n$ with a $\mathbb{Z}$-basis $\{\alpha_i\}_{i=1}^n$. We say that the basis is positive if we have $\langle \alpha_i, \alpha_j \rangle \in \mathbb{N}$ for $1 \leq i \leq j \leq n$. In this part, we study principal subspaces associated with positive bases. Examples we studied in previous two sections are such principal subspaces. Now let us prove a more general result about the map $\psi$ and such principal subspaces. Arakawa, Kawasetsu and Sebag also proved the following result for non-super case by using free vertex algebras in recent preprint [9].

**Theorem 5.5.1.** For a lattice $L$ of rank $n$ with a positive basis, the map $\psi$ is an isomorphism for $W_L$ if and only if its positive basis satisfies $\langle \alpha_i, \alpha_i \rangle = a$, where $a = 0$ or $1$ or $2$, and $\langle \alpha_i, \alpha_j \rangle = b$, where $b = 0$ or $1$.

**Proof.** First let us assume that the positive basis of the lattice $L$ satisfies given conditions. According to Theorem 5.4.5, we know that when $\langle \alpha_i, \alpha_i \rangle = a$, where $a = 0$ or $1$ and $\langle \alpha_i, \alpha_j \rangle = b$, where $b = 0$ or $1$, the map $\psi$ is an isomorphism for the principal subspace. Now the only case we need to consider is the positive basis for which $\langle \alpha_i, \alpha_j \rangle = 2 \delta_{i,j}$. It is not hard to see that $J_\infty(\mathbb{C}[x]/x^2)$ has a basis
\[
\{x_{(m_1)}x_{(m_2)}\ldots x_{(m_k)}|m_{j-1} \leq m_j - 2, k \in \mathbb{N}\}.
\]
Thus $J_\infty(\mathbb{C}[x_1, x_2, \ldots, x_n]/\langle x_1^2, x_2^2, \ldots, x_n^2 \rangle)$ has a basis
\[
\{(x_{i_1})(m_{i_1})(x_{i_2})(m_{i_2})\ldots(x_{i_k})(m_{i_k})\cdots(x_{i_n})(m_{i_n})(m_{k_1})\cdots(x_{i_n})(m_{i_n})(m_{k_n})|m_{j-1} \leq m_j - 2, 1 \leq j \leq k_i - 1\}.
\]
Note that the $C_2$-algebra of $W_L$ is
\[
\mathbb{C}[x_1, \ldots, x_n]/\langle x_1^2, \ldots, x_n^2 \rangle.
\]
Now the map $\psi$ is sending $(x_i)(-1)$ to $(e^{\alpha_i})(-1)1$. According to [85 Corollary 4.14], the image of the basis of $J_\infty(R_{W_L})$ is the basis of $gr(W_L)$. Thus the map $\psi$ is an isomorphism.

Next, let us prove that if the basis does not satisfy given conditions, the map $\psi$ is not an isomorphism. We will consider two cases:

- Suppose that for one simple root $\alpha_i$, we have $\langle \alpha_i, \alpha_i \rangle \geq 3$. Without loss generality,
we prove that $\psi$ is not an isomorphism when lattice $L = \mathbb{Z}\alpha_i$. In this case, from [85, Corollary 4.14], the basis of $gr^F W_L$ is

$$\left\{(e^{\alpha_i})_{(m_1)}(e^{\alpha_i})_{(m_2)} \cdots (e^{\alpha_i})_{(m_k)} \mathbf{1} | m_{j-1} \leq m_j - \langle \alpha_i, \alpha_i \rangle, \ m_k \in \mathbb{Z}_-, \ k \in \mathbb{N}\right\}. \quad (5.4)$$

It is clear that neither $J_\infty(\mathbb{C}[x]/(x^2))$ nor $J_\infty(\bigwedge[x])$ has the same corresponding basis (here $\bigwedge$ denotes the exterior algebra). Indeed, for arc algebra $J_\infty(\mathbb{C}[x]/(x^2))$ we have two monomials of weight 4 with two variables, i.e., $x_{(-1)}x_{(-3)}$ and $x_{(-2)}x_{(-2)}$, and one quotient relation of weight 4, i.e., $x_{(-2)}x_{(-2)} + x_{(-1)}x_{(-3)} = 0$. Therefore, either $x_{(-1)}x_{(-4)}$ or $x_{(-2)}x_{(-2)}$ should be a basis element. But we do not have such corresponding element in (5.4). Similar argument applies to $J_\infty(\bigwedge[x]).$

- Suppose that there exists two distinct roots $\alpha_i, \alpha_j$, where $i < j$ such that $\langle \alpha_i, \alpha_j \rangle \geq 2$. Without loss of generality, we assume $L = \mathbb{Z}\alpha_i \oplus \mathbb{Z}\alpha_j$, then the basis of $J_\infty(W_L)$ is

$$\left\{(x_i)_{(-1-m_1)}(x_i)_{(-1-m_2)} \cdots (x_i)_{(-1-m_k)}(x_j)_{(-1-n_1)}(x_j)_{(-1-n_2)} \cdots (x_j)_{(-1-n_l)} | m_1 - m_2 \geq \langle \alpha_i, \alpha_i \rangle, \ n_1 - n_2 \geq \langle \alpha_j, \alpha_j \rangle, \ m_k \geq l, n_l \in \mathbb{N}\right\}.$$  

Meanwhile, according to [85, Corollary 4.14], the image of this basis under $\psi$ strictly contains the basis of $W_L$. We do not have isomorphism.

Thus we proved the statement. \hfill \Box

### 5.6 New character formulas for $\text{ch}[W_\Gamma]$

If the graph $\Gamma$ is the Dynkin diagram of type $A_k$ or $C_k$ (cycle of length $k$) we expect that the generating series $HS_q(J_\infty(R_\Gamma))$ has much better behaved combinatorial and perhaps even mock modular properties. We now present “sum of tails” formulas for $HS_q(J_\infty(R_{A_k}))$ for several low “rank” cases. To simplify notation we let

$$A_k(q) := HS_q(J_\infty(R_{A_k})).$$

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From Theorem 5.4.2 we have a fermionic formula

\[
A_k(q) = \sum_{n_1, n_2, \ldots, n_k \in \mathbb{N}} \frac{q^{n_1 + n_2 + \cdots + n_k + n_1 n_2 + n_2 n_3 + \cdots + n_{k-1} n_k}}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_k}} ,
\]  

(5.5)

Next formulas are recently given by Jennings-Shaffer and Milas [62].

**Theorem 5.6.1.** We have

- \( A_2(q) = \frac{1}{(1-q)(q)_\infty} \),
- \( A_3(q) = q^{-1} \left( \frac{1}{(q)_\infty} - \frac{1}{(q)_\infty} \right) \),
- \( A_4(q) = \frac{q^{-1}}{(q)_\infty} \sum_{n \geq 1} \frac{q^n}{1-q^n} \),
- \( A_5(q) = \frac{1}{(q)_\infty} \sum_{n \in \mathbb{N}} (q)_n \frac{q^n}{(1-q^{n+1})^2} \),
- \( A_6(q) = \frac{1}{(q)_\infty} \sum_{n,m \in \mathbb{N}} (q)_{n+1} (q)_{m+1} \frac{q^{n+m+nm}}{} \).

Moreover, for cyclic graphs \( C_k \)-graphs we have fermionic formulas for

\[
C_k(q) := HS_q(J_\infty(R_{C_k}))
\]

valid for \( k \geq 3 \)

\[
C_k(q) = \sum_{n_1, n_2, \ldots, n_k \in \mathbb{N}} \frac{q^{n_1 + n_2 + \cdots + n_k + n_1 n_2 + n_2 n_3 + \cdots + n_{k-1} n_k + n_k n_1}}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_k}} .
\]  

(5.6)

Again we have partial results for “bosonic” representations for 3- and 5-cycle graphs [62].

**Proposition 5.6.2.** [62, Proposition 6.1 and Section 7] We have

\[
C_3(q) = \frac{1}{(q)_\infty} \sum_{n \in \mathbb{N}} \frac{q^n}{(q^{n+1})_{n+1}} ;
\]

\[
C_5(q) = \frac{q^{-1}}{(q)_\infty} \sum_{n \in \mathbb{Z}_+} \frac{q^n}{1-q^n} .
\]
5.6.1 Combinatorial interpretation

Next we present combinatorial interpretations of formulas in Theorem 5.6.1 and Proposition 5.6.2. For simplicity, in several formulas we factored out a (power of) Euler factor which can be easily interpreted as the number of (colored) partitions.

**Theorem 5.6.3.** We have:

- $A_2(q)$ counts the number of partitions of $2n$ with all parts either even or equal to 1.
- $qA_3(q)$ counts the number of partitions of $n + 1$ into two kinds of parts with the first kind of parts used in each partition.
- $q(q)_\infty A_4(q)$ counts the total number of parts in all partitions of $n$, which is also sum of largest parts of all partitions of $n$.
- $(q)_\infty^2 A_5(q)$ is the sum of the numbers of times that the largest part appears in each partition of $n$.
- $q(q)_\infty^2 A_6(q)$ counts twice the total number of parts in all partitions of $n$ minus the number of partitions of $n$.
- $(q)_\infty C_3(q)$ counts the number of partitions of $n$ such that twice the least part is bigger than the greatest part.
- $q(q)_\infty C_5(q)$ counts the sum of all parts of all partitions of $n$, also known as $np(n)$.

**Proof.** For $A_2(q)$, observe that $\text{Coeff}_{q^n} A_2(q) = p(1) + p(2) + \cdots + p(n)$, where $p(i)$ is the number of partitions of $i$. The number of 1’s must be even, say $2k$, so we have to compute the number of partitions of $2n - 2k$ where all parts are even. This is given by $p(n - k)$. Then summing over $k$ gives the claim.

The interpretation for the $A_3(q)$ series is clear because we can also write

$$qA_3(q) = \frac{1}{(q)_\infty} \left( \frac{1}{(q)_\infty} - 1 \right).$$

Extracting the coefficient on the right-hand side gives $p_2(n) - p(n)$, where $p_2(i)$ denotes the number of two colored partitions.

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For $A_4(q)$, this can be seen from identity
\[ \sum_{n \in \mathbb{Z}^+} \frac{q^n}{(q)_{\infty}^{1-q^n}} = \sum_{n \in \mathbb{Z}^+} \frac{nq^n}{(q)_n}, \]
which follows by taking the $(x \frac{d}{dx})$ derivative of $\frac{1}{(xq;q)_{\infty}} = \sum_{n \in \mathbb{N}} x^n q^n$. This clearly counts the total number of parts in all partitions of $n$.

The $(q)_{\infty}^2 A_5(q)$ case was already discussed in [62].

For $(q)_{\infty}^2 A_6(q)$, this follows from another identity given in [62]:

\[ \frac{1}{(q)_{\infty}^2} \sum_{n,m \in \mathbb{N}} \frac{q^{n+m+nm}}{(q)_{n+1}(q)_{m+1}} = \frac{q^{-1}}{(q)_{\infty}^2} \left( 2 \sum_{n \in \mathbb{Z}^+} \frac{q^n}{(1 - q^n)(q)_{\infty}} + 1 - \frac{1}{(q)_{\infty}} \right), \]

together with a previous observation that $\sum_{n \in \mathbb{Z}^+} \frac{q^n}{(q)_{\infty}^{1-q^n}}$ counts the total number of parts in all partitions of $n$.

For $(q)_{\infty} C_3(q)$ we use a well-known interpretation for the fifth order mock theta function, and finally for $(q)_{\infty} C_5(q)$ we observe the formula
\[ \left( q \frac{d}{dq} \right) \frac{1}{(q)_{\infty}} = \frac{1}{(q)_{\infty}} \sum_{n \in \mathbb{Z}^+} \frac{nq^n}{1 - q^n} = \sum_{n \in \mathbb{Z}^+} np(n)q^n \]
as claimed.

\[ \square \]

**Remark 5.6.4.** It is interesting to observe that the numerators of $C_3(q)$ and $C_5(q)$ are mock modular forms, and thus $C_3(q)$ and $C_5(q)$ are *mixed* mock. Completion of the Ramanujan fifth order mock theta function $\sum_{n \in \mathbb{N}} \frac{q^n}{(q^{n+1})_{n+1}}$ is well-documented [20]. For $\sum_{n \in \mathbb{Z}^+} \frac{nq^n}{1 - q^n}$ we only have to observe that adding $-\frac{1}{24}$ to the numerator gives $E_2(\tau)$, the weight two quasimodular Eisenstein series, which is known to be mock.
In this Chapter we study q-series identities and Feigin-Stoyanovsky’s principal subspaces [49, 85]. Although our understanding of principal subspaces and their characters has greatly improved with the help of vertex algebra tools (cf. [55, 36, 28, 93]), we found that known character formulas are not sufficient to determine whether the $\psi$ map is injective. The main reason is that in most approaches to character formulas only simple roots are used instead of all positive roots (for some recent result on this subject see for instance [90], [24]). In present Chapter, we try to fill the gap between two approaches to principal subspaces by studying character formulas coming from all positive roots. We will prove the following q-series identities.

**Theorem 6.0.1.** Let $A$ be the Cartan matrix $((\alpha_i, \alpha_j))_{1 \leq i, j \leq n-1}$ of type $A_{n-1}$, $n \geq 2$, that is

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & i = j \\ -1 & |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$m = (m_{1,2}, \ldots, m_{n-1,n}) = (m_{i,j})_{1 \leq i < j \leq n}.$$

(a) We have $(q, x_1, \ldots, x_n)$-series identities:

$$\sum_{m \in \mathbb{N}^{(n-1)/2}} q^{B(m)} \frac{x_1^{\lambda_1} \cdots x_{n-1}^{\lambda_{n-1}}}{\prod_{1 \leq i < j \leq n} (q)_{n_{i,j}}} = \sum_{\mathbf{k} = (k_1, \ldots, k_{n-1}) \in \mathbb{N}^{n-1}} q^{\frac{1}{2} \mathbf{k}^\top A \mathbf{k}} \frac{x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}}{(q)_{k_1}(q)_{k_2} \cdots (q)_{k_{n-1}}},$$

(6.1)
where

$$B(m) = \sum_{1 \leq i_1 < j_1 \leq n} m_{i_1,j_1} + \sum_{1 \leq i_2 < j_2 \leq n} m_{i_2,j_2} + \sum_{1 \leq i_1 < j_1 \leq n, j_1 > i_2 + 1} m_{i_1,j_1}m_{i_2,j_2} + \sum_{1 \leq i_1 < j_1 \leq n, j_1 = j_2} m_{i_1,j_1}m_{i_2,j_2} + \sum_{j < i < i + 1 < j'} m_{i,j,i'},$$

and

$$\lambda_i = \sum_{1 \leq s \leq i, i < \ell \leq n} m_{s,\ell},$$

(b) we have

$$\sum_{m,n \in \mathbb{N}^{n(n-1)/2}} q^{B'(m)} x_1^{\lambda_1} \cdots x_n^{\lambda_n-1} \prod_{1 \leq i < j \leq n} (q)m_{i,j} = \sum_{k=(k_1,\ldots,k_{n-1}) \in \mathbb{N}^{n-1}} q^{\frac{1}{2}k^TAk} x_1^{k_1} \cdots x_n^{n-1} \prod_{k=1}^{n-1} (q)^{k} (q)^{k_1} \cdots (q)^{k_{n-1}}, \quad (6.2)$$

where

$$B'(m) = \sum_{1 \leq i_1 < j_1 \leq n} m_{i_1,j_1}m_{i_2,j_2},$$

and

$$\lambda_i = \sum_{1 \leq s \leq i, i < \ell \leq n} m_{s,\ell}.$$

Example 6.0.2. For sl₃, we have $\lambda_1 = m_{1,2} + m_{1,3}, \lambda_2 = m_{2,3} + m_{1,3}$. Letting new variables $m = n_1, n = n_2$ and $n_1 = m_{1,2}, n_2 = m_{1,3}$ and $n_3 = m_{2,3}$, both equations (6.1) and (6.2) give the following identity

$$\sum_{m,n \in \mathbb{N}} \frac{q^{m^2+n^2-2mn} x^m y^n}{(q)_m(q)_n} = \sum_{n_1,n_2,n_3 \in \mathbb{N}} \frac{x^{n_1+n_3} y^{n_2+n_3} q^{n_1^2+n_2^2+n_3^2+n_1 n_3+n_2 n_3}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}}. \quad (6.3)$$

Notice that both the LHS and RHS of our identities are “stable” with respect to the
rank \( n \). In other words, if we let \( m_{,n} = 0 \) in the theorem, then all identities are valid for \( sl_{n-1} \) (i.e. type \( A_{n-2} \)).

In present Chapter we are concerned with \( q \)-series identities of the form

\[
\sum_{n \in \mathbb{N}^k} \frac{q^{\frac{1}{2}n^\top A_n x_n}}{(q)_{n_1} \cdots (q)_{n_k}} = \sum_{m \in \mathbb{N}^\ell} \frac{q^{B(m)} x_U m}{(q)_{m_1} \cdots (q)_{m_\ell}},
\]

(6.4)

where \( x_n = x_{n_1}^{n_1} \cdots x_{n_k}^{n_k} \), \( A \) is a positive definite integral \( k \times k \) matrix (e.g. Cartan matrix), \( B(m) \) is a quadratic form on \( \ell \)-dimensional space (normally \( \ell \) is bigger than \( k \)) and \( U \) is a \( k \times \ell \) matrix. We say that the \( q \)-series identity (6.4) is equivalent to

\[
\sum_{n \in \mathbb{N}^k} \frac{q^{\frac{1}{2}n^\top \tilde{A}_n x_n}}{(q)_{n_1} \cdots (q)_{n_k}} = \sum_{m \in \mathbb{N}^\ell} \frac{q^{\tilde{B}(m)} x_U m}{(q)_{m_1} \cdots (q)_{m_\ell}},
\]

(6.5)

if for every \( n = (n_1, ..., n_k) \), there is a solution of \( U m = n \) such that

\[
B(m) - \frac{1}{2} n^\top A n = \tilde{B}(m) - \frac{1}{2} n^\top \tilde{A} n.
\]

Under this condition, clearly, (6.4) implies (6.5) and vice-versa as all \( x \) coefficients are equal.

The main idea in the proof of Theorem 6.0.1 is to show that identities (6.1) and (6.2) are in fact equivalent to a pair of identities derived on one hand from the quantum dilogarithm and on the other hand from quiver representations. Since the right-hand side of identities (6.1) and (6.2) is the known (charged) character formula of the principal subspace \( W(\Lambda_0) \) of the affine vertex algebra of type \( A_{n-1} \), as a corollary of Theorem 6.0.1, we also get two new charged character formulas for the principal subspace. Moreover, following the idea in [50] on PBW filtration, we carefully construct a series of filtrations on certain infinite arc algebras of quotient polynomial algebras. From that and Theorem 6.0.1, we obtain the Hilbert series formulas for certain arc algebras. This in particular, clarifies the argument in Theorem 5.2.7, where it was claimed that \( W_{\Lambda_1,0} \) is classically free. Our result also provides a new combinatorial basis of \( W_{\Lambda_1,0} \).
6.1 Principal subspaces of type A

The principal subspace of type $A$, $W_{A_{1,0}}$, was studied by several authors \[49, 26, 28, 55, 98\]. According to \[47\] and \[75\], the $C_2$-algebra $R_{W_{A_{1,0}}}$ of the principal subspace of the affine vertex algebra $L_1(sl_n)$ is isomorphic to

$$\mathbb{C}[E_{i,j} | 1 \leq i < j \leq n]/\langle \sum_{\sigma \in S_2} E_{i_1,j_{\sigma_1},E_{i_2,j_{\sigma_2}} | j_1 > i_2} \rangle,$$

where $E_{i,j}$ can be identified with the roots vectors in $n_+$, and $1 \leq i_1 \leq i_2 \leq n, 1 \leq j_1 \leq j_2 \leq n$. We let $A := R_{W_{A_{1,0}}}$ for sake of brevity. Inside the defining quadratic binomial ideal of $A$ we have the following three types of elements (it is helpful to use graphical interpretation to visualize these elements):

- **Type I**: $E_{i_1,j_1}E_{i_2,j_2} = -E_{i_1,j_2}E_{i_2,j_1}$, where $1 \leq i_1 < j_1 \leq n, 1 \leq i_2 < j_2 \leq n, 1 \leq i_1 < i_2 \leq n, 1 \leq j_1 < j_2 \leq n, j_1 > i_2, j_1 > i_2 + 1.$

- **Type II**: $E_{i_1,j_1}E_{i_2,j_2}$, where $1 \leq i_1 < j_1 \leq n, 1 \leq i_2 < j_2 \leq n, i_1 = i_2, 1 \leq j_1 \leq j_2 \leq n, 1 \leq i_1 < j_1 \leq j_2 \leq n, 1 \leq i_2 < j_2 \leq n, 1 \leq j_1 = j_2 \leq n, 1 \leq i_1 < i_2 \leq n.$

- **Type III**: $E_{i,i+1}E_{j,j'} = -E_{i,j'}E_{j,i+1}$, where $j < i < i + 1 < j'.$

**Remark 6.1.1.** Inside the arc algebra $J_{\infty}(A)$, Type I relations generate further relations coming from $z$-coefficients of

$$E_{i_1,j_1}^+(z)E_{i_2,j_2}^+(z) + E_{i_1,j_2}^+(z)E_{i_2,j_1}^+(z) = 0,$$

66
where $E_{i,j}^+(z) = \sum_{n \in \mathbb{Z}_-} (E_{i,j}(n)) z^{-n-1}$ denotes the holomorphic part of the field associated to $E_{i,j}$. For Type II and III relations, we use similar notation. In order to simplify presentation, we often write $A^+(z)B^+(z) = 0$ to denote set of relations obtained by taking all $z$-coefficients in the expansion.

Let us consider another commutative algebra

$$B = \mathbb{C}[E_{i,j}|1 \leq i < j \leq n]/I,$$

where $I$ is generated by all Type II relations and left-hand side of all relations of type I and III. This way $I$ is a monomial ideal. This algebra and its arc algebra are easier to analyze. We will show that

$$HS_q(J_{\infty}(A)) \leq HS_q(J_{\infty}(B)),$$

where inequality among $q$-series has the obvious meaning: all $q$-coefficients on the left-hand side are less than or equal to the corresponding coefficients on the right-hand side.

Following an idea of E. Feigin on PBW filtration in [50], for $n \geq 3$, we introduce a sequence of increasing filtrations $G^{i,j}$ ($n-2 \geq j \geq i \geq 1$) on arc algebras $J_{\infty}(\mathcal{B})$, where $\mathcal{B}$ is a quotient polynomial algebra, $\mathbb{C}[E_{a,b}|1 \leq a < b \leq n]/I$ by an ideal generated by homogeneous monomials and binomials of weight 2. Although its definition is somewhat technical, the idea behind is very simple - it essentially replaces some binomial quotient relations of $\mathcal{B}$ with monomial ones inside the corresponding associated graded algebra. Eventually, we will end up with a quotient polynomial by a quadratic monomial ideal; the arc algebra associated to it is easier to handle using vertex algebra methods, and its Hilbert series can be computed explicitly [75].

Fix $i$ and $j$ as above. We define the filtration $G^{i,j}$ on $J_{\infty}(\mathcal{B})$ by letting $G^{i,j}_0$ be generated with

$$E_{i,i+2}, E_{i-1,i+2}, \ldots, E_{j,i+2}.$$

We let

$$\mathcal{E}_{i,j} = \{(i, i+2), (i-1, i+2), \ldots, (j, i+2)\},$$

and

$$G^{i,j}_s = \text{span}\{(E_{u,u'})_{(i)} v| i \in \mathbb{Z}_-, (u, u') \notin \mathcal{E}_{i,j}, v \in G^{i,j}_{s-1}\} + G^{i,j}_{s-1}.$$
Here the subscript $s$ is the filtration parameter, and

$$G_{0}^{i,j} = \text{span} \left\{ (E_{i,i+2})_{(k_1)}^{l_1} (E_{i-1,i+2})_{(k_2)}^{l_2} \cdots (E_{j,i+2})_{(k_{i-j+1})}^{l_{i-j+1}} \mid l_r \in \mathbb{N}, k_s \in \mathbb{Z}_{-} \right\}. $$

Then the associated graded algebra of $J_{\infty}(B)$ is defined as

$$gr^{i,j}(J_{\infty}(B)) = G_{0}^{i,j} \oplus \bigoplus_{s>0} G_{s}^{i,j} / G_{s-1}^{i,j}. $$

For better transparency of formulas we abuse notation and use $(E_{a,b})_{-t}$, $1 \leq a < b \leq n$, to denote its representative inside the associated graded algebra $gr^{i,j}(J_{\infty}(B))$.

Next, we introduce a sequence of $(i,j)$ filtrations in a particular order and also construct a commutative algebra at each step of filtration.

- **Step 1**: Let $B_{0,0} := J_{\infty}(A)$. We can define an algebra homomorphism from $P = \mathbb{C}[E_{i,j} | 1 \leq i < j \leq n]$ to $gr^{1,1}(J_{\infty}(A))$ by sending $E_{i,j}$ to its equivalence class. We denote the kernel of this map by $\mathbb{I}_{1,1}$, and define a new algebra

$$B_{1,1} := \mathbb{C}[E_{i,j} | 1 \leq i < j \leq n] / \mathbb{I}_{1,1}. $$

- **Step 2.1**: Define

$$B_{2,1} := \mathbb{C}[E_{i,j} | 1 \leq i < j \leq n] / \mathbb{I}_{2,1}, $$

where $\mathbb{I}_{2,1}$ is the kernel of the algebra homomorphism from $P$ to $gr^{2,1}(J_{\infty}(B_{1,1}))$ by sending $E_{i,j}$ to its equivalence class.

- **Step 2.2**: Define

$$B_{2,2} := \mathbb{C}[E_{i,j} | 1 \leq i < j \leq n] / \mathbb{I}_{2,2}, $$

where $\mathbb{I}_{2,2}$ is the kernel of the map defined in the same manner as above from $P$ to $gr^{2,2}(J_{\infty}(B_{2,1}))$.

We list all generators of $G_{0}^{i,j}$ in each step at stage 2:

**Step 2.1** $G_{0}^{2,1} : E_{2,4} \quad E_{1,4}$
Step 2.2 $G_0^{2,2} : E_{2,4}$.

- **Step** $i.1$: Define
  \[ B_{i,1} := \mathbb{C}[E_{i,j}|1 \leq i < j \leq n]/\mathbb{I}_{i,1} \]
  analogously, where $\mathbb{I}_{i,1}$ is the kernel of the map from $P$ to $gr^{i,1}(B_{i-1,i-1}).$

- **Step** $i.i$: Define
  \[ B_{i,i} := \mathbb{C}[E_{i,j}|1 \leq i < j \leq n]/\mathbb{I}_{i,i} \]
  similarly, where $\mathbb{I}_{i,i}$ is the kernel of the map from $P$ to $gr^{i,i}(B_{i,i-1}).$

We list generators of $G_0^{i,j}$ as following:

- **Step** $i.1$ $G_0^{i,1} : E_{i,i+2} E_{i-1,i+2} \cdots E_{2,i+2} E_{1,i+2}$
- **Step** $i.2$ $G_0^{i,2} : E_{i,i+2} E_{i-1,i+2} \cdots E_{2,i+2}$
  \[ \vdots \]
- **Step** $i.i$ $G_0^{i,i} : E_{i,i+2}.$

Eventually, this procedure terminates at Step $n - 2.n - 2$, and we reached a desired algebra $B_{n-2,n-2}.$

**Claim:** From above construction of $B_{i,j},$ we have a surjective homomorphism

\[
\begin{cases}
J_\infty(B_{i,j}) \twoheadrightarrow gr^{i,j}(J_\infty(B_{i-1,j-1})) & \text{if } j - 1 \in \mathbb{Z}_+ \\
J_\infty(B_{i,j}) \twoheadrightarrow gr^{i,j}(J_\infty(B_{i-1,i-1})) & \text{if } j = 1
\end{cases}
\]

(6.6)

at Step $i.j.$

**Proof.** It is enough to prove the Claim at Stage 1.1 since the result at other Stages can be proved similarly. At Stage 1.1, there is a natural differential algebra epimorphism $\phi_1$ from
\( \text{gr}^{1,1}(J_\infty(A)) \), and this leads to an isomorphism

\[
J_\infty(\mathbb{C}[E_{i,j}|1 \leq i < j \leq n])/J \cong \text{gr}^{1,1}(J_\infty(A)),
\]

where \( J \) is the kernel of map \( \phi_1 \). Note \( J_\infty(B_{1,1}) = J_\infty(\mathbb{C}[E_{i,j}|1 \leq i < j \leq n])/(1_{1,1}) \), and this leads to an isomorphism

\[
J_\infty(\mathbb{C}[E_{i,j}|1 \leq i < j \leq n])/(1_{1,1}) \cong \text{gr}^{1,1}(J_\infty(A))
\]

We proved the Claim.

Observe that given any increasing filtration \( G^{i,j} \) on an algebra \( \mathcal{Q} \), we always have

\[
\text{HS}_q(\text{gr}^{G^{i,j}}(\mathcal{Q})) = \lim_{s \to \infty} \text{HS}_q(G^{i,j}_s) = \text{HS}_q(\mathcal{Q}),
\]

which follows from \( \text{gr}^{G^{i,j}}(\mathcal{Q}) = G^{i,j}_0 \oplus \bigoplus_{s > 0} G^{i,j}_s/G^{i,j}_{s-1} \) by taking Hilbert series on both sides. In particular, we have

\[
\begin{cases}
\text{HS}_q(\text{gr}^{i,j}(J_\infty(B_{i,j-1}))) = \text{HS}_q(J_\infty(B_{i,j-1})) & \text{if } j - 1 \in \mathbb{Z}_+ \\
\text{HS}_q(\text{gr}^{i,j}(J_\infty(B_{i-1,i-1}))) = \text{HS}_q(J_\infty(B_{i-1,i-1})) & \text{if } j = 1
\end{cases}
\]

Hence, we have a chain of inequalities:

\[
\text{HS}_q(J_\infty(B_{n-2,n-2})) \geq \text{HS}_q(J_\infty(B_{n-2,n-3})) \geq \cdots \geq \text{HS}_q(J_\infty(B_{n-2,1})) \geq \cdots \geq \text{HS}_q(J_\infty(A)).
\]

**Example 6.1.2.** Let us consider the case of \( \mathfrak{sl}_4 \). We have

\[
A = \mathbb{C}[E_{1,2}, E_{1,3}, E_{1,4}, E_{2,3}, E_{2,4}, E_{3,4}]/I,
\]

where

\[
I = \langle E_{1,2}^2, E_{1,3}^2, E_{1,4}^2, E_{2,3}^2, E_{2,4}^2, E_{3,4}^2, E_{1,2}E_{1,3}, E_{1,2}E_{1,4}, E_{1,3}E_{1,4}, E_{2,3}E_{2,4}, \\
E_{1,3}E_{2,4} + E_{1,4}E_{2,3}, E_{1,3}E_{2,3}, E_{1,4}E_{2,4}, E_{1,4}E_{3,4}, E_{2,4}E_{3,4} \rangle.
\]
In this case, we have two steps of the filtration. We list the generators of $G_{i,j}^{0}$ for each step.

- **Step 1:** We have

  \[ \text{Step 1.1} \quad G_{0}^{1,1} : E_{1,3} \]

  Here we write down the $G_{0}^{1,1}$ and $G_{1}^{1,1}$ explicitly.

  \[ G_{0}^{1,1} = \text{span} \{ (E_{1,3})_{(l,k)} | l \in \mathbb{N}, k \in \mathbb{Z}^- \} . \]

  \[ G_{1}^{1,1} = \text{span} \{ (E_{1,2})_{(k_1)}(E_{1,3})_{(k_2)}, (E_{1,4})_{(k_1)}(E_{1,3})_{(k_2)}, (E_{2,3})_{(k_1)}(E_{1,3})_{(k_2)}, \]

  \[ (E_{2,4})_{(k_1)}(E_{1,3})_{(k_2)}, (E_{3,4})_{(k_1)}(E_{1,3})_{(k_2)}, |l \in \mathbb{N}, k_1, k_2 \in \mathbb{Z}^- \} + G_{0}^{1,1} , \]

  and similarly we define $G_{s}^{1,1}$, $s \geq 2$.

  The algebra $B_{1,1}$ equals

  \[ \mathbb{C}[E_{1,2}, E_{1,3}, E_{1,4}, E_{2,3}, E_{2,4}, E_{3,4}] / I^\prime , \]

  where

  \[ I^\prime = \langle E_{1,2}^2, E_{1,3}^2, E_{1,4}^2, E_{2,3}^2, E_{2,4}^2, E_{3,4}^2, E_{1,2}E_{1,3}, E_{1,2}E_{1,4}, E_{1,3}E_{1,4}, E_{2,3}E_{2,4}, \]

  \[ E_{1,4}E_{2,3}, E_{1,3}E_{2,3}, E_{1,4}E_{2,4}, E_{1,4}E_{3,3}, E_{2,4}E_{3,4} \rangle . \]

  Indeed, all generating relations of $I$ remain true in $gr^{1,1}(J_{\infty}(A))$ except for $E_{1,3}E_{2,4} = -E_{1,4}E_{2,3}$. At Step 1.1 we have $E_{1,3}E_{2,4} \subset G_{1}^{1,1}$ and $E_{1,4}E_{2,3} \subset G_{2}^{1,1} \setminus G_{1}^{1,1}$. Thus, the relation $E_{1,3}E_{2,4} = -E_{1,4}E_{2,3}$ gives us $E_{1,4}E_{2,3} = 0$ in $gr^{1,1}(A)$.

- **Step 2:** We have

  \[ \text{Step 2.1} \quad G_{0}^{2,1} : E_{2,4} \quad E_{1,4} \]

  \[ \text{Step 2.2} \quad G_{0}^{2,2} : E_{2,4} . \]
It is not hard to see that
\[ B \cong B_{1,1} \cong B_{2,1} \cong B_{2,2}. \]

Therefore, we have a surjective differential algebra homomorphism, \( \phi \), from \( J_\infty(B) \) to \( \text{gr}^{1,1}(J_\infty(A)) \) by sending \( E_{i,j} \) to its equivalent class in \( \text{gr}^{1,1}(J_\infty(A)) \), which implies
\[ HS_q(J_\infty(A)) \leq HS_q(J_\infty(B)). \]

Now we prove this for general \( sl_n \).

**Proposition 6.1.3.** We have
\[ HS_q(J_\infty(A)) \leq HS_q(J_\infty(B)). \]

**Proof.** According to (6.8), we know that
\[ HS_q(J_\infty(B_{n-2,n-2})) \geq HS_q(J_\infty(A)). \]

In order to prove the statement, it is sufficient to show that the ideal of \( B \), i.e., \( I \), belongs to \( \mathfrak{l}_{n-2,n-2} \). Indeed, if it is true, we would have
\[ HS_q(J_\infty(B)) \geq HS_q(J_\infty(B_{n-2,n-2})) \]
since both \( B \) and \( B_{n-2,n-2} \) are quotient polynomial algebras with the same generators. Next, let us show \( I \subset \mathfrak{l}_{n-2,n-2} \).

For Type II relations, it is clear that they all belong to \( \mathfrak{l}_{n-2,n-2} \).

Let us consider Type I relations in \( A \), i.e., \( E_{i_1,j_1}E_{i_2,j_2} = -E_{i_1,j_2}E_{i_2,j_1} \), where \( 1 \leq i_1 < j_1 \leq n, \quad 1 \leq i_2 < j_2 \leq n, \quad 1 \leq i_1 \leq i_2 \leq n, \quad 1 \leq j_1 \leq j_2 \leq n, \quad j_1 > i_2 + 1 \). Following above procedure, we see that these Type I relations belong to \( \mathfrak{l}_{i,j} \) until the step \((j_1-2)(i_1+1)\). Indeed, at that step, \( G_{0}^{i_1-2,i_1+1} \) is generated by
\[ E_{j_1-2,j_1}, E_{j_1-3,j_1}, \ldots, E_{i_1+1,j_1}. \]
Note
\[
E_{i_1,j_1}E_{i_2,j_2} \subset G^{j_1-2,i_1+1}_2 \setminus G^{j_1-2,i_1+1}_1, \text{ while } E_{i_1,j_1}E_{i_2,j_2} \subset G^{j_1-2,i_1+1}_1.
\]

Therefore, we get \(E_{i_1,j_1}E_{i_2,j_2} = 0\) in \(B_{j_1-2,i_1+1}\) at the step \((j_1 - 2)(i_1 + 1)\), and these relations also remain true in \(B_{n-2,n-2}\).

For Type III relations of \(A\), i.e., \(E_{i,i+1}E_{j,j'} = -E_{i,j'}E_{j,i+1}\) where \(j < i < i + 1 < j'\), they give us
\[
E_{i,i+1}E_{j,j'} = 0
\]
in \(B_{i-1,1}\) at the step \((i - 1)(1)\) since we have
\[
E_{i,i+1}E_{j,j'} \subset G^{i-1,1}_2 \setminus G^{i-1,1}_1, \text{ and } E_{i,j'}E_{j,i+1} \subset G^{i-1,1}_1.
\]

Thus, these Type III relations in \(B_{n-2,n-2}\) become \(E_{i,i+1}E_{j,j'} = 0\).

From above arguments and the definition of algebra \(B\), we see that all quotient relations of \(B\) are true in \(B_{n-2,n-2}\). Hence, we obtain the result.

If instead of taking the left-hand side of Type III relations for the generators of the quotient ideal of \(B\), we take the right-hand side of Type III relations, we get another commutative algebra
\[
H := \mathbb{C}[E_{i,j}| 1 \leq i < j \leq n]/I,
\]
where \(I\) is generated by \(E_{i_1,j_1}E_{i_2,j_2}\), where
\[
1 \leq i_1 < j_1 \leq n, \quad 1 \leq i_2 < j_2 \leq n, \quad 1 \leq i_1 \leq i_2 \leq n, \quad 1 \leq j_1 \leq j_2 \leq n, \quad j_1 > i_2.
\]

Following similar filtration procedure as above, we can introduce another family of filtrations, \(G^{i,j}\) which is defined in the same manner as \(G^{i,j}\) just with a different generating set of \(G^{i,j}_0\) (see below) from the one of \(G^{i,j}_0\), and commutative algebras. Without causing confusion, we still use \(gr^{i,j}(J_\infty(H))\) to denote the associated algebra of \(J_\infty(H)\) with respect to the filtration \(G^{i,j}\).
• **Step 1.1:** Define

\[ H_{1,1} := \mathbb{C}[E_{i,j} | 1 \leq i < j \leq n]/\mathbb{T}_{1,1}, \]

where \( \mathbb{T}_{1,1} \) is the kernel of the map from \( P \) to \( gr^{1,1}(J_\infty(A)) \) by sending \( E_{i,j} \) to its equivalence class.

• **Step 1.2:** Define

\[ H_{1,2} := \mathbb{C}[E_{i,j} | 1 \leq i < j \leq n]/\mathbb{T}_{1,2}, \]

where \( \mathbb{T}_{1,2} \) is the kernel of the map from \( P \) to \( gr^{1,2}(J_\infty(H_{1,1})) \) by sending \( E_{i,j} \) to its equivalence class. We list all generators of \( \mathcal{G}_0^{i,j} \) at each step:

**Step 1.1** \( \mathcal{G}_0^{1,1} : E_{2,3} \quad E_{1,3} \)

**Step 1.2** \( \mathcal{G}_0^{1,2} : E_{2,3}. \)

Then we can define \( H_{i,j} \) iteratively. So we only list the generators of \( \mathcal{G}_0^{i,j} \) at each step in the following:

• **Step 2:**

**Step 2.1** \( \mathcal{G}_0^{2,1} : E_{3,4} \quad E_{2,4} \quad E_{1,4} \)

**Step 2.2** \( \mathcal{G}_0^{2,2} : E_{3,4} \quad E_{2,4} \)

**Step 2.3** \( \mathcal{G}_0^{2,3} : E_{3,4}. \)

• **Step \( i \):**

**Step \( i.1 \)** \( \mathcal{G}_0^{i+1} : E_{i+1,i+2} \quad E_{i,i+2} \quad \cdots \quad E_{2,i+2} \quad E_{1,i+2} \)

**Step \( i.2 \)** \( \mathcal{G}_0^{i+2} : E_{i+1,i+2} \quad E_{i,i+2} \quad \cdots \quad E_{2,i+2} \quad E_{1,i+2} \)

\[ \vdots \]

**Step \( i.i+1 \)** \( \mathcal{G}_0^{i+1} : E_{i+1,i+2}. \)
Using similar arguments, we can prove the following result.

**Proposition 6.1.4.** We have

\[ HS_q(J_\infty(A)) \leq HS_q(J_\infty(H)) \]

We also have:

**Proposition 6.1.5.** The Hilbert series of \( J_\infty(B) \) equals

\[
\sum_{m \in \mathbb{N}^{(n-1)/2}} \frac{q^{B(m)}}{\prod_{1 \leq i < j \leq n} (q)_{n_{i,j}}},
\]

and the Hilbert series of \( J_\infty(H) \) equals

\[
\sum_{m \in \mathbb{N}^{(n-1)/2}} \frac{q^{B'(m)}}{\prod_{1 \leq i < j \leq n} (q)_{n_{i,j}}},
\]

**Proof.** First, we consider an integral lattice \( L \) of rank \( \frac{n(n-1)}{2} \) with a basis:

\[ \alpha_{1,2}, \ldots, \alpha_{1,n}, \ldots, \alpha_{n-1,n}. \]

The symmetric bilinear form of this lattice is defined as: \((\alpha_{i_1,j_1}, \alpha_{i_2,j_2}) = 1\) when

- \(1 \leq i_1 < j_1 \leq n, 1 \leq i_2 < j_2 \leq n, 1 \leq i_1 < i_2 \leq n, 1 \leq j_1 < j_2 \leq n, j_1 > i_2 + 1,\)
- \(1 \leq i_1 < j_1 \leq n, 1 \leq i_2 < j_2 \leq n, 1 \leq i_1 = i_2 \leq n, 1 \leq j_1 < j_2 \leq n,\)
- \(1 \leq i_1 < j_1 \leq n, 1 \leq i_2 < j_2 \leq n, 1 \leq j_1 = j_2 \leq n, 1 \leq i_1 < i_2 \leq n,\)
- \(i_1 + 1 = j_1, 1 \leq i_2 \leq i_1 + 1 < j_2 \leq n,\)

\((\alpha_{i,j}, \alpha_{i,j}) = 2\) for all \(\alpha_{i,j}\), and \((\alpha_{i_1,j_1}, \alpha_{i_2,j_2}) = 0\) otherwise. According to [75, Theorem 5.13], we have \( J_\infty(B) \cong \text{gr}^F(W_L) \). It is known that the character of \( W_L \subset V_L \) equals \( (6.9) \). Thus, we have the first identity. The identity for \( J_\infty(H) \) follows along the same lines. \( \square \)
6.2 Quantum dilogarithm

The quantum dilogarithm is defined as $\phi(x) := \prod_{i \in \mathbb{N}} (1 - q^i x)$. Let $x$ and $y$ be non-commutative variables such that $xy = qyx$.

The quantum dilogarithm satisfies an important pentagon identity of Faddeev and Kashaev [44],

$$\phi(y)\phi(x) = \phi(x)\phi(-yx)\phi(y).$$  \hspace{1cm} (6.11)

Using the binomial $q$-series identity we also have

$$\phi(x) = \sum_{n \in \mathbb{N}} (-1)^n q^{\frac{1}{2}n(n-1)} x^n \frac{1}{(q)_n}.$$  \hspace{1cm} (6.12)

We also record another useful form ($j \in \mathbb{Z}_+$)

$$\phi(-q^{\frac{j}{2}} x) = \sum_{n \in \mathbb{N}} q^{\frac{1}{2}n^2 + \frac{j-1}{2} n} x^n \frac{1}{(q)_n}.$$  \hspace{1cm} (6.12)

6.2.1 Warm up: Proof of (6.3) using quantum dilogarithm

We first observe that two sides are equal if and only if Coeff $x^m y^n$ agree. Comparing coefficients on both sides leads to

$$\frac{q^{m^2 + n^2 - mn}}{(q)_m(q)_n} = \sum_{n_1 + n_2 = m, n_2 + n_3 = n} q^{n_1^2 + n_2^2 + n_3^2 + n_1 n_2 + n_2 n_3} \frac{1}{(q)_{n_1}(q)_{n_2}(q)_{n_3}},$$

where, after letting $n_1 = m - n_2$ and $n_3 = n - n_2$, the right hand-side can be rewritten as

$$\sum_{n_2 \in \mathbb{N}} q^{(m-n_2)^2 + (n-n_2)^2 + n_2^2 + (m-n_1)n_2 + (n-n_2)n_2} \frac{1}{(q)_{m-n_2}(q)_{n-n_2}(q)_{n_2}}.$$  \hspace{1cm} (6.13)

After simplifying exponents on both sides we end up with an equivalent identity

$$\frac{1}{(q)_m(q)_n} = \sum_{n_2 \in \mathbb{N}} q^{(n-n_2)(m-n_2)} \frac{1}{(q)_{m-n_2}(q)_{n-n_2}(q)_{n_2}}.$$  \hspace{1cm} (6.14)
This famous identities appeared in a variety of situation and there are several different proofs in the literature [107], [70] (credited to Zwegers), [44], etc.

Using quantum pentagon identity (6.15) in a slightly different form

\[ \phi(-q^{1/2}y)\phi(-q^{1/2}x) = \phi(-q^{1/2}x)\phi(-qyx)\phi(-q^{1/2}y) \]  

(6.15)

together with Euler expansion we get

\[ \sum_{m,n \in \mathbb{N}} \frac{q^{\frac{1}{2}m^2 + \frac{1}{2}n^2}y^m x^n}{(q)_m(q)_n} = \sum_{n_1,n_2,n_3 \in \mathbb{N}} \frac{q^{\frac{1}{2}n_1^2 + \frac{1}{2}n_2(n_2+1)+\frac{1}{2}n_3^2}y^{n_1}x^{n_2}y^{n_3}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}}. \]

Using identities

\[ y^m x^n = x^n y^m q^{-mn}, (yx)^{n_2} = x^{n_2} y^{n_2} q^{-\frac{1}{2}n_2(n_2+1)} \]

we can write

\[ \sum_{m,n \in \mathbb{N}} \frac{q^{\frac{1}{2}m^2 + \frac{1}{2}n^2 - mn}y^m}{(q)_m(q)_n} = \sum_{n_1,n_2,n_3 \in \mathbb{N}} \frac{q^{\frac{1}{2}n_1^2 + \frac{1}{2}n_2^2}x^{n_1}y^{n_2}x^{n_3}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}}. \]

Extracting the term next to \( x^n y^m \) now gives (after letting \( n = n_1 + n_2 \) and \( m = n_2 + n_3 \))

\[ \frac{q^{\frac{1}{2}m^2 + \frac{1}{2}n^2 - mn}}{(q)_m(q)_n} = \sum_{n_2 \in \mathbb{N}} \frac{q^{\frac{1}{2}(n-n_2)^2 + \frac{1}{2}(m-n_2)^2}}{(q)_{n-n_2}(q)_{m-n_2}(q)_{n_2}} \]

which is clearly equivalent to identity (6.13). \( \square \)

In summary, the pentagonal identity for the quantum dilogarithm is equivalent to the \( q \)-series identity formula for the character of the level one principal subspace of \( sl_3 \) (6.3).

6.2.2 General case

Let \( x_i, 1 \leq i \leq n - 1 \), be formal variables. Assume that

\[ x_i x_{i+1} = q x_{i+1} x_i, \]  

(6.16)

and other pairs commute. Then
\[
\phi\left(-\frac{q}{2}x_{n-1}\right) \cdots \phi\left(-\frac{q}{2}x_1\right) = \sum_{k_1, \ldots, k_{n-1} \in \mathbb{N}} q^{k_1^2/2 + \cdots + k_{n-1}^2/2 - k_1k_2 - \cdots - k_{n-2}k_{n-1}} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}. \quad (6.17)
\]

By definition we have two useful formulas:

\[
\phi\left(-q^{\frac{j-i}{2}}x_{j-1}x_{j-2} \cdots x_i\right) = \sum_{m \in \mathbb{N}} (-1)^m q^{m(m-1)/2} (-q^{\frac{j-i}{2}}x_{j-1} \cdots x_i)^m \quad (q)_m
\]
\[
= \sum_{m \in \mathbb{N}} q^{m^2/2} q^{\frac{m(m+1)}{2}} - \frac{m}{2} (x_{j-1} \cdots x_i)^m \quad (q)_m,
\]
\[
(x_{j-1} \cdots x_i)^m = q^{-(j-i)} m(m+1)/2 x_i^m x_{i+1}^m \cdots x_{j-1}^m. \quad (6.18)
\]
\[
(x_{j-1} \cdots x_i)^m = q^{-(j-i)} m^{m+1}/2 x_i^m x_{i+1}^m \cdots x_{j-1}^m. \quad (6.19)
\]

These two formulas combined together give us

\[
\phi\left(-q^{\frac{j-i}{2}}x_{j-1}x_{j-2} \cdots x_i\right) = \sum_{m \in \mathbb{N}} q^{(2-(j-i))m^2/2} x_i^m x_{i+1}^m \cdots x_{j-1}^m. \quad (6.20)
\]

Now for \(\phi\left(-q^{\frac{1}{2}}x_{n-1}\right) \cdots \phi\left(-q^{\frac{1}{2}}x_1\right)\) and formula (6.15) we commute factors such that \(\phi(-q^{\frac{1}{2}}x_i)\) is in front of \(\phi(-q^{\frac{1}{2}}x_{i+1})\). Therefore, we get the quantum dilogarithmic identity (in this particular order!):

\[
\phi\left(-q^{\frac{1}{2}}x_{n-1}\right) \cdots \phi\left(-q^{\frac{1}{2}}x_1\right) = \phi(-q^{\frac{1}{2}}x_1)
\]
\[
= \phi(-q^{\frac{1}{2}}x_1)
\]
\[
= \phi(-qx_2x_1) \phi(-q^{\frac{3}{2}}x_2)
\]
\[
= \cdots \quad (6.21)
\]
\[
= \phi\left(-q^{\frac{n-2}{2}}x_{n-2} \cdots x_1\right) \phi\left(-q^{\frac{n-3}{2}}x_{n-2} \cdots x_2\right) \cdots \phi\left(-q^{\frac{1}{2}}x_{n-2}\right)
\]
\[
= \phi\left(-q^{\frac{n-1}{2}}x_{n-1} \cdots x_1\right) \phi\left(-q^{\frac{n-2}{2}}x_{n-1} \cdots x_2\right) \cdots \phi\left(-qx_{n-1}x_{n-2}\right) \phi(-q^{\frac{1}{2}}x_{n-1}).
\]
Next, we expand for each $j > i$

$$\phi(-q^{j-i} x_{j-1} x_{j-1} \ldots x_i) = \sum_{m_{i,j} \in \mathbb{N}} (-1)^{m_{i,j}} q^{m_{i,j}(m_{i,j}-1)/2} \frac{(-q^{j-i} x_{j-1} x_{j-2} \ldots x_i)^{m_{i,j}}}{(q)_{m_{i,j}}}.$$  \hspace{0.5cm} (6.22)

This allows us to rewrite dilogarithmic identity (6.21) in the form (6.5).

$$\sum_{\mathbf{m} \in \mathbb{N}^{n-1}/2} q^{C(m)} \cdot F = \sum_{k_1, \ldots, k_{n-1} \in \mathbb{N}} \frac{q^{k_1^2/2 + \cdots + k_{n-1}^2/2 - k_1 k_2 - \cdots - k_{n-2} k_{n-1} / (q)_{k_1} \cdots (q)_{k_{n-1}}}}{(q)_{k_1} \cdots (q)_{k_{n-1}}},$$ \hspace{0.5cm} (6.23)

where

$$C(m) = \sum_{m_{i,j} \in \mathbb{N}} \frac{(2 - (j - i)) m_{i,j}^2}{2} \quad \text{and} \quad F = (x_1^{m_{1,2}})(x_1^{m_{1,3}} x_2^{m_{1,3}+m_{2,3}}) \cdots (x_1^{m_{1,n}} \cdots x_{n-1}^{m_{1,n}} x_2^{m_{2,n}} \cdots x_{n-1}^{m_{2,n}} \cdots x_{n-1}^{m_{n-1,n}}).$$

An application of (6.16) to $F$, we get $F = q^{E(m)} x_1^{\lambda_1} \cdots x_n^{\lambda_n}$, where $E(m)$ is some quadratic form and

$$\lambda_i = \sum_{1 \leq s \leq i \leq \ell \leq n} m_{s,\ell}.$$

For identities (6.1) and (6.23), we let

$$m_{i,i+1} := k_i - \sum_{1 \leq s < i} m_{s,i+1} - \sum_{s=i+1} m_{s,\ell} - \sum_{1 \leq s < i \leq \ell+1} m_{s,\ell}.$$

Then we can prove our main result.

**Theorem 6.2.1.** The identity (6.1) and quantum dilogarithmic identity (6.21) are equivalent.

**Proof.** We are going to show that the identity (6.21) is equivalent to the $q$-series identity (6.1).
by induction. We already proved that this is true for $A_2$ in the previous section. Assume that quantum dilogarithmic identity (6.21) and identity (6.1) are equivalent for $A_{n-1}$.

Now we prove the equivalence in the case of $A_n$. In order to prove the equivalence between identity (6.1) and dilogarithmic identity, it suffices to show (with $m_{i,i+1}$ as above) that

$$C(m) + E(m) - (k_1^2/2 + \cdots + k_n^2/2 - k_1k_2 - \cdots - k_{n-1}k_n) = B(m) - \frac{1}{2}k^\top Ak. \quad (6.24)$$

According to our induction hypothesis, if we let $m_{i,n+1} = 0$ where $1 \leq i \leq n$, they are equal. Then we need show that both sides of (6.24) have the same terms which involve $m_{i,n+1}$ ($1 \leq i \leq n$). After expanding both sides of (6.24), it is easy to see the terms of the form $k_i k_j$ ($j > i + 1$) are absent on both sides, and the coefficients of the terms of the form $k_i k_{i+1}$ equal 1 on both sides (so they cancel out). Similarly, terms of the form $k_i^2$ also cancel out.

We are left to analyze 6 possible types of terms involving $m_{i,n+1}$:

- **Type I**: $m_{i_2,j_2+1} m_{i_1,n+1}$, where $j_2 + 1 \leq i_1 - 1$.

- **Type II**: $m_{i_2,j_2+1} m_{i_1,n+1}$, where $i_2 < i_1 \leq j_2 + 1 < n + 1$.

- **Type III**: $m_{i_2,j_2+1} m_{i_1,n+1}$, where $i_1 \leq i_2 < j_2 + 1 < n + 1$, or $i_1 < i_2 < j_2 + 1 \leq n + 1$.

- **Type IV**: $k_i m_{i_1,n+1}$, where $i_1 \geq i + 1$. 

• **Type V:** \(k_i m_{i_1, n+1}\), where \(n \geq i \geq i_1\).

\[
\begin{array}{c}
\bullet \quad i_1 \\
\bullet \quad n + 1 \\
\bullet \quad i \\
\end{array}
\]

• **Type VI:** \(m_{i, n+1}^2\), where \(n \geq i\).

Now we compare terms of each type on the left- and right-hand side of (6.24). We only provide a few details for brevity. First, on the left-hand side of (6.24), straightforward computations with powers of the \(q\)-series give the following coefficients: for terms of Type I and IV coefficients are zero as they are absent from the formula. Similarly, for all Type III terms coefficients are also zero. For Type V and VI the coefficients are \(-1\) and \(1\), respectively. For Type II, contribution comes from two sources:

• a. 

\[
\left(\prod_{s=i_2}^{j_2} x_s\right)^{m_{1,1}} \cdots \left(\prod_{s=i_1}^{n} x_s\right)^{m_{i+1,1}} = q^{-(j_2-i_1)m_{i+1,1} + m_{1,1}} \cdots ,
\]

• b. 

\[
\prod_{s=i_1}^{j_2} q^{m_{s,s+1}} = q^{(j_2-i_1) + m_{1,1} + m_{i+1,1}} \cdots .
\]

Therefore, the overall coefficient of Type II terms is \(1\).

Now we compute coefficients on the right-hand side of (6.24) for the same six types of terms. Again, Type I terms are absent so their coefficients are zero. We list all possible ways, in which we can get Type II terms:

• a. 

\[
\sum_{s=i_1}^{j_2} m_{s,s+1}^2 = 2(j_2 - i_1 + 1)m_{i+1,1} + m_{1,1} + \cdots ,
\]
• b.

\[
\left( \sum_{s=1}^{j_2} m_{s,s+1} \right) m_{i_1,n+1} = -(j_2 - i_1 + 1) m_{i_2,j_2+1} m_{i_1,n+1} + \cdots ,
\]

• c.

\[
\left( \sum_{s=1}^{j_2} m_{s,s+1} \right) m_{i_2,j_2+1} = -(j_2 - i_1 + 1) m_{i_2,j_2+1} m_{i_1,n+1} + \cdots ,
\]

• d.

\[ m_{i_2,j_2+1} m_{i_1,n+1}. \]

Adding these up gives the coefficient of Type II to be 1. Along the same lines Type III and IV coefficients are zero, and Type V and VI coefficients are $-1$ and 1, respectively.

From above argument, we see that both sides of (6.24) have the same terms which involve $m_{i,n+1}$ ($1 \leq i \leq n$). Therefore, two identities are equivalent by induction. \qed

**Corollary 6.2.2.** The Hilbert series of $J_\infty(R_{W(\Lambda_0)})$ is given by

\[
\sum_{m \in \mathbb{N}^{n(n-1)/2}} \frac{q^{B(m)}}{\prod_{1 \leq i < j \leq n} (q)^{n_{i,j}}},
\]

where $W_{\Lambda_{1,0}}$ is the principal subspace of the affine vertex algebra $L_{\hat{sl}_n(1,0)}$.

**Proof.** According to Proposition 6.1.3, Proposition 6.1.5 and Theorem 6.2.1 we know that

\[ H_{\lambda}(J_\infty(R_{W(\Lambda_0)})) \leq \text{ch}[W(\Lambda_0)](q). \]

Since there exists a surjective homomorphism

\[ \psi : J_\infty(R_{W(\Lambda_0)}) \rightarrow gr^F(W(\Lambda_0)), \]

we also have

\[ H_{\lambda}(J_\infty(R_{W(\Lambda_0)})) \geq \text{ch}[W(\Lambda_0)](q). \]
The result follows.

Then we conclude [75]:

**Corollary 6.2.3.** The principal subspace \( W(\Lambda_0) \) of \( sl_n \) is classically free.

Next result is needed in Section 6 and is proven using similar arguments as above.

**Proposition 6.2.4.** For the case of \( A_n \), the coefficients of six types of terms in \( B'(m) - \frac{1}{2}k^\top Ak \) are:

<table>
<thead>
<tr>
<th>Type</th>
<th>Coefficient</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( 2(j_2 - i_1) + 1 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( 2(j_2 - i_2) )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>( i = i_1, i = n )</td>
</tr>
<tr>
<td>5</td>
<td>-2</td>
<td>( i_1 &lt; i &lt; n )</td>
</tr>
<tr>
<td>6</td>
<td>( n - i )</td>
<td></td>
</tr>
</tbody>
</table>

6.3 \( q \)-identities from quiver representations

In this part we discuss identity (6.2) from a perspective of quiver representations following [97]. We will use formulas from [97] and several basic facts about quiver representations. Indecomposable representations of the quiver of type \( A_{n-1}, Q \), are in one-to-one correspondence with positive roots of \( A_{n-1} \), and can be enumerated by segments \([i, j]\), \( 1 \leq i \leq j \leq n - 1 \).

**Theorem 6.3.1.** [97 Corollary 1.5] Let \( Q \) be a quiver of type \( A_{n-1} \). Then for every \( k = (k_1, ..., k_{n-1}) \in \mathbb{N}^{n-1} \) we have an identity

\[
\frac{1}{\prod_{i=1}^{n-1} (q)_{k_i}} = \sum_{\eta} q^{\text{codim}(\eta)} \prod_{1 \leq i \leq j \leq n-1} \frac{1}{(q)_{m_{[i,j]}(\eta)}}
\]

(6.25)

where summation is over all finite-dimensional representations \( \eta \) (up to equivalence) of \( Q \) such that \( \overline{\dim}(\eta) = k \), \( \text{codim}(\eta) \) is the co-dimension of a certain orbit, and \( m_{[i,j]}(\eta) \) indicates the multiplicity of \([i, j]\) in \( \eta \).
Recall another result from [97] for codimensions:

**Lemma 6.3.2.**

\[
\text{codim}(\eta) = \sum_{[I,J] \in \text{ConditionStrands}} m_I \cdot m_J,
\]

where

\[
\text{ConditionStrands} = \{[I, J] : [I, J] \text{ satisfies conditions (1), (2) or (3)}\},
\]

where strand is a pair \(I = [a, b], 1 \leq a \leq b \leq n - 1\) (corresponding to indecomposable rep of \(Q\)) and

1. \(I = [w, x - 1], J = [x, z], w < x \leq z\), e.g.,

\[
\begin{array}{c}
  w \bullet \quad x - 1 \\
  \bullet \quad \bullet \\
  x \quad \bullet \quad \bullet \quad z
\end{array}
\]

2. \(I = [w, y], J = [x, z], w < x \leq y < z\) and the arrows \(a_{x-1}\) and \(a_y\) point in the same direction, e.g.,

\[
\begin{array}{c}
  w \bullet \quad y \\
  \bullet \quad \bullet \\
  x \quad \bullet \quad \bullet \quad z
\end{array}
\]

3. \(I = [x, y], J = [w, z], w < x \leq y < z\) and the arrows \(a_{x-1}\) and \(a_y\) point in different directions, e.g.,

\[
\begin{array}{c}
  w \bullet \quad y \\
  \bullet \quad \bullet \\
  x \quad \bullet \quad \bullet \quad z
\end{array}
\]

For our purpose, we rewrite (6.25) as a family of identities in the shape of (6.5):

\[
\sum_{k=(k_1, \ldots, k_{n-1}) \in \mathbb{N}^{n-1}} x^k \prod_{i=1}^{n-1} \frac{q^{\text{codim}(\eta)}}{q^{\text{codim}(\eta)}} \prod_{1 \leq i \leq j \leq n-1} (q)_{m_{i,j}(\eta)},
\]

and the summation is over all finite-dimensional representation \(\eta\) of \(A_{n-1}\) (up to isomorphism).
Example 6.3.3 (sl₃). Representation η of the $A_2$ quiver of type $\bar{\dim}(\eta) = (m, n)$ is given by $\mathbb{C}^m \to \mathbb{C}^n$. Indecomposable representations are given by $[1,1] := \mathbb{C} \to 0$, $[2,2] := 0 \to \mathbb{C}$, and $[1,2] = \mathbb{C} \to \mathbb{C}$. Condition (1) is $[1,1], [2,2]$ and there are no pairs with conditions two and three. Therefore we have

$$\text{codim}(\eta) = m_{[1,1]}(\eta) \cdot m_{[2,2]}(\eta),$$

and we are summing over all representations η with $\bar{\dim}(\eta) = k = (m, n)$. That means

$$\frac{1}{(q)m(q)n} = \sum_{m_{[1,1]}m_{[2,2]}m_{[1,2]} \in \mathbb{N} \atop m_{[1,1]}+m_{[1,2]}=m} q^{m_{[1,1]}m_{[2,2]}} \frac{1}{(q)m_{[1,1]}(q)m_{[2,2]}(q)m_{[1,2]}},$$

which is precisely formula (8) (we only have to rewrite $m - n_2 = n_1$ and $n - n_2 = n_3$, and use that summation variables are $m_{[1,1]} = n_1$, $m_{[1,2]} = n_2$ and $m_{[2,2]} = n_3$).

Then we can prove the following:

**Theorem 6.3.4.** The identities (6.26) and (6.2) are equivalent for $A_n$ ($n \geq 2$).

**Proof.** We have proved the equivalence for $A_2$. We assume the direction of $A_n$ quiver is always from left to right, i.e., $\bullet \cdots \bullet \cdots \bullet \cdots \bullet$. Because of the orientation, there is no pair with condition (3) in codim($\eta$). Now we prove that these two identities are equivalent for $A_n$ ($n \geq 2$) using induction. Assume that two identities are equivalent for $A_{n-1}$. For the case of $A_n$, by identifying $m_{[i,j]}$ with $m_{i,j+1}$, and letting $m_{i,i+1} (m_{[i,i]})$ be

$$k_i - \sum_{1 \leq s < \ell \leq n+1 \atop i+1 \leq \ell \leq n+1} m_{s,\ell} - \sum_{s=i+1 \leq \ell \leq n+1} m_{s,\ell} - \sum_{1 \leq s < i \leq n} m_{s,\ell},$$

it is enough to show that

$$\text{codim}(\eta) = B'(m) - \frac{1}{2} k^T A k. \quad (6.27)$$

It is clear that we have the same terms involving $k_i k_j$ ($1 \leq i \leq j \leq n$) on both sides of
Note that we also have 6 types of terms that involve
\[ m_{[i_2,j_2]}m_{[i_1,n]} \quad (m_{[i_2,j_2+1]}m_{i_1,n+1}). \]

Then we compute the coefficients of these terms for \( \text{codim}(\eta) \).

- Terms in \( \text{codim}(\eta) \) that involve Type I term are the following:
  a. \( m_{[i_2,i_1-1]}m_{[i_1,i_1]} = -m_{[i_2,i_1-1]}m_{[i_1,n]} + \cdots \),
  b. \( m_{[i_1-1,i_1-1]}m_{[i_1,n]} = -m_{[i_2,i_1-1]}m_{[i_1,n]} + \cdots \),
  c. \( m_{[i_1-1,i_1-1]}m_{[i_1,i_1]} = m_{[i_2,i_1-1]}m_{[i_1,n]} + \cdots \),
  d. \( m_{[i_2,i_1-1]}m_{[i_1,n]} \).

Therefore, the coefficient of each Type I term is 0.

- We list all terms that involve Type II term as following:
  a. \[
  m_{[i_1-1,i_1-1]}m_{[i_1,i_1]} + \sum_{s=i_1}^{j_2-1} m_{[s,s]}m_{[s+1,s+1]} + m_{[j_2,j_2]}m_{[j_2+1,j_2+1]}
  = (2(j_2 - i_1) + 2)m_{[i_2,j_2]}m_{[i_1,n]} + \cdots ,
  \]
  b. \[
  m_{[i_1-1,i_1-1]}m_{[i_1,n]} = -m_{[i_2,j_2]}m_{[i_1,n]} + \cdots ,
  \]
  c. \[
  m_{[i_2,j_2]}m_{[j_2+1,j_2+1]} = -m_{[i_2,j_2]}m_{[i_1,n]} + \cdots ,
  \]
  d. \[
  m_{[i_2,j_2]}m_{[i_1,n]}.
  \]

Therefore, the coefficient of each Type II term is \( 2(j_2 - i_1) + 1 \).

- For Type III terms, we list all possible contributions from \( \text{codim}(\eta) \):
a. 

\[ m_{[i_2-1,i_2-1]} m_{[i_2,i_2]} + \sum_{s=i_2}^{j_2-1} m_{[s,s]} m_{[s+1,s+1]} + m_{[j_2,j_2]} m_{[j_2+1,j_2+1]} \]

\[ = (2(j_2 - i_2) + 2)m_{[i_2,j_2]} m_{[i_1,n]} + \cdots , \]

b. 

\[ m_{[i_2,j_2]} m_{[j_2+1,j_2+1]} = -m_{[i_2,j_2]} m_{[i_1,n]} + \cdots , \]

\[ m_{[i_2-1,i_2-1]} m_{[i_2,j_2]} = -m_{[i_2,j_2]} m_{[i_1,n]} + \cdots ; \]

Thus, the coefficient of each Type III term is \(2(j_2 - i_2)\).

- For Type IV and Type V terms, we have

  a. When \(i_1 > i + 1\), there is no term that involves \(k_i m_{[i_1,n]}\).

  b. When \(i_1 = i + 1\), we have

\[ m_{[i,i]} m_{[i_1,n]} + m_{[i,i]} m_{[i+1,i+1]} = 0(k_i m_{[i_1,n]}) + \cdots . \]

c. When \(i_1 = i\), the only term that involves \(k_i m_{[i_1,n]}\) is

\[ m_{[i,i]} m_{[i+1,i+1]} = -k_i m_{[i_1,n]} + \cdots . \]

d. When \(n > i > i_1\), the following term involves \(k_i m_{[i_1,n]}\):

\[ m_{[i-1,i-1]} m_{[i,i]} + m_{[i,i]} m_{[i+1,i+1]} = -2k_i m_{[i_1,n]} + \cdots ; \]

e. When \(i = n\), the only term that involves with \(k_i m_{[i_1,n]}\) is

\[ m_{[n-1,n-1]} m_{[n,n]} = -k_i m_{[i_1,n]} + \cdots . \]
For Type VI term, we have

$$\sum_{s=i}^{n-1} m_{[s,s]} m_{[s+1,s+1]} = (n - i) m_{[i,n]}^2 + \cdots.$$ 

So the coefficient of $m_{[i,n]}^2$ is $n - i$.

By induction hypothesis and Proposition 6.2.4 we see that,

$$B'(m) - \frac{1}{2} k^\top A k = \text{codim}(\eta).$$

Thus, two $q$-series identities are equivalent. \hfill \Box

**Remark 6.3.5.** It was briefly mentioned in [97] that Keller’s quantum dilogarithm identity for type $A$ quivers [67] is closely related to Theorem 6.3.1. This part can be viewed as a precise clarification of that claim.

### 6.4 Principal subspace of type $B_2$ at level 1

In this and next section we consider principal subspaces of two level one representations for which we have a well-known spinor realization. For more about level one spinor representations for $B^{(1)}_\ell$ and $D^{(1)}_\ell$ affine Kac-Moody Lie algebras we refer the reader to [54].

#### 6.4.1 Spinor representation of $B^{(1)}_\ell$

Level one affine vertex algebra of type $B^{(1)}_\ell$ has a spinor representation via $2\ell + 1$ fermions. We take

$$\mathcal{F}_\ell(Z + 1/2) := \Lambda(a_i(-1/2), a_i(-3/2), \cdots, a_i^*(-1/2), a_i^*(-3/2), \ldots; 1 \leq i \leq \ell)$$

and $\mathcal{F} = \Lambda(e(-1/2), e(-3/2), \ldots)$. Then the even part of

$$\mathcal{F}_\ell(Z + 1/2) \otimes \mathcal{F}$$
is isomorphic to affine vertex algebras $L_{\text{so}(2n+1)}(1,0)$. It is easy to see from the realization that the principal subspace $W(\Lambda_0) \subset L(\Lambda_0)$ is strongly generated by the following fields

$$a_i a_j, \quad 1 \leq i < j \leq \ell \quad : a_i a_j^* : \quad 1 \leq i < j, \quad : a_i e : \quad 1 \leq i \leq \ell,$$

This gives $\ell(\ell - 1)/2 + \ell(\ell - 1)/2 + \ell = \ell^2$ root vectors.

### 6.4.2 Principal subspace of $L_{\text{so}(5)}(1,0)$

For $B_2^{(1)} = C_2^{(1)}$, the principal subspace $W(\Lambda_0)$ is generated by

$$a_1 a_2, \quad a_1 a_2^* : \quad a_1 e : \quad a_2 :$$

corresponding to positive roots $\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_1, \epsilon_2$, respectively. Its $C_2$-algebra, $R_{W_{\Lambda_1,0}}$, is generated by $x_{\epsilon_1+\epsilon_2}, x_{\epsilon_1-\epsilon_2}, x_{\epsilon_2}, x_{\epsilon_1}$, and we have the following relations in $R_{W_{\Lambda_1,0}}$:

$$x_{\epsilon_1+\epsilon_2}^2 = 0, \quad x_{\epsilon_1-\epsilon_2}^2 = 0, \quad x_{\epsilon_1}^2 - x_{\epsilon_1+\epsilon_2} x_{\epsilon_1-\epsilon_2} = 0, \quad x_{\epsilon_2} x_{\epsilon_1+\epsilon_2} = 0, \quad x_{\epsilon_1} x_{\epsilon_1-\epsilon_2} = 0, \quad x_{\epsilon_2} x_{\epsilon_1} = 0.$$

We denote by $A$ the commutative algebra

$$\mathbb{C}[x_{\epsilon_1+\epsilon_2}, x_{\epsilon_1-\epsilon_2}, x_{\epsilon_2}, x_{\epsilon_1}] / \langle x_{\epsilon_1+\epsilon_2}^2, x_{\epsilon_1-\epsilon_2}^2, x_{\epsilon_1}^2 - x_{\epsilon_1+\epsilon_2} x_{\epsilon_1-\epsilon_2}, x_{\epsilon_2} x_{\epsilon_1+\epsilon_2}, x_{\epsilon_1} x_{\epsilon_1-\epsilon_2}, x_{\epsilon_2} x_{\epsilon_1} \rangle.$$

### 6.4.3 Character formula

It is known that the character of the principal subspace of $L_{\text{so}(5)}(1,0)$ is (given by [23]):

$$\text{ch}[W_{\Lambda_1,0}] = \sum_{r_1, r_2, r_3 \in \mathbb{N}} \frac{q^{r_1^2+(r_2+r_3)^2+r_3^2-r_1(r_2+2r_3)}}{(q)_r (q)_r (q)_r r_3). \quad (6.28)$$

This identity is more complicated because $x_{\alpha_1}^3(z) = 0$ but $x_{\alpha_1}^2(z) \neq 0$.

Let us introduce a filtration, $G$, on $J_\infty(A)$ by letting $G_0$ be generated with $x_{\epsilon_1+\epsilon_2}, x_{\epsilon_1-\epsilon_2}$, and

$$G_s = \text{span} \left\{ (x_0)(i) v | u \in \{ \epsilon_1, \epsilon_2 \}, v \in G_{s-1} \right\} + G_{s-1}.$$
Then we have the following lemma.

**Lemma 6.4.1.** The Hilbert series of the arc algebra of

\[
B := \frac{\mathbb{C}[x_{e_1+e_2}, x_{e_1-e_2}, x_{e_2}, x_{e_1}]}{(x_{e_1+e_2}, x_{e_1-e_2}, x_{e_1}, x_{e_2} x_{e_1+e_2}, x_{e_1} x_{e_1-e_2}, x_{e_1+e_2}, x_{e_2}, x_{e_2} x_{e_1})},
\]  

(6.29)

is greater than or equal to \( H_{S_4}(\text{gr}^G(J_\infty(A))) \).

**Proof.** Since both \( J_\infty(B) \) and \( \text{gr}^G(J_\infty(A)) \) are generated by \( x_{e_1+e_2}, x_{e_1-e_2}, x_{e_2}, x_{e_1} \), it is enough to show that the quotient relations of \( J_\infty(B) \) also hold in \( \text{gr}^G(J_\infty(A)) \). Note

\[
(x_{e_1+e_2}, x_{e_1-e_2}, x_{e_1} x_{e_1-e_2}, x_{e_1+e_2}, x_{e_1} x_{e_1+e_2}, x_{e_2}, x_{e_2} x_{e_1}) \partial
\]

are valid in \( \text{gr}^G(J_\infty(A)) \). For the quotient relations of \( J_\infty(A) \), \( (x_{e_1}^2 - x_{e_1+e_2} x_{e_1-e_2}) \partial \), we have

\[
(x_{e_1}^2) \partial \subset G_2 \setminus G_0, \quad \text{while} \quad (x_{e_1+e_2} x_{e_1-e_2}) \partial \subset G_0.
\]

Therefore, quotient relations, \( (x_{e_1}^2 - x_{e_1+e_2} x_{e_1-e_2}) \partial \), in \( J_\infty(A) \) give us \( (x_{e_1}^2) \partial \) in \( \text{gr}^G(J_\infty(A)) \).

The result follows. \( \square \)

**Proposition 6.4.2.** The Hilbert series of \( J_\infty(B) \) is less than or equal to

\[
\sum_{n_1, n_2, n_3, n_4, n_5 \in \mathbb{N}} \frac{q^{n_1^2 + n_2^2 + (n_3 + n_5)^2 + n_3^2 + n_5^2 + (2n_3 + n_5)(n_1) + n_4(n_1 + n_2) + n_3 n_4}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}},
\]

Proof. We will prove that \( J_\infty(B) \) is spanned by a set of monomials whose character is given by the \( q \)-series in the statement. We first observe that we can filter \( J_\infty(B) \) with the number of “particles” of type \( x_{e_2} \). In the zero component of the filtration we have the arc algebra with relations

\[
(x_{e_1+e_2}, x_{e_1-e_2}, x_{e_1} x_{e_1-e_2}, x_{e_1} x_{e_1+e_2}, x_{e_2}, x_{e_2} x_{e_1}) \partial
\]

whose Hilbert series is given by \[75\]

\[
\sum_{n_1, n_2, n_4 \in \mathbb{N}} \frac{q^{n_1^2 + n_2^2 + n_4(n_1 + n_2)}}{(q)_{n_1}(q)_{n_2}(q)_{n_4}}.
\]

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Now we include $x_{\epsilon_2}$ generator and additional relations. It is know that the arc algebra of $\mathbb{C}[x_{\epsilon_2}]/(x_{\epsilon_2}^3)$ is known to admit a combinatorial basis satisfying difference two at the distance two condition as in Rogers-Selberg identities. Therefore, its Hilbert series is given by

$$\sum_{n_3, n_5 \in \mathbb{N}} \frac{q^{(n_3+n_5)^2+n_3^3}}{(q)^{n_3}(q)^{n_5}}.$$ 

This formula can be also explained using arc algeras by letting $a = x_{\epsilon_2}$ and $b = x_{\epsilon_2}$ and considering $(ab, a-b^2, a^2)$, Since $a$ is of weight two, we get contribution $2n_3^2 + 2n_3n_5 + n_5^2$ in the exponent.

To finish the proof we have to analyze two additional relations and their contribution:

$$x_{\epsilon_2}x_{\epsilon_1+\epsilon_2} = 0, \text{ and } x_{\epsilon_2}^2x_{\epsilon_1} = 0.$$ 

The first relation contributes with the boundary condition $(2n_3 + n_5)n_1$ in the $q$-exponent as $(2n_3 + n_5)$ counts the power of $x_{\epsilon_2}$ in $(x_{\epsilon_2}^2)^{n_3}x_{\epsilon_1}^{n_5}$ and we have relation $x_{\epsilon_2}x_{\epsilon_1+\epsilon_2} = 0$ ($n_1$ summation variable corresponds to $x_{\epsilon_1+\epsilon_2}$). For the second relation, using $x_{\epsilon_2}x_{\epsilon_1} \neq 0$ and $x_{\epsilon_2}^2x_{\epsilon_1} = 0$, we get a spanning set of monomials involving $a = x_{\epsilon_2}^2$ and $x_{\epsilon_1}$ to be

$$\{ a(-n_i) \cdots a(-n_1)(x_{\epsilon_1})(m_j) \cdots (x_{\epsilon_1})(m_1) \mid j + 1 \leq n_1, n_s - n_{s-1} \geq 4, m_s - m_{s-1} \geq 2 \},$$

where $j + 1 \leq n_1$ (boundary condition), $n_s - n_{s-1} \geq 4, m_s - m_{s-1} \geq 2$ (difference condition) are coming from $x_{\epsilon_2}^2x_{\epsilon_1} = 0$, $x_{\epsilon_2}^3 = 0$ and $x_{\epsilon_1}^2 = 0$, respectively. Then the character of the space spanned by these monomials is at most

$$\sum_{n_3, n_4 \in \mathbb{N}} \frac{q^{n_3^2+2n_3^2+n_3n_4}}{(q)^{n_3}(q)^{n_4}}.$$ 

If we combining these arguments together, we obtain the result.

\[ \square \]

**Remark 6.4.3.** It is possible to prove the equality in Proposition [6.4.2] but this involves further analysis as in [28] using “simple currents” and “intertwining maps” in disguise.

The Hilbert series of $J_\infty(B)$ is greater than or equal to $\text{ch}[W_{\Lambda_{1,0}}]$. Using Mathematica, we checked that (6.4.2) and the character formula (6.28) agree up to $O(q^{70})$. Thus, we expect the following stronger result using charge variables:

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Conjecture 6.4.4. We have the identity
\[
\sum_{r_1,r_2,r_3 \in \mathbb{N}} \frac{y_1^{r_1} y_2^{r_2} q^{r_3^2 + (r_2 + r_3)^2 + r_2 - r_1 (r_2 + 2r_3)}}{(q)_{r_1} (q)_{r_2} (q)_{r_3}} = \sum_{n_1,n_2,n_3,n_4,n_5 \in \mathbb{N}} \frac{y_1^{n_1+n_2+n_4} y_2^{n_1+2n_3+n_4+n_5} q^{n_1^2+n_2^2+(n_3+n_5)^2+n_4^2+n_3^2+(2n_3+n_5)(n_1)+n_4(n_1+n_2)+n_3n_4}}{(q)_{n_1} (q)_{n_2} (q)_{n_3} (q)_{n_4} (q)_{n_5}}
\]

Provided Conjecture 6.4.4 is true, then we have

Corollary 6.4.5. The $C_2$-algebra of the principal subspace of $L_{\hat{a}05}(1,0)$, $W_{A_1,0}$, is isomorphic to $A$, and the principal subspace is classically free.

Proof. We can use the same filtration $G$ on $J_\infty(R_{W_{A_1,0}})$. Then using the surjectivity of $\psi$ and previous discussion we have

\[
\text{ch}[W_{A_1,0}](q) \leq HS_q(J_\infty(R_{W_{A_1,0}})) = HS_q(gr^G(J_\infty(R_{W_{A_1,0}}))) \\
\leq HS_q(gr^G(J_\infty(A))),
\]

Also from Lemma 6.4.1

\[
HS_q(gr^G(J_\infty(A))) \leq HS_q(J_\infty(B)) = \text{ch}[W_{A_1,0}],
\]

where in the last equation we use Conjecture 6.4.4. Combining these inequalities together, it follows that the character and all intermediate Hilbert series of arc algebras are equal, therefore, $\text{ch}[W_{A_1,0}](q) = HS_q(J_\infty(R_{W_{A_1,0}}))$ and $\psi$ is injective.

We believe that the $q$-series identity (6.4.4) is new and does not come from a simple application of the pentagon identity for the quantum dilogarithm. Perhaps it is a consequence of a more complicated relation which involves both $\phi(x^2)$ and $\phi(x)$. For instance, the coefficients on the left-hand side of the identity can be extracted from the coefficients of

\[
\phi(x)^{-1} \phi(-yq^2) \phi(y^2)^{-1}.
\]

After specialization $y_1 = y_2 = 1$, we expect the following elegant $q$-series identity with
a modular product side.

**Conjecture 6.4.6.**

\[ \sum_{r_1, r_2, r_3 \in \mathbb{N}} \frac{q^{r_1^2 + (r_2 + r_3)^2 + r_3^2 - r_1(r_2 + 2r_3)}}{(q)_{r_1} (q)_{r_2} (q)_{r_3}} = \frac{(-q; q)_{\infty} (-q; q^2)_{\infty}}{(q, q^4, q^5)_{\infty}}. \]

Interestingly, this conjecture would imply

\[ \text{ch}[W_{B_2}(\Lambda_0)](\tau) = \text{ch}[W_{A_2}(\Lambda_0)](\tau) \cdot \text{ch}[W_{A_1}(\Lambda_0)](\tau), \] \hspace{1cm} (6.30)

where the right hand side is the product of characters of the level one principal subspace for $A_2$ and $A_1$ vertex algebras.

### 6.5 Principal subspace of type $D_4$ at level 1

#### 6.5.1 Spinor representation of vertex operator algebra of $D^{(1)}_\ell$

Level one vertex operator algebra of $D^{(1)}_\ell$ has a spinor representation via $2\ell$ fermions. We take

\[ \mathcal{F}_\ell(\mathbb{Z} + 1/2) = \Lambda(a_i(-1/2), a_i(-3/2), \ldots, a_i^*(-1/2), a_i^*(-3/2), \ldots; 1 \leq i \leq \ell) \]

and $\mathcal{F} = \Lambda(e(-1/2), e(-3/2), \ldots)$. Then the even part of

\[ \mathcal{F}_\ell(\mathbb{Z} + 1/2) \]

is isomorphic to $L_{\mathfrak{so}_{2\ell}}(1, 0)$. The principal subspace $W_{\Lambda_{1,0}} \subset L_{\mathfrak{so}_{2\ell}}(1, 0)$ is strongly generated by

\[ : a_i a_j :, \quad 1 \leq i < j \leq \ell, \quad : a_i^* a_j^* :, \quad 1 \leq i < j \leq \ell, \]

all together $\ell(\ell - 1) = \ell^2$ root vectors.

For $D^{(1)}_4$, the principal subspace is generated by

\[ : a_i a_j : a_i^* a_j^* : (1 \leq i < j \leq 4), \]
corresponding to $\epsilon_i + \epsilon_j, \epsilon_i - \epsilon_j$, respectively. And its $C_2$-algebra, $R_{W_{\Lambda_{1,0}}}$, is generated by $x_{\epsilon_i+\epsilon_j}, x_{\epsilon_i-\epsilon_j} (1 \leq i < j \leq 4)$, which we denote by $W_{ij}$ and $V_{ij}$, respectively. And we have the following relations in the quotient of $R_{W_{\Lambda_{1,0}}}$:

\[
\begin{align*}
W_{12}W_{23}, & \quad W_{12}W_{24}, \quad W_{13}W_{34}, \quad W_{23}W_{34}, \quad W_{12}W_{13}, \quad W_{12}W_{14}, \\
W_{23}W_{23}, & \quad W_{23}W_{24}, \quad W_{24}W_{24}, \quad W_{34}W_{34}, \quad W_{13}W_{23}, \quad W_{14}W_{24}, \quad W_{14}W_{34}, \\
W_{24}W_{34}, & \quad -W_{13}W_{24} = W_{23}W_{14} = W_{12}W_{34}, \quad W_{12}V_{23}, \quad W_{12}V_{24}, \quad W_{13}V_{34}, \\
W_{23}V_{34}, & \quad W_{12}V_{13}, \quad W_{13}V_{12}, \quad W_{12}V_{14}, \quad W_{14}V_{12}, \quad W_{23}V_{24}, W_{24}V_{23}, \\
W_{13}V_{14}, & \quad W_{14}V_{13}, \quad W_{12}V_{12} = W_{13}V_{13} = W_{14}V_{14}, \quad W_{23}W_{23} = W_{24}W_{24}, \\
- W_{13}V_{24} = W_{23}V_{14} = W_{12}V_{34}, & \quad -W_{14}V_{23} = W_{24}V_{13}, \quad V_{12}V_{12}, \quad V_{12}V_{13}, \quad V_{12}V_{14}, \\
V_{23}V_{23}, & \quad V_{23}V_{24}, \quad V_{24}V_{24}, \quad V_{34}V_{34}, \quad V_{13}V_{23}, \quad V_{14}V_{24}, \quad V_{14}V_{34}, \quad V_{24}V_{34}, \\
- V_{13}V_{24} = V_{23}V_{14}, & \quad W_{13}V_{23} + W_{23}V_{13} = W_{24}V_{14} + W_{14}V_{24}.
\end{align*}
\]

We denote by $D$ the algebra $\mathbb{C}[W_{ij}, V_{ij}|1 \leq i < j \leq 4]/I$, where $I$ is generated by above quadratic relations. We expect that the algebra $D$ is the $C_2$-algebra of $R_{W_{\Lambda_{1,0}}}$.

### 6.5.2 Character formula and quantum dilogarithm

We first assume

\[x_1x_2 = qx_2x_1, \quad x_2x_3 = qx_3x_2, \quad x_2x_4 = qx_4x_2.\]

Then by using properties of quantum dilogarithm we have

\[
\begin{align*}
\phi(-q^{\frac{1}{2}}x_1)&\phi(-q^{\frac{1}{2}}x_3)\phi(-q^{\frac{1}{2}}x_2)\phi(-q^{\frac{1}{2}}x_1) \\
&=\phi(-q^{\frac{1}{2}}x_1)\phi(-q^2x_2)\phi(-q^2x_2)\phi(-q^2x_2)\phi(-q^2x_2)\phi(-q^2x_2)\phi(-q^2x_2)\phi(-q^2x_2).
\end{align*}
\]

It is equivalent with

\[
\sum_{(m,n) \in \mathbb{N}^4} \frac{q^{B(m,n)}x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}x_4^{\lambda_4}}{\prod_{1 \leq i < j \leq n}(q)^{m_{ij}}(q)^{n_{ij}}} = \sum_{k \in \mathbb{N}^4} \frac{q^{k_1k_2k_3k_4}x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}}{(q)^{k_1}(q)^{k_2}(q)^{k_3}(q)^{k_4}}.
\]
where

- $\mathcal{D}$ is the Cartan matrix of type $D_4$,
- $(\mathbf{m}, \mathbf{n}) = (m_{ij}, n_{ij}|1 \leq i < j \leq n)$,
- \[
\begin{align*}
\lambda_1 &= m_{12} + m_{13} + m_{14} + n_{14} + n_{13} + n_{12}, \\
\lambda_2 &= m_{23} + m_{14} + m_{24} + n_{24} + n_{23} + n_{13} + 2n_{12} + n_{14}, \\
\lambda_3 &= m_{34} + m_{14} + m_{24} + n_{23} + n_{13} + n_{12}, \\
\lambda_4 &= n_{34} + n_{14} + n_{24} + n_{23} + n_{13} + n_{12}, \\
\end{align*}
\]
- \[
B(\mathbf{m}, \mathbf{n}) = m_{12}^2 + m_{12}m_{13} + m_{12}m_{14} + m_{12}n_{12} + m_{12}n_{13} + m_{12}n_{14} + m_{13}^2 \\
+ m_{13}m_{14} + m_{13}m_{23} + m_{13}m_{24} + 2m_{13}n_{12} + m_{13}n_{13} + m_{13}n_{14} \\
+ m_{13}n_{23} + m_{13}n_{24} + m_{14}^2 + m_{14}m_{24} + m_{14}m_{34} + m_{14}n_{12} + m_{14}n_{13} \\
+ m_{14}n_{23} + m_{23}^2 + m_{23}m_{24} + m_{23}n_{12} + m_{23}n_{23} + m_{23}n_{24} + m_{24}^2 \\
+ m_{24}m_{34} + m_{24}n_{12} + m_{24}n_{23} + m_{34}^2 + m_{34}n_{12} + m_{34}n_{13} + m_{34}n_{23} \\
+ n_{12}^2 + n_{12}n_{13} + n_{12}n_{14} + 2n_{12}n_{23} + n_{12}n_{24} + n_{12}n_{34} + n_{13}^2 \\
+ n_{13}n_{14} + n_{13}n_{23} + n_{13}n_{34} + n_{14}^2 + n_{14}n_{23} + n_{14}n_{24} + n_{14}n_{34} + n_{23}^2 \\
+ n_{23}n_{24} + n_{23}n_{34} + n_{24}^2 + n_{24}n_{34} + n_{34}^2.
\]

Note the right-hand side of above identity is the character of the principal subspace of $L_{\mathfrak{so}}(1, 0)$. Then we have:

**Proposition 6.5.1.** The following two statements are equivalent:

1. The Hilbert series of $J_\infty(R_{\mathfrak{W}(\Lambda_0)})$ equals

\[
\sum_{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^n} \frac{q^{B(\mathbf{m}, \mathbf{n})}}{\prod_{1 \leq i < j \leq n} (q)_{m_{ij}} (q)_{n_{ij}}^\alpha}.
\]
The principal subspace of $L_{\hat{so}_8}(1,0)$ is classically free.

**Remark 6.5.2.** For algebra $D$, we can choose filtration $G^1$ and $G^2$ by letting $G^1_0$ and $G^2_0$ be generated with $\{V_{23}, V_{24}\}$ and $\{V_{14}, W_{24}\}$, respectively. And let

$$G^1_s = \text{span} \{(W_{ij})_n v, (V_{ij})_n v | n \leq 1, V_{ij} \neq V_{23}, \text{ or } V_{24}, v \in G^1_{s-1}\} + G^1_{s-1},$$

and

$$G^2_s = \text{span} \{(W_{ij})_n v, (V_{ij})_n v | n \leq 1, V_{ij} \neq V_{14}, W_{ij} \neq W_{24}, v \in G^2_{s-1}\} + G^2_{s-1}.$$

Then following the similar argument in Proposition 6.1.3, it is not hard to see that

$$HS_q(J_\infty(D))$$

is less than or equal to

$$\sum_{(m,n) \in \mathbb{N}^{12}} q^{B'(m,n)} \prod_{1 \leq i < j \leq n} (q)^{m_{ij}(q)n_{ij}},$$

where $B'(m,n)$ differs from $B(m,n)$ by $n_{12}n_{23}$ and $n_{12}m_{13}$, i.e.,

$$B(m,n) - B'(m,n) = n_{12}n_{23} + n_{12}m_{13}.$$

The method we have used to prove the classically freeness in the case of $sl_n$ fails here, since the “filtration procedure” does not give us a satisfactory upper bound of $HS_q(J_\infty(D))$.

In [52], a characterization of the $C_2$-algebra of the affine vertex algebra of type $C$ at non-negative integer level $k$ was given. For orthogonal series, a certain quotient of the $C_2$-algebra of the affine vertex algebra of type $D$ at non-negative level was also determined. We hope to use their results to further investigate the freeness of the affine vertex algebra of type $C$ and $D$ in the future.
6.6 Principal subspaces of type $C$ at level $-\frac{1}{2}$

6.6.1 Feingold-Frenkel (FF) construction

In this part we closely follow [53]. Consider the affine symplectic algebra $\widehat{sp}(2l)$. Let $\mathfrak{h}$ be the standard Cartan subalgebra and $\Delta \subset \mathfrak{h}^*$ the corresponding root system such that

$$\Delta = \{\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i : 1 \leq i, j \leq l\}.$$ 

Let

$$\alpha_1 = \epsilon_1 - \epsilon_2, ..., \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = 2\epsilon_l$$

be the simple roots, and the set of positive roots is $\{\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i : 1 \leq i < j \leq l\}$. Denote the root vectors in $sp(2l)$ by $x_\alpha$, where $\alpha \in \Delta$, and $x_\alpha(n)$ the corresponding generator in $\widehat{sp}(2l)$.

Let us recall the FF-construction of $\widehat{sp}(2l)$ via affine Weyl algebra $\text{Weyl}(Z)$, where $Z = Z$ or $Z + \frac{1}{2}$. Now we introduce the Weyl algebra (or $\beta\gamma$ system in the physics literature). Consider the Lie algebra generated by $\beta_i(n)$ and $\gamma_i(m)$ where $m, n \in Z$ such that

$$[\beta_i(n), \gamma_j(m)] = \delta_{i,j} \delta_{m+n,0},$$

$$[\beta_i(n), \beta_j(m)] = 0, \ [\gamma_i(n), \gamma_j(m)] = 0.$$ 

Denote the corresponding Fock space for $\text{Weyl}(Z)$ by $F^Z_{\beta\gamma}$.

**Theorem 6.6.1. (Feingold-Frenkel)** For $l \geq 2$, the assignment

$$X_{\epsilon_i+\epsilon_j}(z) =: \beta_i(z)\beta_j(z):$$

$$X_{2\epsilon_i}(z) = -\frac{1}{2} : \beta_i(z)\beta_i(z):$$

$$X_{\epsilon_i-\epsilon_j}(z) =: \beta_i(z)\gamma_j(z):$$

$$X_{-2\epsilon_i}(z) = \frac{1}{2} : \gamma_i(z)\gamma_i(z):$$

defines a representation $\widehat{sp}(2l)$ on the space $F^Z_{\beta\gamma}$. Moreover, as $\widehat{sp}(2l)$-modules

$$F^{Z+1/2}_{\beta\gamma} = L(-\frac{1}{2}\Lambda_0) \oplus L(-\frac{3}{2}\Lambda_0 + \Lambda_1), \quad F^Z_{\beta\gamma} = L(-\frac{1}{2}\Lambda_l) \oplus L(-\frac{3}{2}\Lambda_l + \Lambda_{l-1}).$$
If $l = 1$ (the case of $g = sl_2$) the representation is given by

$$X_\alpha(z) =: \beta(z)\beta(z) : \quad X_{-\alpha}(z) =: \gamma(z)\gamma(z) :$$

(6.31)

where $[\beta(n), \gamma(m)] = \delta_{m+n,0}$, and other commutators are zero.

Clearly, $L(-\frac{1}{2}\Lambda_0)$ has a natural vertex operator algebra structure. It is known that there are precisely four non-isomorphic irreducible $L(-\frac{1}{2}\Lambda_0)$-modules in the category $\mathcal{O}$: $L(-\frac{1}{2}\Lambda_0)$, $L(-\frac{3}{2}\Lambda_0 + \Lambda_1)$, $L(-\frac{1}{2}\Lambda_l)$ and $L(-\frac{3}{2}\Lambda_l + \Lambda_{l-1})$. Thus (by restriction to the subalgebra) each $W(\lambda)$ is a $W(-\frac{1}{2}\Lambda_0)$-module, where

$$\lambda \in S := \{-\frac{1}{2}\Lambda_0, -\frac{3}{2}\Lambda_0 + \Lambda_1, -\frac{1}{2}\Lambda_l, -\frac{3}{2}\Lambda_l + \Lambda_{l-1}\},$$

and $W(\lambda)$ is the principal subspace of $L(\lambda)$.

### 6.6.2 The case $g = sl_2$

If $g = sl_2$ the Weyl algebra construction is still applicable, however, the structure of the principal subspace is far from being interesting.

**Proposition 6.6.2.** For $g = sl_2$, we have

$$W(-\frac{1}{2}\Lambda_0) \cong \mathbb{C}[x].$$

**Proof.** Follows directly from (6.31). \qed

### 6.6.3 The case $g = sp_4$

In this section, we denote $W(-\frac{1}{2}\Lambda_0)$ by the principal subspace of $sp_4$-module $L(-\frac{1}{2}\Lambda_0)$.

**Proposition 6.6.3.** The $C_2$-algebra of $W(-\frac{1}{2}\Lambda_0)$ is isomorphic to $\mathbb{C}[x, y, z, w]/\langle xy - z^2 \rangle$, where we send $(X_{2\epsilon_1})(-1)\mathbf{1}$, $(X_{2\epsilon_2})(-1)\mathbf{1}$, $(X_{\epsilon_1+\epsilon_2})(-1)\mathbf{1}$ and $(X_{\epsilon_1-\epsilon_2})(-1)\mathbf{1}$ to $x$, $y$, $z$ and $w$, respectively.

**Proof.** Suppose $R_{W(-\frac{1}{2}\Lambda_0)} \cong \mathbb{C}[x, y, z, w]/I$. It is clear, from Theorem 6.6.1 that $\langle xy - z^2 \rangle \subset I$. 

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Assume $p$ is a homogeneous element of weight $d$ in $I$, $\sum_{m+n+s+t=d}c_{m,n,s,t}x^m y^n z^s w^t$. The polynomial $p$ comes from a normally product polynomial relation in $W(-\frac{1}{2}\Lambda_0)$

$$P = P(X_{2e_1}(z), \partial(X_{2e_1}(z)), \cdots, X_{2e_2}(z), \partial(X_{2e_2}(z)), \cdots, X_{e_1+e_2}(z), \partial(X_{e_1+e_2}(z)) \cdots X_{e_1-e_2}(z), \partial(X_{e_1-e_2}(z)) \cdots) = 0.$$ (6.32)

By using the relation $xy = z^2$ repeatedly, the summand of $p$ can be written as

$$c_{m', s', t'} x^{m'} z^{s'} w^{t'} \quad \text{or} \quad c_{n'', s'', t''} y^{n''} z^{s''} w^{t''}$$

which comes from

$$c_{m', s', t'} : X_{2e_1}(z)^{m'} X_{e_1+e_2}(z)^{s'} X_{e_1-e_2}(z)^{t'} : + \cdots$$

or

$$c_{n'', s'', t''} : X_{2e_1}(z)^{n''} X_{e_1+e_2}(z)^{s''} X_{e_1-e_2}(z)^{t''} : + \cdots$$

in $P$; by using free fields realization, we rewrite it as

$$c : \beta_1(z)^{2m'+s'+t'} \beta_2(z)^{s'} \gamma_2(z)^{t'} : + \cdots$$

or

$$c' : \beta_1(z)^{s''+t''} \beta_2(z)^{2n''+s''} \gamma_2(z)^{t''} : + \cdots.$$

Since $c : \beta_1(z)^{2m'+s'+t'} \beta_2(z)^{s'} \gamma_2(z)^{t'}$ : has to be cancelled within $P$, the only summand in $P$ can help us achieve it is

$$c' : \beta_1(z)^{s''+t''} \beta_2(z)^{2n''+s''} \gamma_2(z)^{t''} :$$

satisfying

$$\begin{cases}
2m' + s' + t' = s'' + t'' \\
2n'' + s'' \\
t' = t''
\end{cases}.$$ (6.33)

Note $m', s', t', n'', s'', t'' \in \mathbb{N}$. It is not hard to see that (6.33) is valid only when $m' = n'' = 0$, which implies $p \in \langle xy - z^2 \rangle$. The result follows.
Theorem 6.6.4. The principal subspace $W(-\frac{1}{2}\Lambda_0)$ is classically free, and its character is

$$\frac{1}{(1-q)(q)_\infty^3}.$$ 

Proof. Let $\hat{\mu}$ be the morphism

$$\hat{\mu} : gr(W(-\frac{1}{2}\Lambda_0)) \to gr(F_{\beta-\gamma})$$

induced from the embedding from $W(-\frac{1}{2}\Lambda_0)$ to $F_{\beta-\gamma}$. In this case, the $C_2$-algebra $R_{F_{\beta+1/2}}$ is isomorphic to $C[\beta_1, \beta_2, \gamma_1, \gamma_2]$. From Proposition 6.6.3, we have $R_W(-\frac{1}{2}\Lambda_0) \cong C[x, y, z, w]/\langle xy - z^2 \rangle$. Define a map $\mu : R_W(-\frac{1}{2}\Lambda_0) \to R_{F_{\beta-\gamma}}$ by sending $x, y, z$ and $w$ to $\beta_1\beta_2, \beta_1\beta_2, \beta_1\beta_2$ and $\beta_1\gamma_2$, respectively. Clearly, the $C_2$-algebra $R_W(-\frac{1}{2}\Lambda_0)$ is isomorphic to the image of the map $\mu$ which we denote by $W$. We have the following commutative diagram

$$
\begin{array}{ccc}
J_\infty(R_W(-\frac{1}{2}\Lambda_0)) & \xrightarrow{gr(W(-\frac{1}{2}\Lambda_0))} & J_\infty(W) \\
\downarrow & & \downarrow \\
J_\infty & & J_\infty(W)
\end{array}
$$

Note $J_\infty(\mu)$ and $\hat{\mu}$ are isomorphisms. The map $\psi$ has to be isomorphic.

Since the scheme $	ext{Spec}(C[x, y, z]/\langle xy = z^2 \rangle)$ is a surface with double rational singular point at origin, according to [86][Proposition 4.3], we have $[C[x, y, z]/\langle xy = z^2 \rangle](q) = \frac{1}{(1-q)(q)_\infty^2}$. Thus the character of $W(-\frac{1}{2}\Lambda_0)$ is

$$\frac{1}{(1-q)(q)_\infty^3}.$$ 

We can also derive the Hilbert series of $J_\infty(C[x, y, z]/\langle xy = z^2 \rangle)$ by using combinatorial method. Let ordered monomial in $J_\infty(C[x, y, z]/\langle xy = z^2 \rangle)$ be of the form:

$$y(-n)^{a_n}z(-n)^{b_n}x(-n)^{c_n} \cdots y(-2)^{a_2}z(-2)^{b_2}x(-2)^{c_2}x(-1)^{a_1}z(-1)^{b_1}x(-1)^{c_1}.$$ 

Then we have a complete lexicographic ordering on the set of ordered monomials in the sense of Section 2.4.
Lemma 6.6.5 ([104]). The leading terms, \(lt\), of the relations \((xy - z^2)_\partial\) are given by:

\[
LT(T^{2i-2}(xy - z^2)) = z(-i)^2 \quad i \in \mathbb{Z}_+
\]

and

\[
LT(T^{2i-1}(xy - z^2)_{-2i-1}) = x(-i-1)y(-i) \quad i \in \mathbb{Z}_+.
\]

Lemma 6.6.6. The Hilbert series of \(J_\infty(\mathbb{C}[x, y]/(xy - z^2))\) agrees with the Hilbert series of

\[
\mathbb{C}[x(-1), y(-1), z(-1), x(-2), y(-2), z(-2), ...]/I
\]

where \(I\) is given by

\[
I = \langle z(-i)^2, x(-i-1)y(-i), i \in \mathbb{Z}_+ \rangle
\]

Proof. Use Lemma 6.6.5 and Groebner’s bases as in [104] Proposition 17.2.

Lemma 6.6.7. [Combinatorial Lemma] Fix \(m \in \mathbb{N}\). Let \(C_2(m, n)\) denote the number of two colored partition \(\pi = \lambda \cup \mu\) of \(n\), where \(\lambda = (\lambda_1, ..., \lambda_k)\) and \(\mu = (\mu_1, \mu_2, ..., \mu_\ell)\) are ordinary partitions, subject to \(\mu_i \neq \lambda_j + m\) for every \(i\) and \(j\). Then the generating function of \(C_2(m, n)\) is given by

\[
\sum_{n \in \mathbb{N}} C_2(m, n)q^n = \prod_{i=1}^{m} \frac{1}{1-q^i} \prod_{i \in \mathbb{Z}_+} \left( \frac{1}{1-q^i} + \frac{1}{1-q^{i+m}} - 1 \right).
\]

(6.34)

In particular, for \(m = 1\), we get

\[
\frac{1}{1-q} \prod_{i \in \mathbb{Z}_+} \frac{1}{(1-q^{2i})(q)_{\infty}}.
\]

Proof. We observe that for any \(i\), the factor \(\frac{1}{1-q^i}\) appears precisely twice inside (6.34). Since there is no condition on \(\mu\) for all parts of size \(\leq m\) we have unrestricted contribution from the \(\mu\)-color giving \(\prod_{i=1}^{m} \frac{1}{(1-q^i)}\) inside the infinite product. When \(\mu\) has parts of size \(m+1\) there is an extra condition that there cannot be simultaneously any parts of \(\lambda\) of size 1. This is expressed as \(\frac{1}{1-q} + \frac{1}{1-q^{m+1}} - 1\). Similar argument applies for \(\lambda\) parts of size \(i\) and \(\mu\) parts of size \(m+i\). Putting everything together gives the claim.

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For \( m = 1 \) this simplifies using \( \frac{1}{1 - q^i} + \frac{1}{1 - q^{i+1}} - 1 = \frac{1 - q^{2i+1}}{(1 - q^i)(1 - q^{i+1})} \). \( \square \)

Next we obtain

**Lemma 6.6.8.**

\[
HS_q(\mathbb{C}[x, y, z]/(xy - z^2)) = \frac{1}{(1 - q)(q^2)}.
\]

In particular, monomials in \( x(-i), y(-j), z(-k) \) without segments of type \( I_1 \) and \( I_2 \), form a basis of the arc algebra.

**Proof.** We need to count partitions in three colors such that neither \( I_1 \) nor \( I_2 \) holds. We use the fact that all permutation which do not satisfy the \( I_1 \) condition are \( z \)-partition with distinct parts whose generating function is \( \prod_{i \in \mathbb{Z}^+} (1 + q^i) \). The previous Lemma (with \( m = 1 \)), is now used to determine all monomials such that the \( I_2 \) condition on \( x \) and \( y \)-colors does not hold. Putting this together gives

\[
\prod_{i \in \mathbb{Z}^+} \frac{(1 + q^i)}{(1 - q)(1 - q^{2i})(q^\infty)} = \prod_{i \in \mathbb{Z}^+} \frac{1}{(1 - q)(1 - q^{2i-1})(1 - q^{2i})(q^\infty)} = \frac{1}{(1 - q)(q^\infty)^2}
\]

for the Hilbert series, as required. \( \square \)

### 6.6.4 Presentation of principal subspaces

**Proposition 6.6.9.** We have

\[ \text{Spec}(J_\infty(R_W(-\frac{1}{2}A_0)/I)) \subset SS(W(-\frac{1}{2}A_0) \subset \text{Spec}(J_\infty(R_W(-\frac{1}{2}A_0))), \]

where \( I = \langle X_{\epsilon_i+\epsilon_j}X_{\epsilon_k-\epsilon_j} - X_{\epsilon_i-\epsilon_j}X_{\epsilon_k+\epsilon_j} | i \leq j, k < j \rangle \).

**Proof.** We define a map \( \mu \) from \( R_{W(-\frac{1}{2}A_0)} \) to \( R_{x^{2i+1/2}} \) by sending \( X_{\epsilon_i+\epsilon_j}, X_{\epsilon_i-\epsilon_j}, X_{2\epsilon_i} \) to \( \beta_i\beta_j, \beta_i\gamma_j, -\frac{1}{2}\beta_i\beta_i \). The image of \( \mu \) is \( \mathbb{C}[\beta_i\beta_j, \beta_i\gamma_j, -\frac{1}{2}\beta_i\beta_i] \) with following three types of relations:

1. \( \beta_i\beta_j\beta_k\gamma_l = \beta_{\sigma(i)}\beta_{\sigma(j)}\beta_{\sigma(k)}\gamma_l \), where \( \sigma \in S_3, i \leq j, k < l, \sigma(i) \leq \sigma(j) \) and \( \sigma(k) < l \).
II. $\beta_i \beta_j \beta_k \beta_l = \beta_{\sigma(i)} \beta_{\sigma(j)} \beta_{\sigma(k)} \beta_{\sigma(l)}$, where $\sigma \in S_4$, $i \leq j$, $k \leq l$, $\sigma(i) \leq \sigma(j)$ and $\sigma(k) < \sigma(l)$.

III. $\beta_i \beta_j \beta_k \gamma_j = \beta_i \gamma_j \beta_k \beta_j$, where $i \leq j$ and $k < j$.

By $\beta$-$\gamma$ realization, the preimages of type I and II relations are also quotient relations of $R_{W(-\frac{1}{2}\Lambda_0)}$. Note: $\beta_i(z)\beta_j(z)\beta_k(z)\gamma_j(z) : - : \beta_i(z)\gamma_j(z)\beta_k(z)\beta_j(z) : - : \beta_i(z)\partial\beta_k(z) :$. Thus $X_{\epsilon_i+\epsilon_j} X_{\epsilon_k-\epsilon_j} - X_{\epsilon_i-\epsilon_j} X_{\epsilon_k+\epsilon_j} \neq 0$ in $R_{W(-\frac{1}{2}\Lambda_0)}$. The kernel of $\mu$ is generated by

$$\{X_{\epsilon_i+\epsilon_j} X_{\epsilon_k-\epsilon_j} - X_{\epsilon_i-\epsilon_j} X_{\epsilon_k+\epsilon_j} | i \leq j, k < j \}.$$

The result follows from the similar argument in Theorem 6.6.4.

6.7 Principal subspaces of type $A$ at level $-1$

In this section, we let $W(-\Lambda_0)$ be the principal subspace of $L_{\tilde{sl}_n}(-1,0)$. We denote the $(i,j)$-th entry of $sl_n$ by $E_{i,j}$. It was shown in [11] that $L_{\tilde{sl}_n}(-1,0) n \geq 4$ is not necessarily classically free. We will prove that $W(-\Lambda_0)$ is classically free.

Firstly, we have the following free fields realization for $W(-\Lambda_0)$.

**Lemma 6.7.1.** [4] We have the following embedding from $W(-\Lambda_0)$ to $\beta$-$\gamma$ system:

$$E_{i,j}(z) \rightarrow: \beta_i(z)\gamma_j(z) :.$$

Then we characterize the $C_2$-algebra of $W(-\Lambda_0)$.

**Proposition 6.7.2.** We have $R_{W(-\Lambda_0)} \cong \mathbb{C}[E_{i,j}|1 \leq i < j \leq n]/I$, where $I$ is generated by

$$\{E_{i,j}E_{k,l} = E_{i,j}E_{k,l} | \max \{i,k\} < \min \{j,l\}\}.$$

**Proof.** According to the free fields realization of $W(-\Lambda_0)$, it is not hard to see that

$$E_{i,j}(z)E_{k,l}(z) - E_{i,l}(z)E_{k,j}(z) = 0, \quad \text{if} \quad \max \{i,k\} < \min \{j,l\}. \quad (6.35)$$

Thus, it is enough to show $I \subset \langle E_{i,j}E_{k,l} = E_{i,l}E_{k,j} | \max \{i,k\} < \min \{j,l\}\rangle$. Given any $p \in I$,
it is of the form \( p = \sum_{1 \leq i_1, \ldots, i_m \leq n} E_{i_1,j_1} E_{i_2,j_2} \cdots E_{i_m,j_m} \), and it comes from the relation
\[
P = \sum_{1 \leq i_1, \ldots, i_m \leq n} E_{i_1,j_1}(z) E_{i_2,j_2}(z) \cdots E_{i_m,j_m}(z) + \text{higher order terms} = 0 \quad (6.36)
\]
in \( W(-\Lambda_0) \). By using relation (6.35) iteratively, (6.36) can be rewritten as
\[
P = \sum_{1 \leq i_1, \ldots, i_m \leq n} c \cdot E_{i_1,j_1}(z) E_{i_2,j_2}(z) \cdots E_{i_m,j_m}(z) + \text{higher order terms} = 0, \quad (6.37)
\]
where \( c \) is some constant, and if \( \max \{i_s, i_t\} < \min \{j_s, j_t\} \) \((1 \leq s < t \leq m)\), we have \( j_s < j_t \).
We claim that there is at most one summand in the first sum of (6.37). We prove this claim by contradiction. Suppose there are two distinct summands, i.e., \( c \cdot E_{i_1,j_1}(z) E_{i_2,j_2}(z) \cdots E_{i_m,j_m}(z) \) and \( c' \cdot E_{k_1,l_1}(z) E_{k_2,l_2}(z) \cdots E_{k_m,l_m}(z) \). Since these two terms will cancel each other out, we have \( i_p = k_p \) for \( 1 \leq p \leq m \) and \( \{j_1, \ldots, j_m\} = \{l_1, \ldots, l_m\} \). Let \( t \) be the smallest number such that \( j_p \neq l_p \).
Without loss of generality, we assume \( j_p > l_p \). Then there must exist some number \( r \) such that \( r > t \) and \( j_r = l_t \) which implies \( \max \{i_p, i_r\} < \min \{j_p, j_r\} \). This contradicts our previous assumption. Therefore, we can only have at most one summand in the first sum of (6.37). If we have one summand, its coefficient must equal 0. This means that \( p \in I \).

**Theorem 6.7.3.** The principal subspace \( W(-\Lambda_0) \) is classically free.

**Proof.** We define a map from \( R_W(-\Lambda_0) \) to \( R_{\mathbb{R}^2_{x^2 + y^2}} \) by sending \( E_{i,j} \) to \( \beta_{i} \beta_{j} \). By Proposition 6.7.2, \( R_W(-\Lambda_0) \) is isomorphic to the image of this map. The result follows from the same argument as in Theorem 6.6.4.

**6.7.1 Character formula of the principal subspace of \( L_{\hat{\mathfrak{sl}}_4}(-1,0) \)**

According to Proposition 6.7.2 the \( C_2 \)-algebra of the principal subspace of \( L_{\hat{\mathfrak{sl}}_4}(-1,0) \), \( W_{\Lambda_3}(\Lambda_0) \) is
\[
R_{W_{\Lambda_3}(\Lambda_0)} = \mathbb{C}[x,y,z,w,u,v]/\langle xy - zw \rangle.
\]
Here we are interested in \( xy - zw = 0 \). For \( J_{\infty}(\mathbb{C}[x,y,z,w]/\langle xy - zw \rangle) \) we define
ordered monomial to be of the form

\[ x(-n)^{a_n} z(-n)^{b_n} w(-n)^{c_n} y(-n)^{d_n} \cdots x(-2)^{a_2} z(-2)^{b_2} w(-2)^{c_2} y(-2)^{d_2} \]
\[ x(-1)^{a_1} z(-1)^{b_1} w(-1)^{c_1} y(-1)^{d_1}. \]

With a complete lexicographic ordering on the set of ordered monomials in the sense of Section 2.4 the relevant leading terms in \( \langle xy - zw \rangle \) are

\[ x(-i)^{t_i} y(-i)^{t_i} (i \in \mathbb{Z}_+) \text{ and } z(-i - 1)^{t_i} w(-i)^{t_i} (i \in \mathbb{Z}_+) \].

So we have to count all 4-colored partitions without these two types of segments. Using Lemma 6.6.7 for \( m = 0 \) and \( m = 1 \), we get for the Hilbert series of \( J_\infty(\mathbb{C}[x, y, z, w]/\langle I_3, I_4 \rangle) \):

\[ \prod_{i \in \mathbb{Z}_+} \frac{(1 + q^i)^2}{1 - q^i} \frac{1}{(1 - (q)(q^2)_\infty)(q)_\infty} = \frac{(-q)_\infty}{(1 - q)(q^2)_\infty(q^3)_\infty} = \frac{1}{(1 - q)(q^3)_\infty}. \]

(cf. \( m = 0 \), Lemma 6.6.7 gives \( \prod_{i \in \mathbb{Z}_+} \left( \frac{1}{1 - q^i} + \frac{1}{1 - q^i} - 1 \right) = \frac{(-q)_\infty}{(q)_\infty} \).)

**Proposition 6.7.4.** The set \( \mathcal{G} = \{ T^i (xy - zw) | i \in \mathbb{N} \} \) is a Gröbner basis of \( (xy - zw)_\partial \).

**Proof.** Note all leading terms of the elements in \( \mathcal{G} \) are coprime with each other. Then given any two elements \( g_1, g_2 \in \mathcal{G} \), \( S \)-polynomial \( S(g_1, g_2) \) reduces to 0 modulo \( \mathcal{G} \). The result follows. \( \square \)

**Corollary 6.7.5.**

\[ \frac{\text{ch}[W_{C_2}(-\frac{1}{2}\Lambda_0)]}{(q^2)_\infty^2} = \text{ch}[W_{A_4}(-\Lambda_0)] \]

**Proof.** This follows immediately from the description of

\[ \text{gr}(W_{A_4}(-\Lambda_0)) = J_\infty(\mathbb{C}[x, y, z, w, u, v]/(xy - zw)). \]

\( \square \)
CHAPTER 7

Virasoro and N=1 superconformal vertex algebras

7.1 N = 1 superconformal vertex algebras

In this section we consider rational N = 1 vertex superalgebras $L_{c_{2,4k}}^{N=1}$ ($k \in \mathbb{Z}_+$) associated to N = 1 superconformal (2, 4k)-minimal models [2]. Here the central charge is

$$c_{2,4k} = \frac{3}{2} \left( 1 - \frac{2(4k-1)^2}{8k} \right).$$

According to [82, 84], we know that the normalized character of $L_{c_{2,4k}}^{N=1}$ (without the $q^{-c/24}$ factor) is:

$$\text{ch}[L_{c_{2,4k}}^{N=1}](q) = \prod_{n \neq 0, \pm 1(4k)} \frac{1}{1 - q^{\frac{n}{2}}},$$

$$= \sum_{m_1, \ldots, m_{k-1} \in \mathbb{N}} \frac{(-q^{\frac{n}{2}})^{N_1} q^{\frac{n}{2} N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_{k+1} + N_{(s+1)/2} + N_{(s+3)/2} + \cdots + N_k}}{(q)_{m_1}(q)_{m_2} \cdots (q)_{m_{k-1}}}.$$

And the fermionic character formula is the generating function (cf. [82])

$$\text{ch}[L_{c_{2,4k}}^{N=1}](q) = \sum_{n=0}^{\infty} D_{k,1}(n) q^{\frac{n}{2}}$$

of the number of partitions of $D_{k,1}(n)$ of $\frac{n}{2}$ in the form $\frac{n}{2} = b_1 + \cdots + b_m$ ($b_j \in \frac{1}{2} \mathbb{Z}_+$), where $b_1, \ldots, b_m$ satisfy the following conditions:

- no half-odd integer is repeated,
- $b_j \geq b_{j+1}$, $b_m \in \frac{3}{2} \mathbb{Z}_+$,
- $b_j - b_{j+k-1} \in \mathbb{Z}_+$ if $b_j \in \frac{1}{2} + \mathbb{Z}$,
- $b_j - b_{j+k-1} > 1$ if $b_j \in \mathbb{Z}$.

Since $N = 1$ vertex superalgebra $L_{c_{2,4}}^{N=1}$ is isomorphic to $\mathbb{C}$, we only need consider $L_{c_{2,4k}}^{N=1}$, where $k > 1$. First let us find the $C_T$-algebra of $L_{c_{2,4k}}^{N=1}$. According to [84, Section 4], we know
that the null vector in universal algebra which survives inside the $C_2$-algebra is $L^{k-1}_{(-2)} G_{(-\frac{3}{2})} \mathbf{1}$. Moreover, if we let $G_{(-\frac{1}{2})}$ act on the null vector, we get another null vector which survives in the $C_2$-algebra, i.e., $L^k_{(-2)} \mathbf{1}$. These two null vectors in the vacuum algebra generate the whole quotient ideal of $R_{L^N_{2,4k}}$. Thus $R_{L^N_{2,4k}}$ is isomorphic to superalgebra $\mathbb{C}[l, g]/\langle l^k, l^{k-1}g \rangle$, where $g$ is an odd element.

We are going to prove that $\psi$ is an isomorphism. We identify $l, g$ with $l(-2), g(-\frac{3}{2})$, respectively, inside the arc superalgebra.

It is clear that $J_\infty(\mathbb{C}[l, g]/\langle l^k, l^{k-1}g \rangle)$ is isomorphic to

$$\mathbb{C}[l(-2-i), g(-\frac{3}{2}-j) | i, j \in \mathbb{N}]/\langle l(z)^k, l(z)^{k-1}g(z) \rangle,$$

where $l(z) = \sum_{n \in \mathbb{N}} l(-2-n) z^n$, $g(z) = \sum_{n \in \mathbb{N}} g(-\frac{3}{2}-n) z^n$ and $\langle l(z)^k, l(z)^{k-1}g(z) \rangle$ is the ideal generated by the Fourier coefficients of $l(z)^k, l(z)^{k-1}g(z)$. We define ordered monomial in $J_\infty(\mathbb{C}[l, g]/\langle l^k, l^{k-1}g \rangle)$ to be a monomial of the form

$$l(-2-n)^{a_1} g(-\frac{3}{2}-n)^{b_1}l(-1-n)^{a_2} g(-\frac{3}{2}-n+1)^{b_2} \cdots l(-2)^{a_{n+1}} g(-\frac{3}{2})^{b_{n+1}},$$

where $n \in \mathbb{N}$. Then we have a complete lexicographic ordering on all ordered monomials according to Section 2.4.

We know that all ordered monomials constitute a spanning set of the arc superalgebra. Following the similar argument in Section 4.2 we can make use of the quotient relation to impose some conditions on the spanning set to get a smaller spanning set. Firstly, since all variables $g(k)$’s are odd, no two $g(k)$ can appear in the ordered monomial. The leading term of any coefficient of $z^{nk}$ in $l(z)^k$ is $l(-2-n)^k$. Thus $l(-2-n)^k$ should not appear as a segment of any element in spanning set. Similarly we can list further leading terms in the quotient:

- Leading term of the coefficient of $z^{nk}$ in $l(z)^{k-1}g(z)$:

$$l(-2-n)^{k-1}g(-\frac{3}{2}-n).$$

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• Leading term of the coefficient of \( z^{n(k-1-i)+(n-1)i+n} \) in \( l(z)^{k-1}g(z) \):

\[
l(-2 - n)^{k-1-i}g(-\frac{3}{2} - n)l(-2 - n + 1)^i \quad (i = 1, \cdots, k - 1).
\]

Now we obtain a smaller spanning set, where above three type leading terms can not appear inside any ordered monomial. More precisely, any element in this spanning set is of the form

\[
w(b_1)w(b_2) \cdots w(b_m),
\]

where \( b_i \geq b_{i+1}, \ w(a) = l(a) \) if \( a \in \mathbb{Z} \) and \( w(a) = g(a) \) if \( a \in \frac{1}{2}\mathbb{Z} \). And the fact that \( g(a) \) is odd implies that no half-odd-integer is repeated in \( \{b_1, b_2, \cdots, b_m\} \). Moreover, we have the condition

\[
b_j - b_{j-k+1} > 1, \quad \text{if} \quad b_j \in \mathbb{Z},
\]

because

\[
l(-2 - n)^k, \quad l(-2 - n)^k g(-\frac{3}{2} - n),
\]

\[
l(-2 - n)^{k-1-i}g(-\frac{3}{2} - n)l(-2 - n + 1)^i \quad (i = 1, \cdots, k - 1)
\]

are leading terms of some elements in the quotient ideal. We also have a condition

\[
b_j - b_{j-k+2} \geq 1 \quad \text{if} \quad b_j \in \frac{1}{2}\mathbb{Z},
\]

because

\[
g(-\frac{3}{2} - n)l(-2 - n + 1)^{k-1}
\]

is the leading term of some element in the quotient ideal. So we have

\[
HS_q(J_\infty(\mathbb{C}[l, g]/\langle l^k, l^{k-1}g \rangle)) \leq \sum_{n=0}^{\infty} D_{k,1}(n)q^{\frac{n}{2}} = \text{ch}[gr(L_{c_{2,4k}}^{N=1})](q).
\]

Meanwhile the surjectivity of \( \psi \) implies that

\[
HS_q(J_\infty(\mathbb{C}[l, g]/\langle l^k, l^{k-1}g \rangle)) \geq \text{ch}[gr(L_{c_{2,4k}}^{N=1})](q).
\]
Thus $HS_q(J_{\infty}(\mathbb{C}[l,g]/\langle t^k, t^{k-1}g \rangle)) = \text{ch}[\text{gr}(L^{N=1}_{c_{2,4k}})](q)$, and $\psi$ is an isomorphism. It implies that above spanning set is a basis of the arc superalgebra. The image of basis of arc superalgebra under map $\psi$ is a basis of $\text{gr}(L^{N=1}_{c_{2,4k}})$.

We have following result which is a super-analog of [104, Theorem 16.13]:

**Theorem 7.1.1.** Let $p' > p \geq 2$ satisfy that $\frac{p'-p}{2}$ and $p$ are coprime positive integers. Let $L^{N=1}_{c_{p,p'}}$ be the simple $N = 1$ vertex superalgebra associated with $N = 1$ superconformal $(p,p')$-minimal model of central charge $c_{p,p'} = \frac{3}{2}(1 - \frac{2(p'-p)^2}{pp'})$. Then the map $\psi$ is an isomorphism if and only if $(p,p') = (2, 4k)$, $(k \in \mathbb{Z}_+)$.

**Proof.** We first consider the $C_2$-algebra of $L^{N=1}_{c_{p,p'}}$. We let

$$|c_{p,p'}| = \frac{(p - 1)(p' - 1)}{4} + \frac{1 + (-1)^{pp'}}{8} \in \mathbb{N}.$$

When $p$ and $p'$ are both even, according to [84, Section 4], there are two null vectors which survive in $R_{V^{N=1}_{c_{p,p'}}}$, i.e., $L^{|c_{p,p'}|}_{(-2)} \mathbf{1}$ and $L^{|c_{p,p'}|-1}_{(-2)} \mathbf{G}_{-\frac{3}{2}}$. They generate the quotient ideal of $R_{V^{N=1}_{c_{p,p'}}}$ in the vacuum algebra. In this case, the $C_2$-algebra $R_{L^{N=1}_{c_{p,p'}}}$ is isomorphic to

$$\mathbb{C}[l,g]/\langle l^{|c_{p,p'}|}, l^{|c_{p,p'}|-1}g \rangle.$$

When $p$ and $p'$ are both odd, again from [84, Section 4], the null vector $L^{|c_{p,p'}|}_{(-2)} \mathbf{1}$ generates the quotient ideal of $R_{L^{N=1}_{c_{p,p'}}}$. The $C_2$-algebra is isomorphic to

$$\mathbb{C}[l,g]/\langle l^{|c_{p,p'}|} \rangle.$$

Suppose $p$ and $p'$ are both odd. Then

$$HS_q(J_{\infty}(\mathbb{C}[l,g]/\langle l^{|c_{p,p'}|} \rangle)) = HS_q(J_{\infty}(\mathbb{C}[g]))HS_q(J_{\infty}(\mathbb{C}[l]/\langle l^{|c_{p,p'}|} \rangle)),$$

It is clear that $HS_q(J_{\infty}(\mathbb{C}[g])) = \prod_{i \in \mathbb{Z}_+} (1 + q^{i+1/2})$. According to [48], we get

$$HS_q(J_{\infty}(\mathbb{C}[l]/\langle l^{|c_{p,p'}|} \rangle)) \cong \text{ch}[L_{V^{ir}}(c_{2,(p-1)(p'-1)_{-1}+1}, 0)](q),$$

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where $L_{Vir}(c_2, (p-1)(p'-1)+1, 0)$ is the simple Virasoro vertex algebra coming from $(2, \frac{(p-1)(p'-1)}{2} + 1)$-minimal model. By using the character formula of $L_{Vir}(c_{q,q'}, 0)$ from [48], the Hilbert series of $J_{\infty}(\mathbb{C}[l, g]/\langle l^{c_{p,p'}}, l^{c_{p,p'}-1}g \rangle)$ is

$$\prod_{i \in \mathbb{Z}_+} \frac{(1 + q^{i+\frac{1}{2}})}{(1 - q^{i})} \sum_{j \in \mathbb{Z}} \left( q^{j(p-1)(p'-1) + 2j + \frac{(p-1)(p'-1)}{2} - 1} - q^{(2j+1)\left(\frac{(p-1)(p'-1)}{2} + 1\right)j + 1} \right). \quad (7.1)$$

Meanwhile, by [82] the character of $L_{N=1}^{c_{p,p'}}$ is

$$\text{ch}[L_{c_{p,p'}}^{N=1}](q) = \prod_{i \in \mathbb{Z}_+} \frac{(1 + q^{i-\frac{1}{2}})}{(1 - q^{i})} \sum_{j \in \mathbb{Z}} \left( q^{\frac{j(p'p' + p - p)}{2}} - q^{\frac{(j+1)(p'p' + 1)}{2}} \right). \quad (7.2)$$

By comparing (7.1) and (7.2) we get that $\psi$ is not an isomorphism in this case.

Let $p$ and $p'$ be both even. Suppose $(p, p') \notin \{(2, 4k) \mid k \in \mathbb{Z}_+\}$ and $\psi$ is an isomorphism for $L_{c_{p,p'}}^{N=1}$. Then

$$HS_q(J_{\infty}(\mathbb{C}[l, g]/\langle l^{c_{p,p'}}, l^{c_{p,p'}-1}g \rangle)) = \text{ch}[L_{c_{p,p'}}^{N=1}](q).$$

On the other hand, we have shown that

$$HS_q(J_{\infty}(\mathbb{C}[l, g]/\langle l^k, l^{k-1}g \rangle)) = \text{ch}[L_{c_{2,4k}}^{N=1}](q), \quad (k \in \mathbb{Z}_+).$$

Therefore, the character of $L_{c_{p,p'}}^{N=1}$ must coincide with the character of $L_{c_{2,4k}}^{N=1}$ for some $k$. Note (7.2) is also true when $p$ and $p'$ are both even, and it is easy to verify from the numerator that no two $N = 1$ minimal vertex algebras have the same character. This is a contradiction. Thus the statement is proved.

\[\square\]

### 7.2 Extended Virasoro vertex algebras

For a simple Virasoro vertex algebra $L_{Vir}(c_{2,2k+1}, 0)$ coming from $(2, 2k + 1)$-minimal model, according to [48], we know that $R_{L_{Vir}(c_{2,2k+1})} \cong \mathbb{C}[x]/(x^k)$, and $\psi$ is an isomorphism.
Let $p$ and $p'$ be two positive coprime integers satisfying $p > p' \geq 2$. It is easy to see that $\psi$ is an isomorphism if and only if $(p, p') = (2, 2k + 1)$ (see [104, Theorem 16.13]). Recently, the authors displayed the kernel of $\psi$ [105, Theorem 1] for the $c = \frac{1}{2}$ Ising model vertex algebra $L_{\text{Vir}}(c_{3,4}, 0)$, based on a new fermionic character formula of $L_{\text{Vir}}(c_{3,4}, 0)$.

If we consider extended Virasoro vertex algebras associated with minimal model which is not necessarily a $(2, 2k + 1)$-minimal model, we might still have that $\psi$ is isomorphism. Our discussion is heavily motivated by [61], where the combinatorics of (super)extensions of $(3, p)$-minimal vertex algebras were discussed.

**Example 7.2.1.** For the free fermion model $\mathcal{F} = L_{\text{Vir}}(c_{3,4}, 0) \oplus L_{\text{Vir}}(c_{3,4}, \frac{1}{2})$, $\psi$ is clearly an isomorphism as discussed in Proposition 4.1.2.

**Example 7.2.2.** The $L_{c_{2,8}}^{N=1}$ minimal vertex superalgebra has the following realization:

$$L_{c_{2,8}}^{N=1} \cong L_{\text{Vir}}(c_{3,8}, 0) \oplus L_{\text{Vir}}(c_{3,8}, \frac{3}{2}).$$

This realization is called extended algebra and was studied in [61]. The map $\psi$ is not an isomorphism in the case of $L_{\text{Vir}}(c_{3,8}, 0)$. But we have shown that for the extended algebra of $L_{\text{Vir}}(c_{3,8}, 0)$, the map $\psi$ is an isomorphism. This model was analyzed from a different perspective in [77].

**Example 7.2.3.** Next, let us consider $V = L_{\text{Vir}}(c_{3,10}, 0) \oplus L_{\text{Vir}}(c_{3,10}, 2)$. It is well-known that

$$L_{\text{Vir}}(c_{2,5}, 0) \otimes L_{\text{Vir}}(c_{2,5}, 0) \cong L_{\text{Vir}}(c_{3,10}, 0) \oplus L_{\text{Vir}}(c_{3,10}, 2).$$

We let $\omega_1$ and $\omega_2$ be conformal vectors of the first factor and the second factor of

$$L_{\text{Vir}}(c_{2,5}, 0) \otimes L_{\text{Vir}}(c_{2,5}, 0).$$

Then the isomorphism map $f$ sends $\omega_1 + \omega_2$ to the conformal vector $\omega$ of $L_{\text{Vir}}(c_{3,10}, 0)$ and $\omega_1 - \omega_2$ to the lowest weight vector $\phi$ of $L_{\text{Vir}}(c_{3,10}, 2)$. Since we know that

$$J_\infty(R_{L_{\text{Vir}}(c_{2,5}, 0)}) \cong J_\infty(C[x]/(x^2)) \cong gr(L_{\text{Vir}}(c_{2,5}, 0)),$$
the map $\psi$ is an isomorphism for $V$, i.e.,

$$J_\infty(R_V) = J_\infty(R_{L_{Vir}(c(2,5),0)} \otimes R_{L_{Vir}(c(2,5),0)}) = J_\infty(C[x,y]/\langle x^2, y^2 \rangle) \cong gr(V).$$

For $L_{Vir}(c(3,0), 0) \oplus L_{Vir}(c(3,0), 2)$, its $C_2$-algebra is isomorphic to

$$\mathbb{C}[u,v]/\langle uv, u^2 + v^2, u^3, v^3 \rangle$$

after we identify $x + y, x - y$ in $\mathbb{C}[x,y]/\langle x^2, y^2 \rangle$ with $u$ and $v$, respectively.

**Remark 7.2.4.** We also know from [61] that the normalized parafermionic character of $V = L_{Vir}(c(3,0), 0) \oplus L_{Vir}(c(3,0), 2)$ is given by

$$\text{ch}[V](q) = \sum_{n_1, n_2, m_1 \in \mathbb{N}} q^{(n_1 + n_2 + m_1)(n_1 + n_2) + n_2(n_2 + m_1) + m_1^2 + m_1 + n_1 + 2n_2} \frac{(q)_{n_1}(q)_{n_2}(q)_{m_1}}{(q)_{n_1 + n_2}(q)_{m_1}}.$$ 

Next, let us consider the arc algebra

$$J_\infty(\mathbb{C}[u,v]/\langle u^2, v^3, uv \rangle),$$

where weights of $u$ and $v$ are both 2. Clearly, it has the following spanning set:

$$u_{(-n_1)} \cdots u_{(-n_N)}v_{(-m_1)} \cdots v_{(-m_M)}$$

subject to constraints:

(a) *difference two condition at distance 1*: $n_i \geq n_{i+1} + 2$,

(b) *difference two condition at distance 2*: $m_i \geq m_{i+2} + 2$,

(c) *boundary condition*: $n_N \geq 2 + M$,

where conditions (a), (b), (c) are coming from $(u^2)_\partial$, $(v^3)_\partial$, $(uv)_\partial$ in the quotient ideal of the arc algebra. Meanwhile according to Proposition [5.2.1] and Theorem [5.4.5], we know that

$$J_\infty(\mathbb{C}[u]/\langle u^2 \rangle) \cong gr(W_{\Lambda_1,0}),$$

$$J_\infty(\mathbb{C}[v]/\langle v^3 \rangle) \cong gr(W_{\Lambda_2,0}),$$

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\[ J_\infty(\mathbb{C}[u, v]/\langle uv \rangle) \cong gr(W_{\Gamma}), \]

where \( \Gamma \) is the graph \( o - o \). Using three realizations of arc algebras and Gordon-Andrews character formulas from [49, 27], it is not hard to see that the above spanning set, subject to constraints (a)-(c), would produce a basis of the arc algebra \( J_\infty(\mathbb{C}[u, v]/\langle u^2, v^3, uv \rangle) \) whose Hilbert series is given by

\[
\sum_{n_1, n_2, m_1 \in \mathbb{N}} q^{(n_1+n_2+m_1)(n_1+n_2)+n_2(n_2+m_1)+m_1^2+m_1+n_1+2n_2}(q)_{n_1}(q)_{n_2}(q)_{m_1}.
\]

On the other hand, the normalized character formula for \( V = L_{Vir}(c(2,5), 0) \otimes L_{Vir}(c(2,5), 0) \) is

\[
\text{ch}[V](q) = \sum_{n_1, n_2 \in \mathbb{N}} q^{n_2+n_2+n_2+n_2}(q)_{n_1}(q)_{n_2}.
\]

Thus we have Hilbert series identities:

\[
HS_q(J_\infty(\mathbb{C}[x, y]/\langle x^2, y^2 \rangle)) = HS_q(J_\infty(\mathbb{C}[u, v]/\langle uv, u^2 + v^2, u^3, v^3 \rangle))
\]

\[
= HS_q(J_\infty(\mathbb{C}[u, v]/\langle u^2, v^3, uv \rangle))
\]

and

\[
\sum_{n_1, n_2, m_1 \in \mathbb{N}} q^{(n_1+n_2+m_1)(n_1+n_2)+n_2(n_2+m_1)+m_1^2+m_1+n_1+2n_2}(q)_{n_1}(q)_{n_2}(q)_{m_1} = \sum_{n_1, n_2 \in \mathbb{N}} q^{n_2+n_2+n_2+n_2}(q)_{n_1}(q)_{n_2}.
\]
CHAPTER 8

Equivariant oriented cohomology of Bott-Samelson varieties

8.1 Equivariant oriented cohomology theory

In this section, we define some notations, and collect some basic notions and facts about equivariant oriented cohomology theory.

Let $G$ be a split semisimple linear algebraic group over a field $k$, with rank $n$. Let $T$ be a split maximal torus of $G$ and $B \subset G$ be a Borel subgroup. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots, and $\Sigma$ be the set of roots. Let $P_i$ be the minimal parabolic subgroup corresponding to the simple root $\alpha_i$. The Weyl group $W$ of $G$ is generated by $\{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$ where $s_{\alpha_i}$ is the reflection corresponding to $\alpha_i$. Note that $W$ can be identified with $N_G(T)/T$. Sometimes we will understand $s_i = s_{\alpha_i}$ as an element in $G$. We denote the group of characters of $T$ by $\Lambda$. For each positive integer $l$, denote $[l] = \{1, 2, \ldots, l\}$.

Let $F$ be a formal group law over the commutative ring $R$. Examples include the additive formal group law $F_a = x + y$ over $\mathbb{Z}$, and the multiplicative formal group law $F = x + y - \beta xy$ over $\mathbb{Z}[\beta, \beta^{-1}]$.

Definition 8.1.1. Let $R[x_\lambda] := R[x_\lambda \mid \lambda \in \Lambda]$ be the power series ring. Let $J_F$ be the closure of the ideal generated by $x_0$ and $x_{\lambda + \mu} - F(x_\lambda, x_\mu)$, $\lambda, \mu \in \Lambda$. We define the formal group algebra $R[\Lambda]_F$ to be the quotient

$$R[\Lambda]_F = R[x_\lambda]/J_F.$$

Indeed, $R[\Lambda]_F$ is non-canonically isomorphic to the formal power series ring with $n$ variables. For simplicity, we denote $S = R[\Lambda]_F$. Note that by definition, $x_{-\lambda}$ is the formal inverse of $x_\lambda$, that is, $F(x_\lambda, x_{-\lambda}) = 0$. Since any formal group law $F$ is always of the form

$$F(x, y) = x + y + a_{11}xy + \text{higher order terms, } a_{11} \in R,$$

so it is not difficult to see that $x_{-\lambda} = -x_\lambda + x_\lambda^2 f(x_\lambda)$ for some $f(t) \in R[[t]]$. Therefore, $\frac{x_\lambda}{x_{-\lambda}}$
is an invertible element in $S$.

**Example 8.1.2.** 1. Let $F_a$ be an additive formal group law, then we have a ring isomorphism

$$R[[\Lambda]]_{F_a} \cong S_R(\Lambda)^\wedge, \quad x_\lambda \mapsto \lambda,$$

where $S_R(\Lambda)$ is the symmetric algebra of $\Lambda$ and the completion is done at the augmentation ideal.

2. Let $R[\Lambda]$ be the group algebra $\left\{ \sum_j a_j e^{\lambda_j} | a_j \in R, \lambda_j \in \Lambda \right\}$. Then we have isomorphism

$$R[[\Lambda]]_{F_m} \cong R[\Lambda]^\wedge, \quad x_\lambda \mapsto \beta^{-1}(1 - e^\lambda),$$

where the completion $^\wedge$ is done at the augmentation ideal.

Throughout this paper, we assume that the root datum of $G$ together with the formal group law $F$ satisfy the regularity condition of [34, Definition 4.4]. For example, this is satisfied if 2 is regular in $R$. Please consult loc.it. for more details. In particular, $x_\alpha$ is regular in $S$, for any root $\alpha$ of $G$. The Weyl group action on $\Lambda$ induces an action of $W$ on $R[[\Lambda]]_F$ by $s_\alpha(x_\lambda) = x_{s_\alpha(\lambda)}$. In particular, we have

**Lemma 8.1.3.** [32, Corollary 3.4] For any $v, w \in W$, any root $\alpha$ of $G$ and $p \in S$, we have

$$\frac{vs_\alpha w(p) - vw(p)}{x_{v(\alpha)}} \in S.$$

**Proof.** According to [32, Corollary 3.4], we know that $s_\alpha w(p) - w(p)$ is uniquely divisable by $x_\alpha$. In other word,

$$\frac{s_\alpha w(p) - w(p)}{x_\alpha} \in S.$$

Then

$$v \left( \frac{s_\alpha w(p) - w(p)}{x_\alpha} \right) = \frac{vs_\alpha w(p) - vw(p)}{x_{v(\alpha)}} \in S.$$

In particular, taking $w = v = e$, we see that $x_\alpha | (p - s_\alpha(p))$. We can then define the
Demazure operator $\Delta_\alpha : S \to S$ by

$$\Delta_\alpha(p) = \frac{p - s_\alpha(p)}{x_\alpha}. \quad (8.1)$$

**Remark 8.1.4.** By direct calculation, we have the following formulas: for $p, q \in S$,

$$s_\alpha \Delta_\alpha(p) = -\Delta_{-\alpha}(p) \quad (8.2)$$

$$\Delta_\alpha(pq) = \Delta_\alpha(p)q + p\Delta_\alpha(q) - \Delta_\alpha(p)\Delta_\alpha(q)x_\alpha. \quad (8.3)$$

Let $h_T$ be an equivariant oriented cohomology theory of Levine-Morel. Roughly speaking, it is an additive contravariant functor $h_T$ from the category of smooth projective $T$-varieties to the category of commutative rings with units, satisfying the following axioms: existence of push-forwards for projective morphisms, homotopy invariance and the projective bundle axioms [33, Section 2]. The Chern classes are defined. Moreover, there exists a formal group law $F$ over $R = h_T(pt)$ such that if $L_1$ and $L_2$ are locally free sheaves of rank one, then

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

It is proved in [33, Theorem 3.3] that

$$S = R[[\Lambda]]_F \cong h_T(pt), \quad x_\lambda \mapsto c_1(L_\lambda),$$

where $L_\lambda$ is the associated line bundle. As an immediate consequence, we see that if the variety $X$ is a finite set of points (with trivial $T$-action), then

$$h_T(X) = F(X; S),$$

where the latter is the set of all maps from $X$ to $S$. It has a $S$-basis $f_x, x \in X$, and is a ring with product defined by

$$f_x \cdot f_y = \delta_{x,y} f_x, \quad \text{and unit } \sum_{x \in X} f_x.$$
By functoriality, if \( p : X \to Y \) is a \( T \)-equivariant map between two finite discrete sets of points on which \( T \) acts trivially, then

\[
p^*(f_y) = \sum_{x \in f^{-1}(y)} f_x, \quad p_*(f_x) = f_{p(x)}.
\] (8.4)

We recall the definition of the characteristic map. Let \( X \) be a \( T \)-variety on which \( B \) acts on the right, and the \( T \) and \( B \) actions are commutative. Moreover, suppose the quotient \( X/B \) exists and \( X \to X/B \) is a \( T \)-equivariant principal bundle. Following [32, Section 10.2], we can define a ring homomorphism

\[
c : S = h_T(pt) \to h_T(X/B), \quad x_\lambda \mapsto c_1(\mathcal{L}_\lambda).
\]

It is called the characteristic map.

Let \( \alpha \) be a simple root with corresponding minimal parabolic subgroup \( P_\alpha \). Consider the fiber product \( X' = X \times^B P_\alpha \), then \( X' \) is a \( T \)-equivariant principal \( P_\alpha \)-bundle over \( X/B \). Denote \( p : X'/B \to X/B \), and there is a zero section

\[
\sigma : X/B \to X'/B, \quad x \mapsto (x, 1).
\]

Similar as [32, Section 10.5], we have

\[
h_T(X'/B) \cong h_T(X/B)[\xi]/(\xi^2 - y\xi), \quad \xi = \sigma_*(1), \quad y = p^*\sigma^*\xi.
\] (8.5)

The following properties can be proved similarly as the non-equivariant version in [32, Section 10].

**Lemma 8.1.5.** Denote \( c : S \to h_T(X/B) \) and \( c' : S \to h_T(X'/B) \). For each \( \lambda \in \Lambda \), denote the associated line bundles on \( X/B \) and \( X'/B \) by \( \mathcal{L}_\lambda \) and \( \mathcal{L}'_\lambda \), respectively.

1. We have \( \sigma^*\xi = c_1(\mathcal{L}_{-\alpha}) = c(x_{-\alpha}) \).
2. \( y = p^*\sigma^*\xi = p^*c(x_{-\alpha}) \).
3. For any \( u \in S \), we have

\[
\sigma^* c'(u) = c(u), \quad c'(u) = p^* c(s_\alpha(u)) + p^* c(\Delta_{-\alpha}(u)) \cdot \xi.
\]

### 8.2 Bott-Samelson Varieties

In this section, we collect some facts about Bott-Samelson varieties.

**Definition 8.2.1.** For any sequence \( I = (i_1, i_2, \ldots, i_l) \) with \( 1 \leq i_j \leq n \), we define the variety \( \hat{X}_I \) to be

\[
\hat{X}_I = \hat{P}_{i_1} \times^B \hat{P}_{i_2} \times^B \cdots \times^B \hat{P}_{i_l} / B.
\]

Here the right \( B \)-action is given by right multiplication on the last coordinate. If \( I = \emptyset \), then we set \( \hat{X}_\emptyset = \text{pt} \). The variety \( \hat{X}_I \) is called the Bott-Samelson variety corresponding to \( I \). It has an obvious \( T \)-action by left multiplication on the first coordinate.

Since \( P_i / B \cong \mathbb{P}^1 \), so we have a sequence of \( \mathbb{P}^1 \)-bundles:

\[
\begin{array}{cccccc}
\hat{X}_I & \xrightarrow{\sigma_l} & \hat{X}_{(i_1, \ldots, i_{l-1})} & \xrightarrow{\sigma_{l-1}} & \cdots & \xrightarrow{\sigma_2} & \hat{X}_{(i_1)} & \xrightarrow{\sigma_1} & \text{pt},
\end{array}
\]

where \( \sigma_i, 1 \leq i \leq l \) are the zero sections. Multiplication of all factors of \( \hat{X}_I \) induces a map

\[
q_I : \hat{X}_I \to G / B.
\]

Denote by \( \pi_i : G / B \to G / P_i \) the canonical map, and denote \( I' = (i_1, \ldots, i_{l-1}) \). We then have the following Cartesian diagram:

\[
\begin{array}{ccc}
\hat{X}_I & \xrightarrow{q_I} & G / B \, . \\
| p \downarrow & & \downarrow \pi_{\alpha_l} \\
\hat{X}_{I'} & \xrightarrow{\pi_{\alpha_l} q_{I'}} & G / P_{\alpha_l}
\end{array}
\]

So we have the base-change formula

\[
q_{I*} p^* = \pi_{\alpha_l}^*(\pi_{\alpha_l} q_{I'})_*.
\]
The operator
\[ \pi^* \alpha \pi : h_T(G/B) \to h_T(G/B) \]
is called the push-pull operator.

Denote by \( c_I : S \to \hat{X}_I \) the characteristic map. The following proposition describes the \( R \)-algebra structure of equivariant oriented cohomology of Bott-Samelson varieties.

**Proposition 8.2.2.** \([32, Section 11.3]\) We have the following presentation
\[ h_T(\hat{X}_I) \cong h_T(pt)[\eta_1, \eta_2, \ldots, \eta_l]/(\{\eta_j^2 - y_j \eta_j | j = 1, \ldots, l\}), \]
where
\[ y_j = p^* c_{(i_1, \ldots, i_{j-1})}(x_{-\alpha_{ij}}), \quad \eta_j = p^* \sigma_j(1), \]
with \( p^* \) the pull-back from \( h_T(\hat{X}_{(i_1, \ldots, i_j)}) \) to \( h_T(\hat{X}_I) \).

For ordinary oriented cohomology, this theorem is proved in \([32]\). The idea of the proof is to apply the projective bundle formula to the sequence of \( \mathbb{P}^1 \)-bundle (8.6). One can check that all the arguments hold in the equivariant setting, which can be used to prove Proposition 8.2.2.

For each subset \( L \subseteq [l] \), define
\[ \eta_L = \prod_{j \in L} \eta_j \in h_T(\hat{X}_I). \]

**Corollary 8.2.3.** The \( S \)-module \( h_T(\hat{X}_I) \) is free with basis \( \{\eta_L | L \in \mathcal{P}_l\} \).

Since in Proposition 8.2.2, the \( y_j \) does not belong to the coefficient ring \( h_T(pt) \), so the presentation of \( h_T(\hat{X}) \) is not satisfactory. To get a polynomial presentation of it, we follow the idea in \([32, Theorem 11.4]\).

**Lemma 8.2.4.** For any sequence \( I = (i_1, \ldots, i_l) \), we have
\[ c_I(u) = \sum_{L \subseteq [l]} \theta_{I,L}(u) \eta_L, \quad u \in S, \]

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where \( \theta_{l,L} = \theta_1 \cdots \theta_l \) with \( \theta_j = \begin{cases} \Delta_{-\alpha_{i_j}}, & \text{if } j \in L, \\ s_{i_j}, & \text{otherwise.} \end{cases} \)

**Proof.** We prove it by induction on \( l \). If \( l = 1 \), from Lemma 8.1.5, we have

\[
\mathbf{c}_{i_1}(u) = p^* \mathbf{c}_{\emptyset}(s_{i_1}(u)) + p^* \mathbf{c}_{\emptyset}(\Delta_{-\alpha_{i_1}}(u)) \cdot \eta_1.
\]

Note that the characteristic map \( \mathbf{c}_{\emptyset} : S \to h_T(pt) \) is the identity map. So it holds.

Now assume the conclusion holds for \( I' := (i_1, \ldots, i_{l-1}) \). Denote the cannonical projection from \( \hat{X}_I \) to \( \hat{X}_{I'} \) by \( p \). By Lemma 8.1.5 we have

\[
\mathbf{c}_{I}(u) = p^* \mathbf{c}_{I'}(s_{i_l}(u)) + p^* \mathbf{c}_{I'}(\Delta_{-\alpha_{i_l}}(u)) \cdot \eta_l
= \sum_{L \subset [l-1]} \theta_{l-1,L}(s_{i_l}(u)) \eta_L + \sum_{L \subset [l-1]} \theta_{l-1,L}(\Delta_{-\alpha_{i_l}}(u)) \eta_L \cdot \eta_l
= \sum_{L \subset [l]} \theta_{l,L}(u) \eta_L.
\]

\[\square\]

**Proposition 8.2.5.** [32, Theorem 11.4] The ring \( h_T(\hat{X}_I) \) is a quotient of the polynomial ring \( S[\eta_1, \eta_2, \ldots, \eta_l] \) modulo the relations

\[
\eta_j^2 = \sum_{L \subset [j-1]} \theta_{j-1,L}(x_{-\alpha_{i_j}}) \eta_L \eta_j, \quad j \in [l].
\]

**Proof.** Denote \( K = (i_1, \ldots, i_{j-1}) \) and \( p : \hat{X}_I \to \hat{X}_K \). By definition of \( y_j \) and Lemma 8.2.4 we have

\[
y_j = p^* \mathbf{c}_K(x_{\alpha_{i_j}}) = p^* \left( \sum_{L \subset [j-1]} \theta_{j-1,L}(x_{-\alpha_{i_j}}) \eta_L \right) = \sum_{L \subset [j-1]} \theta_{j-1,L}(x_{-\alpha_{i_j}}) \eta_L.
\]

The statement then follows from the fact that \( \eta_j^2 = y_j \eta_j \). \[\square\]

**Example 8.2.6.** For \( SL(4) \) whose simple roots are \( \alpha_1, \alpha_2, \alpha_3 \), let us consider Bott-Salmelson \( \hat{X}_I = P_{\alpha_1} \times B P_{\alpha_2} \times B P_{\alpha_3}/B \). Then \( h_T(\hat{X}_I) \) is a polynomial algebra generated by \( \eta_1, \eta_2, \eta_3 \) with following quotient relations:

\[
\eta_1^2 = x_{-\alpha_1} \eta_1.
\]
\begin{align*}
\eta_2^2 &= x_{-a_1-a_2}\eta_1 + \frac{x_{-a_2} - x_{a_1-a_2}}{x_{-a_1}}\eta_1\eta_2, \\
\eta_3^2 &= x_{a_1-a_2-a_3}\eta_3 + \frac{x_{a_3-a_2} - x_{2a_1-a_2-a_3}}{x_{-a_1}}\eta_1\eta_3 + \frac{x_{a_3} - x_{a_1+a_2-a_3}}{x_{-a_1-a_2}}\eta_2\eta_3 \\
&+ \left(\frac{x_{a_3-a_2-a_3}}{x_{a_2x_{-a_1}}} - \frac{x_{a_3} - x_{a_2-a_1-a_3}}{x_{a_1-a_2x_{-a_1}}}\right)\eta_1\eta_2\eta_3.
\end{align*}

Let us consider some geometry information of \(\hat{X}\), and the \(T\)-fixed points. We fix some notations first. For any \(L \subseteq [l]\), define

\[(\hat{X}_I)_L = \{g_1, g_2, \ldots, g_l \in \hat{X}_I \mid g_j \in B \text{ if } j \notin L, \text{ and } g_i \notin B \text{ if } i \in L\} \subset \hat{X}_I,\]

and

\[v^L_j = \prod_{k \in L \cap [j]} s_{ik}, \quad v^L = v^L_1 = \prod_{k \in L} s_{ik}.\]

The following lemma will be used in the proof of Theorem 8.3.6.

**Lemma 8.2.7.** If \(I = (i_1, ..., i_l)\) is a sequence such that \(i_j\) are all distinct, then for any \(L \subseteq [l]\) and \(j \in L^c\), \(v^L_{j-1}(x_{-a_{ij}})\) are all distinct.

**Proof.** Suppose \(j_1, j_2 \in L^c\) and \(j_1 < j_2\). Then \(L \cap [j_1] \subseteq L \cap [j_2]\). There are two cases.

Case 1: \(L \cap [j_1] = L \cap [j_2]\). Then

\[v^L_{j_1-1}(x_{-a_{j_1}}) = \prod_{k \in L \cap [j_1]} s_{ik}(x_{-a_{j_1}}),\]

and

\[v^L_{j_2-1}(x_{-a_{j_2}}) = \prod_{k \in L \cap [j_2]} s_{ik}(x_{-a_{j_2}}) = \prod_{k \in L \cap [j_1]} s_{ik}(x_{-a_{j_2}}).\]

They are not equal since \(\alpha_{j_1} \neq \alpha_{j_2}\).

Case 2. \(L \cap [j_1] \not\subseteq L \cap [j_2]\). Denote \(M = (L \cap [j_2]) \setminus (L \cap [j_1])\). Then

\[v^L_{j_1-1}(x_{-a_{j_1}}) = \prod_{k \in L \cap [j_1]} s_{ik}(x_{-a_{j_1}}), \quad v^L_{j_2-1}(x_{-a_{j_2}}) = \prod_{k \in L \cap [j_1]} s_{ik}(\prod_{k' \in M} s_{ik'}(x_{-a_{j_2}})).\]
By definition of the Weyl group action,

\[ \prod_{k' \in M} s_{i_{k'}^j}(x - \alpha_{i_{k'}^j}) = x \prod_{k' \in M} s_{i_{k'}^j}(-\alpha_{i_{k'}^j}) = x - \alpha_{i_{k'}^j} + \sum_{k' \in M} c_{k'} \alpha_{i_{k'}^j}, \quad c_{k'} \in \mathbb{Z}, \]

which is different from \( x - \alpha_{i_{k'}^j1} \), since the set \( \{-\alpha_{i_{k'}^j}, -\alpha_{i_{k'}^j1}, +\alpha_{i_{k'}^j}, |k' \in M\} \) is linearly independent. Thus \( v_{j_1-1}^L(x - \alpha_{i_{j_1}^j}) \) and \( v_{j_2-1}^L(x - \alpha_{i_{j_2}^j}) \) are not equal to each other. \( \Box \)

The following lemma recalled from [106, Proposition 2.6], provides some geometric information of the Bott-Samelson variety, which is useful for our computation.

**Lemma 8.2.8.** 1. The set \( \hat{X}_T^L \) of \( T \)-fixed points in \( \hat{X}_1 \), consists of \( 2^l \) points

\[ [g_1, g_2, \ldots, g_l] \]

where \( g_j \in \{e, s_{ij}\} \). Here we think of \( s_{ij} \) as in \( W \rightarrow N_G(T)/T \) and pick a preimage for \( s_{ij} \) in \( N_G(T) \subset G \). Consequently, we have bijection of sets from the power set \( \mathcal{P} := \mathcal{P}(\{l\}) \) to \( \hat{X}_1^T \),

\[ L \mapsto pt_L := [g_1, \ldots, g_l], \quad g_j = \begin{cases} s_{ij}, & \text{if } j \in L, \\ e, & \text{if } j \not\in L. \end{cases} \]

2. The set \( (\hat{X}_1)_L \) is a \( T \)-orbit containing the fixed point \( pt_L \), and isomorphic to the affine space of dimension \( |L| \). The variety \( \hat{X}_1 \) has a decomposition \( \bigsqcup_{L \in E} (\hat{X}_1)_L \).

3. Suppose \( L, L' \subset [l] \). then \( pt_L \in \overline{(\hat{X}_1)_{L'}} \) if and only if \( L \subset L' \). The weights of the \( T \)-action on the tangent space of \( (\hat{X}_1)_{L'} \) at \( pt_L \) are

\[ \{-v_j^L(\alpha_{ij})|j \in L'\}. \]

**Example 8.2.9.** For the \( A_2 \)-case, consider \( \hat{X}_{(1,2)} = P_1 \times^B P_2/B \). There are four \( T \)-fixed points, denoted by \{00, 01, 10, 11\}, corresponding to \{\{e, e\}, \{e, s_2\}, \{s_1, e\}, \{s_1, s_2\}\}, or \{0, \{2\}, \{1\}, \{1, 2\}\} as subsets of \{2\}. The weights of the tangent spaces of \( \hat{X}_{(1,2)} \) at the four points are:

- 00: \( -\alpha_1, -\alpha_2 \)
- 01: \( -\alpha_1, \alpha_2 \)
- 10: \( \alpha_1, -\alpha_1 - \alpha_2 \)
- 11: \( \alpha_1, \alpha_1 + \alpha_2 \).
We denote the set of functions on $\mathcal{E}_I = \hat{X}_I^T$ with values in $S$ by $F(\mathcal{E}_I; S)$. It is a free $S$-module with basis $f_L, L \in \mathcal{E}_I$ defined by $f_L(L') = \delta_{L,L'}$, and have a ring structure given by

$$f_L \cdot f_{L'} = \delta_{L,L'} f_L.$$ 

Moreover, we have

$$h_T((\hat{X}_I)^T) \cong F(\mathcal{E}_I; S),$$

where the total Chern class of the tangent space at the fixed point $L, \text{pt}_L$, corresponds to the basis element $f_L$ up to a scalar.

Denote $j^I : \hat{X}_I^T \to \hat{X}_I$. For each $L \subset [l]$, denote by $j_L^I$ the embedding of $\text{pt}_L$ into $\hat{X}_I$. Sometimes we will drop the superscript $I$ for simplicity. Then

$$j^*(f) = \sum_{L \subset [l]} j^I_L(f) f_L, \quad f \in h_T(\hat{X}_I).$$

Denote

$$x_{I,L} = \prod_{1 \leq j \leq l} v_j^L(x_{-\alpha_{ij}}). \quad (8.8)$$

We have

**Lemma 8.2.10.** For any $L \subset [l]$, we have $j^* j_* (f_L) = x_{I,L} f_L$.

**Proof.** This follows from [33 Section 2.A8], and Lemma 8.2.8 concerning the weights of the tangent space of $\hat{X}_I$ at the point $L$. \hfill $\square$

**Example 8.2.11.** Following Example 8.2.9 with $I = (\alpha_1, \alpha_2)$, we have

$$x_{I,00} = x_{-\alpha_1} x_{-\alpha_2}, \quad x_{I,10} = x_{\alpha_1} x_{-\alpha_1 - \alpha_2}, \quad x_{I,01} = x_{-\alpha_1} x_{\alpha_2}, \quad x_{I,11} = x_{\alpha_1} x_{\alpha_1 + \alpha_2}.$$ 

8.3 Restriction to $T$-fixed points

In this section, we compute the restriction formula of the $\eta_L$ basis. We first compute the restriction formula of the image of the characteristic map.
Lemma 8.3.1. Let $I$ be a sequence of length $l$, and $c_I : S \to \hat{X}_I$ be the characteristic map, then

$$j^*c_I(u) = \sum_{L \subset [l]} v^L(u) f_L.$$  

Proof. We prove it by induction on the length $l$ of $I$. If $I = (i_1)$, then there are two points in $(P_{i_1}/B)^T$, corresponding to $e$ and $s_1$ (or $\emptyset$ and $[1]$ as subsets of $[1]$). Then from [33 Section 10] we have

$$j^*c_I(u) = uf_e + s_{i_1}(u)f_{s_{i_1}}.$$  

So the conclusion follows.

Now assume it holds for all sequences of length $\leq l - 1$, and assume $I = (i_1, \ldots, i_l)$. Denote $I' = (i_1, \ldots, i_{l-1})$ and $\sigma : \hat{X}_{I'} \to \hat{X}_I$ the zero section. By induction assumption, for each $L' \subset [l-1]$, we have

$$(j_{I'}^L)^*c_{I'}(u) = v_{l-1}^{L'}(u). \tag{8.9}$$  

Concerning $L \subset [l]$, we have two cases:

Case 1: $l \in L$. In this case, $pt_L \notin \sigma(\hat{X}_{I'})$, so

$$(j_L^I)^* \circ \sigma_* = 0. \tag{8.10}$$  

Moreover, we have the following commutative diagram

$$
\begin{array}{ccc}
pt & \xrightarrow{j_L^I} & \hat{X}_I \\
j_{L\setminus\{l\}} & \searrow & \\
 & \xrightarrow{p} & \hat{X}_{I'},
\end{array}
$$  

that is, $p \circ j_L^I = j_{L\setminus\{l\}}^I$, so

$$(j_L^I)^* \circ p^* = (j_{L\setminus\{l\}}^I)^*. \tag{8.11}$$
Denote \( \xi = \sigma_{\ast}(1) \), then by Lemma 8.1.5, we have

\[
(j^I_L)^\ast \circ c_I(u) = (j^I_L)^\ast[p^\ast c_{I'}(s_i(u)) + p^\ast c_{I'}(\Delta_{-\alpha_i}(u)) \cdot \xi]
\]
\[
= (j^I_L)^\ast p^\ast c_{I'}(s_i(u)) + (j^I_L)^\ast p^\ast c_{I'}(\Delta_{-\alpha_i}(u)) \cdot (j^I_L)^\ast(\sigma_{\ast}(1))
\]
\[
\overset{\sharp_1}{=} (j^I_{L\setminus\{l\}})^\ast c_{I'}(s_i(u))
\]
\[
\overset{\sharp_2}{=} v^L_{l-1} \circ s_i(u) = v^L_l(u).
\]

Here the identity \( \sharp_1 \) follows from (8.10) and (8.11), and \( \sharp_2 \) follows from (8.9).

Case 2: \( l \not\in L \). In this case, we can view \( L \subset [l - 1] \), so we have commutative diagrams:

\[
\begin{array}{ccc}
\text{pt} & \xrightarrow{j^I_L} & \hat{X}_I, \\
\downarrow & & \downarrow \\
\hat{X}_I' & \xrightarrow{p} & \hat{X}_I',
\end{array}
\]

\[
\begin{array}{ccc}
\text{pt} & \xrightarrow{j^I_L'} & \hat{X}_I, \\
\downarrow & & \downarrow \\
\hat{X}_I' & \xrightarrow{j^I_L'} & \hat{X}_I',
\end{array}
\]

so \( p \circ j^I_L = j^I_{L'} \) and \( \sigma \circ j^I_L = j^I_L \). The latter implies that

\[
(j^I_L)^\ast(\sigma_{\ast}(1)) = (j^I_L)^\ast(\sigma_{\ast}(1)) \overset{\text{Lem} 8.1.5}{=} (j^I_L)^\ast c_{I'}(x_{-\alpha_i}).
\]

Therefore,

\[
(j^I_L)^\ast(c_I(u)) = (j^I_L)^\ast[p^\ast c_{I'}(s_i(u)) + p^\ast c_{I'}(\Delta_{-\alpha_i}(u)) \cdot \xi]
\]
\[
= (j^I_L)^\ast p^\ast c_{I'}(s_i(u)) + (j^I_L)^\ast p^\ast c_{I'}(\Delta_{-\alpha_i}(u)) \cdot (j^I_L)^\ast(\sigma_{\ast}(1))
\]
\[
= (j^I_L')^\ast c_{I'}(s_i(u)) + (j^I_L')^\ast c_{I'}(\Delta_{-\alpha_i}(u)) \cdot (j^I_L')^\ast c_{I'}(x_{-\alpha_i})
\]
\[
= (j^I_L')^\ast c_{I'}(s_i(u)) + \frac{u - s_i(u)}{x_{-\alpha_i}} x_{-\alpha_i}
\]
\[
= (j^I_L')^\ast c_{I'}(u)
\]
\[
= v^L_{l-1}(u) = v^L_l(u).
\]

The proof is finished.

Before computing the restriction formula of \( \eta_L \), we first consider an example.

**Example 8.3.2.** Consider the case of \( A_2 \). Let \( \{\alpha_1, \alpha_2\} \) be the set of simple roots. We
consider the Bott-Samelson variety $\tilde{X}_I = P_1 \times^B P_2 / B$ for $I = (1, 2)$. Following Example 8.2.9, there are four torus-fixed points, denoted by $P_2 = \{00, 01, 10, 11\}$. Similarly, denote $(P_1 / B)^T$ by $P_1 = \{0, 1\}$. Denote $j^I : \mathcal{E}_I \hookrightarrow \tilde{X}_I$ and $j^1 : P_1 \hookrightarrow (P_1 / B)^T$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
P_1 \times^B P_2 / B & \xleftarrow{j^I} & P_2 = \{00, 01, 10, 11\} \\
\sigma_2 \downarrow p_2 & & \downarrow p_2' \\
P_1 / B & \xleftarrow{j^1} & P_1 = \{0, 1\} \\
\sigma_1 \downarrow p_1 & & \downarrow p_1 \\
pt & & pt
\end{array}
\]

Here $\sigma_i$ are the zero sections, $p'_2$ is induced by the projection map $p_2$, so it maps 00, 01 to 0, and 10 and 11 to 1. Moreover, by definition, $j^1_0 = \sigma_1$ and $\sigma_2 \circ j^1_i = j^I_i$ for $i = 0, 1$. We have

\[\eta_1 = p'_2 \sigma_1, \quad \eta_2 = \sigma_2, \]

and

\[h_T((\tilde{X}_I)^T) = S\{f_{00}, f_{01}, f_{10}, f_{11}\}, \quad h_T((P_1 / B)^T) = S\{f_0, f_1\}.\]

Denote $c_1 : S \rightarrow h_T(P_1 / B)$.

First of all, from the definition of $p'_2$ and \([8.4]\), we know

\[p'_2^*(f_0) = f_{00} + f_{01}, \quad p'_2^*(f_1) = f_{10} + f_{11}.\]

Moreover, since $j^1_0$ coincides with $\sigma_1$ and $j^1_1(pt) \not\in \sigma_1(pt)$, so $(j^1)^* \sigma_1 = 0$ and

\[(j^1)^* \sigma_1(1) = (j^1_0)^* \sigma_1(1) = \sigma_1^* \sigma_1(1) = x_{-\alpha_1} f_0,
\]

where the last identity follows from the fact that the tangent space of $P_1 / B$ at 0 has weight $-\alpha_1$. Hence,

\[(j^1)^*(\eta_1) = (j^1)^* p'_2 \sigma_1(1) = p'_2^*(j^1)^* \sigma_1(1) = p'_2^*(x_{-\alpha_1} f_0) = x_{-\alpha_1} (f_{00} + f_{01}). \quad (8.13)\]
We then compute \((j^I)^*(\eta_2)\), by using the identity

\[
(j^I)^*(\eta_2) = \sum_{x \in \mathcal{P}_3} (j^{I_x}_\ast)^*(\eta_2) f_x.
\]

Since \(01, 11 \notin \sigma_2(P_1/B)\), we have \((j^I_{01})^*(\eta_2) = (j^I_{11})^*(\eta_2) = 0\). From Lemma 8.1.5 we know that \(\sigma_2^*\sigma_{2*}(1) = c_1(x_{-\alpha_2})\). So

\[
(j^I_{00})^*(\eta_2) = (j^I_{00})^*\sigma_{2*}(1) = (j^I_{00})^*\sigma_2^*\sigma_{2*}(1) = (j^I_{00})^*(c_1(x_{-\alpha_2})) \equiv x_{-\alpha_2},
\]

where \(\equiv\) follows from Lemma 8.3.1. Similarly, from \(j^I_{10} = \sigma_2 \circ j^I_1\), we have

\[
(j^I_{10})^*(\eta_2) = (j^I_{10})^*\sigma_{2*}(1) = (j^I_{10})^*\sigma_2^*\sigma_{2*}(1) = (j^I_{10})^*(c_1(x_{-\alpha_2})) = s_1(x_{-\alpha_2}) = x_{-\alpha_1-\alpha_2}.
\]

Therefore,

\[
(j^I)^*(\eta_2) = x_{-\alpha_2} f_{00} + x_{-\alpha_1-\alpha_2} f_{10}. \tag{8.14}
\]

Now we compute the restriction formula of \(\eta_L\).

**Theorem 8.3.3.** Let \(I\) be a sequence of length \(l\). For any two subsets \(L, M \subset [l]\) denote \(L^c = [l] \setminus L\) and

\[
a_{L,M} = \prod_{k \in L} \psi_{k-1}^M(x_{-\alpha_{i_k}}).
\]

Then

\[
j^*(\eta_L) = \sum_{M \subset L^c} a_{L,M} f_M.
\]

**Proof.** We first consider \(L = \{k\}\), and prove the following identity

\[
j^*(\eta_k) = \sum_{M \subset L^c} \psi_{k-1}^M(x_{-\alpha_{i_k}}) f_M.
\]

Denote \(I_k = (i_1, \ldots, i_k)\) and similarly denote \(I_{k-1}\). Firstly, we compute \((j^I_{\{k\}})^*\sigma_k^*(1)\) for each
\( M \subset [k], \) with \( \sigma_k : \hat{X}_{I_{k-1}} \to \hat{X}_I \). If \( k \in M \), then the point \( j_M^I(\text{pt}) \not\in \sigma_k(\hat{X}_{I_{k-1}}) \), so

\[(j_M^I)^* \sigma_k(1) = 0.\]

If \( k \not\in M \), then \( M \subset [k - 1] \), \( v^M_k = v^M_{k-1} \), and we have the following commutative diagram

\[
\begin{array}{ccc}
\text{pt} & \xrightarrow{j_M^I} & \hat{X}_I \\
\downarrow & & \downarrow \sigma_k \\
\hat{X}_{I_{k-1}} & \xrightarrow{j_{M}^{I_{k-1}}} & \\
\end{array}
\]

Therefore

\[(j_M^I)^* \sigma_k(1) = (j_{M}^{I_{k-1}})^* \sigma_k(1) \overset{\text{Lem. 8.3.2}}{=} (j_M^I)^* c_{I_{k-1}}(x - \alpha_{I_k}) \overset{\text{Lem. 8.3.1}}{=} v^M_{k-1}(x - \alpha_{I_k}). \tag{8.15}\]

Here \( c_{I_{k-1}} \) is the characteristic map on \( \hat{X}_{I_{k-1}} \). Then we consider the following commutative diagram

\[
\begin{array}{ccc}
(\hat{X}_I)^T & \xrightarrow{j'} & \hat{X}_I \\
\downarrow & & \downarrow p \\
(\hat{X}_I)^T & \xrightarrow{j^I} & \hat{X}_I \\
\end{array}
\]

We have

\[
(j')^*(\eta_k) = (j')^* p^*(\sigma_k(1)) = p^*(j_M^I)^* \sigma_k(1) \\
= p^* \left[ \sum_{M \subset [k]} (j_M^I)^* \sigma_k(1) f_M \right] \\
\overset{\text{8.15}}{=} p^* \left[ \sum_{M \subset [k-1]} v^M_{k-1}(x - \alpha_{I_k}) f_M \right] \\
\overset{\text{8.4}}{=} \sum_{M \subset [k-1]} v^M_{k-1}(x - \alpha_{I_k}) \sum_{M' \subset \{k+1, \ldots, l\}} f_{M \cup M'} \\
= \sum_{M \subset ([l] \setminus \{k\})} v^M_{k-1}(x - \alpha_{I_k}) f_M.
\]

So the case \( L = \{k\} \) is proved.
Now for a general subset $L \subset [l]$, we have

$$j^*(\eta_L) = \prod_{k \in L} j^*(\eta_k) = \prod_{k \in L} \sum_{M \subset ([l] \setminus \{k\})} v_{k-1}^M (x_{-\alpha_{ik}}) f_M = \sum_{M \subset L^c} \prod_{k \in L} v_{k-1}^M (x_{-\alpha_{ik}}) f_M.$$

For $I$ of length $l$ and $L \subset [l]$, note the difference between

$$a_{[l],L} = \prod_{1 \leq k \leq l} v_{k-1}^L (x_{-\alpha_{ik}}), \quad x_{I,L} = \prod_{1 \leq k \leq l} v_{k}^L (x_{-\alpha_{ik}}).$$

They are only related when $L = [l]$, in which case we have

$$a_{[l],[l]} = \prod_{1 \leq k \leq l} v_{k-1}^L (x_{-\alpha_{ik}}) = \prod_{1 \leq k \leq l} v_{k}^L (x_{-\alpha_{ik}}), \quad x_{I,[l]} = \prod_{1 \leq k \leq l} v_{k}^L (x_{-\alpha_{ik}}).$$

**Corollary 8.3.4.** The map $j^*: h_T(\hat{X}_I) \to h_T(\hat{X}_I^T)$ is an injection.

**Proof.** It follows from Theorem 8.3.3 that

$$j^*(\eta_L) = \sum_{M \subset L^c} a_{L,M} f_M.$$

So if we order $\{j^*(\eta_L)|L \subset [l]\}, \{f_M|M \subset [l]\}$ by inclusion of subsets $L' \subset L$, then the transition matrix from $f_M$ to $j^*(\eta_L)$ will be skew-triangular. Moreover, the entries on the skew-diagonal will be

$$a_{L,L^c} = \prod_{k \in L} v_{k-1}^L (x_{-\alpha_{ik}}),$$

which is regular in $S$. Therefore, $j^*$ is injective.

**Theorem 8.3.5.** Let $I$ be a sequence of length $l$. Then

$$\text{im } j^* \subset \left\{ \sum_{L \subset [l]} a_{L,f_L} \frac{a_{L_1} - a_{L_2}}{v_{k-1}^{L_1} (x_{-\alpha_{ik}})} \in S, \forall L_1, L_2 \text{ such that } L_1 = L_2 \sqcup \{k\} \right\}.$$

Here $\sqcup$ denotes the disjoint union.

**Proof.** Denote the right hand side by $\Psi$. We first show that $\Psi$ is a ring. It is clearly additively
closed. For the multiplication, consider

\[ f = \sum_{L \subset [l]} a_L f_L, \quad g = \sum_{L' \subset [l]} b_{L'} f_{L'} \in \Psi, \]

then

\[ fg = \sum_{L, L' \subset [l]} \delta_{L, L'} a_L b_{L'} f_L = \sum_{L \subset [l]} a_L b_L f_L. \]

For any \( L_1, L_2 \) such that \( L_1 = L_2 \sqcup \{k\} \), by definition we have \( v_{k-1}^{L_1} = v_{k-1}^{L_2} \), so \( v_{k-1}^{L_1} (x_{-\alpha_k}) = v_{k-1}^{L_2} (x_{-\alpha_k}) \). Therefore,

\[ a_{L_1} b_{L_1} - a_{L_2} b_{L_2} = (a_{L_1} - a_{L_2}) b_{L_1} - (b_{L_2} - b_{L_1}) a_{L_2}, \]

is divisible by \( v_{k-1}^{L_1} (x_{-\alpha_k}) \). We have \( fg \in \Psi \).

We then show that \( \text{im} j^* \subset \Psi \). Since \( j^* \) is multiplicative, it suffices to show

\[ j^* (\eta_m) = \sum_{L \subset [l] \setminus \{m\}} v_{m-1}^{L} (x_{-\alpha_m}) f_L \]

belongs to the RHS. Suppose \( L_1 = L_2 \sqcup \{k\} \). Clearly \( k \neq m \). If \( k > m \), then by definition we have \( v_{m-1}^{L_1} = v_{m-1}^{L_2} \). Thus \( v_{m-1}^{L_1} (x_{-\alpha_m}) = v_{m-1}^{L_2} (x_{-\alpha_m}) \), which implies that \( j^* (\eta_m) \in \Psi \).

If \( k < m \), denote

\[ L_1 \cap [m-1] = \{j_1 < j_2 < \cdots < j_t < k < j_{t+1} < \cdots < j_s\}, \]

\[ L_2 \cap [m-1] = \{j_1 < j_2 < \cdots < j_t < \hat{k} < j_{t+1} < \cdots < j_s\}, \]

(in other words, \( k \) is omitted in \( L_2 \)). Then

\[ v_{m-1}^{L_1} (x_{-\alpha_m}) - v_{m-1}^{L_2} (x_{-\alpha_m}) = s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_s}} (x_{-\alpha_m}) - s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_t}} s_{i_{j_{t+1}}} \cdots s_{i_{j_s}} (x_{-\alpha_m}) \]

\[ = v_{k-1}^{L_1} s_{i_k} (s_{i_{j_{t+1}}} \cdots s_{i_{j_s}})(x_{-\alpha_m}) - v_{k-1}^{L_1} (s_{i_{j_{t+1}}} \cdots s_{i_{j_s}})(x_{-\alpha_m}). \]

According to Lemma 8.1.3, this is divisible by \( v_{k-1}^{L_1} (x_{-\alpha_k}) \).
We can strengthen the conclusion in some cases. The proof essentially uses the fact that the transition matrix from \( f_L, L \subset \llbracket l \rrbracket \) to \( j^*(\eta_M), M \subset \llbracket l \rrbracket \) is skew-triangular, following from Theorem 8.3.3.

**Theorem 8.3.6.** If \( I = (i_1, \ldots, i_l) \) with \( i_j \) all distinct, then we have equality in Theorem 8.3.5.

**Proof.** It suffices to show that \( \Psi \subset \text{im} j^* \). Suppose

\[
f = \sum_{L \subset \llbracket l \rrbracket} a_L f_L \in \Psi, \quad \text{with } a_L = 0 \text{ unless } L = \emptyset,
\]

then for any \( k \in \llbracket l \rrbracket \), \( a_\emptyset = a_\emptyset - a_{\{k\}} \) is divisible by \( v_{k-1}^0(x - \alpha_{i_k}) = x - \alpha_{i_k} \). Since \( x_{\alpha_{i_j}}, 1 \leq j \leq l \) are all distinct, by [35, Lemma 2.7], we see that \( a_\emptyset \) is divisible by \( \prod_{k \in \llbracket l \rrbracket} x - \alpha_{i_k} \). Note that by Theorem 8.3.3,

\[
j^*(\eta_{\llbracket l \rrbracket}) = \prod_{k \in \llbracket l \rrbracket} x - \alpha_{i_k} f_\emptyset,
\]

so \( f \) is a multiple of \( j^*(\eta_{\llbracket l \rrbracket}) \), i.e., \( f \in \text{im} j^* \).

Assume the conclusion holds for any \( f \) that can be written as a linear combination of \( f_L \) with \( |L| \leq t - 1 \). Now let

\[
f = \sum_{L \subset \llbracket l \rrbracket} a_L f_L \in \Psi, \quad \text{with } a_L = 0 \text{ unless } |L| \leq t.
\]

Let \( L_0 \) be a subset of \( \llbracket l \rrbracket \) of cardinality \( t \). For any \( k \in L_0^c \), we have \( a_{L_0 \cup \{k\}} = 0 \), so \( v_{k-1}^{L_0}(x - \alpha_{i_k})|a_{L_0} \). Now from Theorem 8.3.3, we know

\[
j^*(\eta_{L_0^c}) = \sum_{M \subset L_0} a_{L_0^c, M} f_M, \quad a_{L_0^c, L_0} = \prod_{j \in L_0^c} v_{j-1}^{L_0}(x - \alpha_{i_j}).\]

By Lemma 8.2.7, we have that \( v_{j-1}^{L_0}(x - \alpha_{i_j}) \) are all distinct for \( j \in L_0^c \). By [35, Lemma 2.7], we know that \( a_{L_0^c, L_0}|a_{L_0} \). Write \( a_{L_0} = c_{L_0} a_{L_0^c, L_0} \) with \( c_{L_0} \in S \). Therefore,

\[
f' := f - \sum_{|L_0| = t} c_{L_0} j^*(\eta_{L_0^c}) = \sum_{|L| < t} a'_L f_L,
\]

By induction hypothesis, \( f' \in \text{im} j^* \). Therefore, \( f \in \text{im} j^* \). The proof is finished. \( \square \)
Remark 8.3.7. Let $T_i$ be the subtorus of rank 1 corresponding to $\alpha_i$, i.e., $T_i = (\ker \alpha_i)^{\circ}$ where $\alpha_i$ is viewed as a character $T \to k^*$. If $I = (i_1, ..., i_l)$ is a sequence such that $i_j$ are all distinct, it is not difficult to see that for any $1 \leq k \leq l$,

$$\hat{X}_T^{T_{i_k}} = \{[g_1, ..., g_l] | g_j B \in \{B, s_{i_j} B\} \forall j \neq k\},$$

and

$$\hat{X}_{T'}^{T'} = \{[g_1, ..., g_l] | g_j B \in \{B, s_{i_j} B\} \forall j\}$$

if $T'$ is any subtorus of corank 1 different from $T_{i_j}, j = 1, ..., l$. In other words, for any subtorus of corank 1, the irreducible component of the invariant subvariety has dimension at most one. This corresponds to the so-called Goresky-Kottwitz-MacPherson (GKM) condition. In other words, in this case, the Bott-Samelson variety is a GKM space. This corresponds to the conclusion of Theorem 8.3.6.

On the other hand, if $P_{i_j}$ are not distinct, the space $\hat{X}_I$ will not be GKM. For instance, if $I = (1, 2, 1)$, the $T_1$-fixed subspace contains the following subset

$$\{[g_1, e, g'_1]| g_i, g'_i \in P_1\},$$

so the dimension condition is not satisfied. For more detailed discussion of GKM spaces, see [57, 58].

8.4 Push-forward to cohomology of flag varieties

In this section, we compute the push-forward of the basis $\eta_L$ along the canonical map $q_I : \hat{X}_I \to G/B$, which generalizes the computation of Bott-Samelson classes in [33].

Recall that the set of $T$-fixed points of $G/B$ is in bijection to $W$, so we have

$$h_T((G/B)^T) \cong \bigoplus_{w \in W} S.$$ 

Denote by $f_w \in h_T(W)$ the basis element corresponding to $w \in W$. Denote $i : W \to G/B$ to be the embedding, and denote $pt_e = (i|_e)_*(1) \in h_T(G/B)$. Let $\pi_i : G/B \to G/P_i$ be the canonical map, and denote $A_i = \pi_i^* \circ \pi_{i*} : h_T(G/B) \to h_T(G/B)$. For any sequence $I$, denote
by $I^{\text{rev}}$ the sequence obtained by reversing $I$.

**Proposition 8.4.1.** (Lemma 7.6) For any sequence $I$, we have $q_{I^*}(1) = A_{I^{\text{rev}}(\text{pt}_e)}$.

The following is an easy generalization of Proposition 8.4.1.

**Theorem 8.4.2.** Let $I$ be a sequence of length $l$ and $1 \leq k \leq l$. Denote by $I_k$ the subsequence of $I$ obtained by removing the $k$-th term from $I$. Then $(q_I)_*(\eta_k) = A_{I_k^{\text{rev}}(1)}$.

**Proof.** Denote the sequence by $I = (i_1, ..., i_l)$. For any $k \leq l$, denote

$$P_1 \times^B P_2 \times^B \ldots \times^B P_k/B \xrightarrow{q_k} G/B$$

$$P_1 \times^B P_2 \times^B \ldots \times^B P_{k-1}/B \xrightarrow{q_{k-1}} G/B.$$

Note that $q_I = q_l$ and $q_k \circ \sigma_k = q_{k-1}$. Denote by $p$ the composition of $p_{k+1}, \ldots, p_l$. By using the base change formula from diagram (8.7), we have

$$q_{I^*}(\eta_k) = q_{l*} p^* (\sigma_{k*}(1))$$

$$= q_{l*} p_{l-1}^* \cdots p_{k+1}^* \sigma_k(1)$$

$$= (\pi_{\alpha_{i_1}} \cdots \pi_{\alpha_{i_{l-k}}}) (\pi_{\alpha_{i_{l-k+1}}} \cdots \pi_{\alpha_{i_{l-1}}}) q_{l-1*} p_{l-1}^* \cdots p_{k+1}^* \sigma_k(1)$$

$$= A_{i_1} A_{i_{l-1}} \cdots A_{i_{k+1}} q_{l-1*}(1)$$

$$= A_{i_1} A_{i_{l-1}} \cdots A_{i_{k+1}} A_{i_{k-1}} \cdots A_{i_1}(\text{pt}_e) = A_{I_k^{\text{rev}}(\text{pt}_e)}.$$

To compute $q_{I^*}(\eta_L)$ for general $L \subset [l]$, we need the following lemma.

**Lemma 8.4.3.** For any $L \subset [l]$, we have

$$\eta_L = \sum_{L_1 \subset [l]} \frac{a_{L,L_1}}{x_{I,L_1}} j_*(f_{L_1}),$$

where $a_{L,L_1}$ are defined in Theorem 8.3.3. Note that the coefficients in this formula belong to $Q := S[\frac{1}{x_\alpha} | \alpha \in \Sigma]$. 

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Proof. By Corollary 8.3.4 we know that $j^*(\eta_L)$ becomes a basis of $Q \otimes_S h_T(W)$. In other words, $j^*$ induces an isomorphism

$$j^*: Q \otimes_S h_T(\hat{X}_I) \to Q \otimes_S h_T(W).$$

Moreover, by Lemma 8.2.10 we know that $j_*$ is the inverse of the $j^*$ (after tensoring with $Q$). Therefore, $j_*(f_L)$ is a $Q$-basis of $Q \otimes_S h_T(\hat{X}_I)$. Denote

$$\eta_L = \sum_{L_1 \subseteq [l]} b_{L,L_1}j_*(f_{L_1}), \quad b_{L,L_1} \in Q.$$

Then by Theorem 8.3.3 and Lemma 8.2.10 we have

$$\sum_{L_2 \subseteq L^c} a_{L,L_2}f_{L_2} = j^*(\eta_L) = \sum_{L_1 \subseteq [l]} b_{L,L_1}j_*j^*(f_{L_1}) = \sum_{L_1 \subseteq [l]} b_{L,L_1}x_{I,L_1}f_{L_1}.$$ 

Therefore, $b_{L,L_1} = \frac{a_{L,L_1}}{x_{I,L_1}}$. 

The following is the main result of this section, which computes the push-forward of $\eta_L$ to the cohomology of $G/B$.

**Theorem 8.4.4.** For any sequence $I = (i_1, \ldots, i_l)$, we have

$$i^*\eta_L = \sum_{L_1 \subseteq L^c} \frac{a_{L,L_1}}{x_{I,L_1}}x^\Pi(f_{v_{L_1}}), \quad x^\Pi := \prod_{\alpha < 0} x_{\alpha} \in S.$$ 

Note that a priori the coefficients of $f_{v_{L_1}}$ belong to $S$.

**Proof.** Consider the following commutative diagram

$$\begin{array}{ccc}
\hat{X}_I^T & \xrightarrow{j} & \hat{X}_I \\
\downarrow q' & & \downarrow q \\
W & \xrightarrow{i} & G/B.
\end{array}$$

Note that by definition, $q'$ maps the point corresponding to $L \subseteq [l]$ to $v_L \in W$. Therefore,

$$q'_*(f_L) = f_{v_L} \in h_T(W).$$
Firstly, we have
\[ i^* q_I \star j_*(f_L) = i^* i_* q'_I (f_L) = i^* i'_* (f_L) = v^L (x_{\Pi}) f_{\nu L}, \]
where the last identity follows from \([33, \text{Corollary 6.4}]\). Consequently, by Lemma\(8.4.3\) we have
\[ i^* q_I \star (\eta_L) = i^* q_I \sum_{L_1 \subset L} a_{L, L_1} j^*(f_{L_1}) = \sum_{L_1 \subset L} a_{L, L_1} \cdot v^L_1 (x_{\Pi}) f_{\nu L_1}. \]

\[ (8.16) \]

\[ i^* q_{L^c} (\eta_L) = \sum_{L_1 \subset L^c} \prod_{\alpha < 0} v^L_1 (x_\alpha) f_{\nu L_1}. \]

\[ (8.17) \]

**Remark 8.4.5.** In case \( \eta_L = \eta_0 \) or \( \eta_k \), as in Proposition\(8.4.1\) and Theorem\(8.4.2\) one can express \( q_{I_*} (\eta_L) \) as the operators \( A_i \) applied on \( \text{pt}_e \). By using the method of formal affine Demazure algebra, started in \([68, 69]\) and continued in \([34, 35, 33]\), one will obtain a restriction formula of \( i^* q_{I_*} (\eta_L) \). Roughly speaking, there is an algebra \( \mathbf{D}_F \) generated by algebraic analogue of the push-pull operators \( A_i \), whose dual is isomorphic to \( h_T(G/B) \). The algebra \( \mathbf{D}_F \) acts on \( h_T(G/B) \), via two actions (denoted by \( \bullet \) and \( \odot \) in \([71]\)). Both actions will give restriction formulas of \( A_I (\text{pt}_e) \). Indeed, by using the two actions, one will obtain two different, but equivalent formulas, one of which coincides with the one given by Theorem\(8.4.4\).

**Corollary 8.4.6.** Let \( I \) be any sequence of length \( l \). For any \( L \subset [l] \), denote by \( q_L : \hat{X}_L \to G/B \). Then \( q_{I_*} (\eta_L) = q_{L^c} (1) \).

**Proof.** From Theorem\(8.4.4\) we have
\[ i^* q_{I_*} (\eta_L) = \sum_{L_1 \subset L} a_{L, L_1} \prod_{\alpha < 0} v^L_1 (x_\alpha) f_{\nu L_1}, \]

\[ (8.16) \]
\[ i^* q_{L^c} (1) = \sum_{L_1 \subset L^c} \prod_{\alpha < 0} v^L_1 (x_\alpha) f_{\nu L_1}. \]

\[ (8.17) \]

By definition
\[ x_{I, L_1} = \prod_{j \in I} v^L_1 (x_{-\alpha_j}), \quad x_{L^c, L_1} = \prod_{j \in L^c} v^L_1 (x_{-\alpha_j}), \quad a_{L, L_1} = \prod_{j \in L} v^L_1 (x_{-\alpha_j}). \]

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Since $L \cap L_1 = \emptyset$, so for any $j \in L$, $v_j^{L_1} = v_j^{L_1}$, and we have

$$x_{I,L} = \prod_{j \in L} v_j^{L_1}(x_{-\alpha_j}) \prod_{j \in L} v_j^{L_1}(x_{-\alpha_j})$$

$$= \prod_{j \in L} v_j^{L_1}(x_{-\alpha_j}) \prod_{j \in L} v_j^{L_1}(x_{-\alpha_j})$$

$$= x_{L,L} a_{L,L}.$$

Therefore, $i^* q_I^* (\eta_L) = i^* q_{L^c}^*(1).$ By [33, Theorem 8.2], we know $i^*$ is injective. So $q_I^* (\eta_L) = i^* q_{L^c}^*(1).$  

By using this result, we can derive the Chevalley formula for equivariant oriented cohomology. For each $w \in W$, we fix a reduced sequence $I_w$, then the Bott-Samelson class $\zeta_{I_w}$ is defined to be the push-forward class along the map $q_{I_w} : \hat{X}_{I_w} \to G/B$, i.e., $\zeta_{I_w} = q_{I_w}^*(1)$. It is proved in [33, Proposition 8.1] that $\{ \zeta_{I_w} | w \in W \}$ is a basis of $h_T(G/B)$. Denote the characteristic maps from $h_T(pt)$ to $G/B$ and to $\hat{X}_{I_w}$ by $c'$ and $c_{I_w}$, respectively.

By definition, $c_{I_w} = q_{I_w}^* c'$. 

**Corollary 8.4.7** (Chevalley Formula). For any $u \in h_T(pt)$, we have

$$c'(u) \cdot \zeta_w = \sum_{L \subset [\ell(w)]} \theta_{I,L}(u) \zeta_L,$$

where $\zeta_L = q_{L^c}^*(1)$ and $\theta_{I,L}(u)$ was defined in Lemma 8.2.4.

**Proof.** We have

$$q_{I_w}^*(c_{I_w}(u)) = q_{I_w}^*(c_{I_w}(u) \cdot 1) = q_{I_w}^*(q_{I_w}^*(c'(u)) \cdot 1) = c'(u) \zeta_{I_w},$$

where the last identity follows from the projection formula. Then Lemma 8.2.4 and Corollary 8.4.6 imply that

$$q_{I_w}^*(c_{I_w}(u)) = \sum_{L \subset [\ell(w)]} \theta_{I_w,L}(u) q_{I_w}^*(\eta_L)$$

$$= \sum_{L \subset [\ell(w)]} \theta_{I_w,L}(u) q_{L^c}^*(1)$$

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\[
= \sum_{L \subseteq [\ell(w)]} \theta_{L_w, L} (u) \zeta_L^c.
\]

The conclusion then follows. \qed
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