Microstate geometries & hidden symmetries

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MICROSTATE GEOMETRIES & HIDDEN SYMMETRIES

by

Qinglin Li

A Dissertation
Submitted to the University at Albany, State University of New York
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ABSTRACT

In this thesis, we study microscopic states of black holes and their hidden symmetries.

The first part of this thesis focuses on analyzing microscopic degrees of freedom responsible for the black hole entropy. Specifically, we use semiclassical approximation to count geometries describing a large class of microstates of four-dimensional supersymmetric black holes with AdS$_2 \times$S$^2$ asymptotics. We also evaluate various charges of the microstates.

The second part of this thesis focuses on hidden symmetries of microstate geometries and more general supergravity solutions. Specifically, we analyze the Klein–Gordon and Dirac equations on AdS$_p \times$S$^p$, AdS$_3 \times$S$^3 \times$S$^3$, and on superstrata geometries which describe microscopic states of five–dimensional black holes. We also identify all symmetries of dynamical equations associated with these spaces.
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CHAPTER 1

Introduction

Black holes provide important insights into quantum theory of gravity that unifies general relativity and quantum mechanics. Although the idea of a black hole, a massive objects whose gravitational attraction is so strong that even light cannot escape, goes back to work of John Michell and Pierre–Simon Laplace [1] in XVIII century, the first quantitative description of a black hole was given by Schwarzschild in 1916 [2]. The first quantum properties of black holes were uncovered half a century later, with the discovery of the Bekenstein–Hawking entropy [3, 4, 5, 6] and the Hawking radiation [5].

The thought experiments proposed in [3, 4] demonstrate that every black hole carries entropy $S$ that is proportional to the area of the event horizon, so according to statistical mechanics, the hole should contain $e^S$ microscopic states. However, the no–hair theorems [7] assert that a state of a black hole is completely determined by three parameters: mass, electric charge and angular momentum. Explanation of this discrepancy is the main challenge in the statistical interpretation of that entropy [8]. Furthermore, a semi–classical calculation by Hawking found black hole can evaporate via quantum mechanical pair production [5], producing thermal radiation which is described by a mixed state of matter. This leads to the information paradox [9]: an evolution of a pure state of matter that forms a black hole into a mixed state of thermal radiation.

String theory, the most promising candidate of quantum theory of gravity, has made significant progress towards understanding the quantum physics of black holes and towards construction of the microstate geometries that provide the resolution of the information paradox. The first counting of microscopic states describing macroscopic black holes was accomplished in [10], and the result was in a perfect agreement with the Bekenstein–Hawking entropy. The agreement has been later extended to subleading orders, which are described by the higher derivative terms on the gravity side [11]. These results provide the statistical explanation for the entropy of black holes.

To fully resolve the information paradox one needs to study dynamics of the micro-
scopic states, which goes beyond counting. In particular, one must demonstrate that the radiation from an individual microstate is described by a pure state that is not fully thermal. The first step in this direction involves an explicit construction of the individual microstates. In particular, for a supersymmetric two–charge system, all microstate geometries were constructed in [12, 13]. Unfortunately, the corresponding black holes have a horizon at the Planck scale, so it is necessary to extend to the three charge system, the D1–D5–P system, which possesses the macroscopic horizon. Some early examples of the three–charge geometries were constructed in [14], and in the last decade new classes of microstates were found in [15, 16, 17, 18]. These superstrata are the families of smooth supergravity solutions contributing to the entropy of the five–dimensional three–charge black holes. Some interesting dynamical properties of superstrata are studies in chapter 4 of this thesis.

The superstrata describe solutions with AdS$_3 \times$S$^3$ asymptotics, where AdS is a space–time with a negative cosmological constant, and S is a sphere. In chapters 2 and 3 we study regular geometries with AdS$_2 \times$S$^2$ asymptotics [19], which contribute to the entropy of four–dimensional black holes. Specifically, we count these states and compare their properties with features of D–branes on AdS$_2 \times$S$^2$.

To get quantitative description of the Hawking radiation, one must understand the dynamics of particles and fields in the vicinity of black holes and their microstates. Such dynamics is described by partial differential equations, which may not be amenable to analytical treatment. However, four–dimensional black holes have extensive symmetries, which allow one to solve equations for all dynamical fields [20, 21, 22], and it is interesting to extend such symmetries to black holes in higher dimensions and their microstates. We analyze hidden symmetries responsible for separation of dynamical equations on various backgrounds in chapters 4 and 5.

This thesis has the following organization.

In chapter 2, we quantize microstate geometries constructed in [19]. After reviewing a covariant method for quantizing gravity solutions, we apply it to the bubbling geometries of AdS$_2 \times$S$^2$ and find very nice canonical structures. Our results give interesting predictions for the Conformal Field Theory dual to this space. Chapter 2 is based on a published article [23].
In chapter 3, we analyze the dynamical properties of the D–branes on AdS$_2 \times S^2$ geometry and regular supergravity solutions. In particular, we compare energy and angular momenta of various excitations of AdS$_2 \times S^2$.

In chapter 4, we analyze hidden symmetries of superstrata and their implications for dynamics of various fields. Specifically, we demonstrate that the Dirac equation on such spaces is not separable.

In chapter 5, we analyze hidden symmetries of maximally symmetric space and their products. Once all such symmetries are classified, one can identify the combinations that survive when one of the elements in the product is replaced by a black hole. In particular, we present all Killing tensors for AdS$_p \times S^p$ and AdS$_3 \times S^3 \times S^3$ spacetimes that solve equations of motion of string theory.

All our results are summarized in chapter 6, and some technical details are presented in appendices.
CHAPTER 2
Minisuperspace quantization of bubbling geometries of $\text{AdS}_2 \times S^2$

2.1 Introduction

Gauge theory and supergravity are connected by the AdS/CFT correspondence [24], which allows one to describe supergravity configuration in terms of microscopic degrees of freedom in field theories on the boundary. The full understanding of the map between the bulk and the gravity sides for $\text{AdS}_2$/CFT$_1$ is still missing, although impressive progress has been reported in [25]. Recently, regular BPS geometries with $\text{AdS}_2 \times S^2 \times T^6$ asymptotics were constructed in [19, 26], and one can hope to use them to get insight into the structure of the dual field theory. In higher dimensions, where the AdS/CFT duality is well–understood, the gravity solutions, known as bubbling geometries [28] and fuzzballs [12, 13, 29, 30], can be mapped into the field–theoretic degrees of freedom via quantization of the moduli space [31, 32, 33]. In this chapter we gain some insights into the field theory dual to $\text{AdS}_2$ by applying the techniques developed in [31, 32, 33] to the geometries constructed in [28].

According to the AdS/CFT dictionary, $\text{AdS}_2 \times S^2$ corresponds to the ground state of the dual fields living on the boundary of the AdS space. The light excitations of the dual fields correspond to strings moving on $\text{AdS}_2 \times S^2$, and such strings which turn out to be integrable [34]. Excitations with intermediate energies are mapped into probe D branes whose counterparts in six–and ten–dimensional theories are called giant gravitons [35]. At higher energies the gravitational backreaction of branes must be taken into account, and the corresponding solutions of supergravity were constructed in [28]. The higher dimensional counterparts of such gravity solutions have been constructed in [12, 13, 28], and in [31, 32, 33], the spaces of these supersymmetric geometries were quantized using the method of Crnković–Witten [36] and Zukerman [37], and a perfect agreement with the expectations from the field theory side [38, 39] was found. Additional applications of this method have also been developed in [40, 41]. In the four–dimensional case we can use the similar quantization to get insights into the dual theory.

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1Related geometries with flat asymptotics have been discussed in [27].
The organization of this chapter is as follows. In section 2.2 we review the approach to quantize supergravity solutions directly, which is called Crnković–Witten–Zukerman covariant method. Two examples employing the this method have been reviewed in section 2.3 whose results exactly coincide with the ones obtained from the dual quantum field descriptions. In section 2.4, we apply this method to the bubbling geometries for $\text{AdS}_2 \times \text{S}^2$, and the symplectic form is found. A short discussion is given in the last section.

### 2.2 Crnković–Witten–Zukerman covariant method

A method for quantizing Lagrangian theories was proposed by Crnković and Witten [36] and Zukerman [37]. As a lagrangian theory, Type IIB SUGRA theory is equipped a symplectic form $\Omega$ defined over the full phase space, which contains the information of the commutation relations. In this chapter we restrict $\Omega$ on the moduli space of solutions, which is a subspace of the full phase space and is what we would like to quantize. Therefore, this method also can be called “on–shell quantization”. One can write the symplectic form as an integral over the Cauchy surface $\Sigma$:

$$\Omega = \int d\Sigma_i J^i. \quad (2.1)$$

We call $J^i$ as symplectic current, and its explicit expression is given by:

$$J^i = \sum_i \delta \left[ \frac{\partial L}{\partial (\partial_t \phi_i)} \right] \wedge \delta \phi_i, \quad (2.2)$$

where $\delta$ denotes the exterior derivative operator and $\phi_i$ runs over all fields in the action. Once the symplectic form is known as

$$\Omega = \frac{1}{2} \omega^{-1}_{ij} dq_i \wedge dp_j, \quad (2.3)$$

one then can encode the Poisson bracket

$$\{q_i, p_j\}_{P.B.} = \omega_{ij}. \quad (2.4)$$

The commutators are directly obtained from above brackets by the Dirac prescription.
In order to know about this method clearly, let us review a simple example, that is the chiral sector of a 2–dimensional free boson whose action is given by

$$S = \frac{1}{2} \int dt dx \left( \dot{\phi}^2 - \phi'^2 \right).$$

(2.5)

Reading the Lagrangian from this action and applying it into the formula (2.2) and (2.1), one then can obtain

$$\Omega = \int_{t=\text{constant}} dx \delta \pi(t, x) \wedge \delta \phi(t, x), \quad \pi = \dot{\phi},$$

(2.6)

The symplectic form here is on the full phase space of the theory. The fluctuations around the solutions can be represented by the two one–forms $\delta \pi(t, x)$ and $\delta \phi(t, x)$.

As mentioned at the beginning of this chapter, we are only interested in the quantization of moduli space of solution. Thus here we only consider the space of left–moving solutions, which are parametrized via an arbitrary function:

$$\mathcal{M} = \{ \phi(t, x) = f(t + x), f \text{ arbitrary} \}$$

(2.7)

As the symplectic form is confined on the moduli space of solutions, one can write

$$\omega = \Omega|_{\mathcal{M}} = \int du \delta f'(u) \wedge \delta f(u).$$

(2.8)

The Poisson bracket then can be extracted

$$\{ f(u_1), f'(u_2) \} = \delta(u_1 - u_2).$$

(2.9)

The quantization thus is achieved by promoting the Poisson bracket to Dirac bracket

$$[ f(u_1), f'(u_2) ] = i\hbar \delta(u_1 - u_2).$$

(2.10)

According to this commutation relation, the Fock space then can be constructed as the usual
operation:
\[ \hat{f}(u) = \int_0^\infty \frac{dp}{2\pi} \frac{1}{\sqrt{2p}} e^{ipu} \alpha_p + h.c. \]
\[ [\alpha_p, \alpha_p^\dagger] = 2\pi \hbar \delta(p - p') \] (2.11)

So far the standard quantization of the chiral boson sector has been recovered. Here all the calculation has been done in the moduli space of solution.

2.3 Review of minisuperspace quantization of supergravity solutions

Thanks to AdS/CFT we can describe the microstates of black holes in terms of microscopic degrees of freedom living in conformal field theories. There is a question: is that possible to derive the full quantum structure directly from gravity side? The answer is yes. The role of direct quantization from gravity side is extremely useful when the dual description is unknown, for instance the one–dimensional quantum mechanics dual to AdS\(_2\). In this section we present two examples using the method introduced in previous section to quantize two regular horizonless supergravity solutions which describe the excitations of AdS\(_3\)×S\(^3\) and AdS\(_5\)×S\(^5\), respectively.

2.3.1 Quantization of the gravity solutions to D1–D5 system

In this section, we would like to consider the subspace of the full phase space, the moduli space of solutions to D1–D5 system. These solutions are parametrized by closed curves \( F(v) \) on a four–dimensional space, which have been presented in appendix A.1. In current case, the phase space is that of Type IIIB SUGRA(supergravity) whose the Einstein–frame action can be written as
\[ S = \frac{1}{(2\pi)^7 g_s^2} \int \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^\phi |F_3|^2 \right], \] (2.12)
where \( g_s \) is string coupling constant and \( R \) is Ricci scalar. Here \( F_3 \) is three–from field strength that can be written as the exterior derivative of two–form potential: \( F_3 = dC \). On the constant time surface, the dynamical degrees of freedom are given by the spatial components of the metric \( g_{ab} \), the potential \( C_{ab} \) and dilaton \( \phi \). Defining \( q \equiv \{ g_{ab}, C_{ab}, \phi \} \), the symplectic form,
according to (2.1) and (2.2), of the theory then can be expressed as

$$\Omega \sim \int_{t=\text{constant}} d^3 x \sum_q \delta \Pi_q(x, t) \wedge \delta q(x, t), \quad \Pi_q = \frac{\partial L}{\partial \dot{q}^q}. \quad (2.13)$$

Here in order to distinguish the spacetime differential $dx^\mu$ we use $\delta$ to denote the differential in the space of fields. According to different fields involved, the symplectic form can be written as covariant formalism which is an integral of the symplectic currents contributed by different fields over a Cauchy surface $\Sigma$:

$$\Omega = \frac{1}{(2\pi)^2 g_s} \int d^3 \Sigma \mu \left( J^\mu_G + J^\mu_F + J^\mu_\phi \right), \quad (2.14)$$

where $J^\mu_G, J^\mu_F; J^\mu_\phi$ are gravity, two–form $C$ and dilaton symplectic currents, respectively. The first step to derive the gravity current is to add the Gibbons–Hawking boundary term [42], which is equivalent to removing the second derivatives from the Einstein–Hilbert action. The consequence of this operation is given the so–called $\Gamma\Gamma\Gamma\Gamma$ Lagrangian

$$L = \sqrt{-g^i k} \left[ \Gamma^m_{il} \Gamma^l_{km} - \Gamma^l_{ik} \Gamma^m_{lm} \right]. \quad (2.15)$$

Performing (2.2), one can obtain the gravity current

$$J^l_G = -\delta \Gamma^d_{mn} \wedge \delta(\sqrt{-g}g^{mn}) + \delta \Gamma^m_{mn} \wedge \delta(\sqrt{-g}g^{lm}). \quad (2.16)$$

The other two currents can be found directly by computing as (2.2) from the action (2.12)

$$J^\mu_F = -\delta \left( \sqrt{-g}e^{-\phi} F^{\mu[\alpha\beta]} \right) \wedge C_{\alpha\beta},$$

$$J^\mu_\phi = -\delta \left( \sqrt{-g} \partial^\mu \phi \right) \wedge \delta \phi. \quad (2.17)$$

Once these currents have been figured out, we then can extract the Poisson bracket from symplectic form, and thus the commutator.

Fortunately, there is one constraint which can simplify the derivation of symplectic form. [33] pointed out that the subspace we are interested in is not arbitrary but the one that consists of time–independent solutions. It turns out that for the current case this fact allows us to determine the expression of symplectic form except a constant coefficient.
Applying the standard general relativity formula for the asymptotically flat spacetime, the energy of the D1–D5 microstate geometries (see appendix A.1) can be found

\[ H|_\mathcal{M} = \frac{RV_4}{g_s^2} \left( \frac{Q_5}{L} \int_0^L |F'(v)|^2 dv + Q_5 \right), \tag{2.18} \]

which should agree with the total mass of the D branes

\[ E_{tot} = g_s^{-1} (N_1 R + N_5 RV_4). \tag{2.19} \]

This consistency condition predicts the form the Poisson bracket as

\[ \{ F_i(v), F'_j(\tilde{v}) \} = \alpha \delta_{ij} \delta(v - \tilde{v}), \tag{2.20} \]

where \( \alpha \) is a constant. This bracket corresponds to a symplectic form which is fixed up to a proportionality coefficient

\[ \Omega = \frac{1}{2\alpha} \int \delta F'_i(v) \wedge \delta F_i(v) dv. \tag{2.21} \]

To determine the value of \( \alpha \), we have to do some explicit computation on symplectic currents.

So far we have already known the form of the symplectic form but one constant, which allows us to consider a simple situation. Recall that the supergravity solutions are parametrized by profiles \( F(v) \) in four–dimensional space. Thus we can make two assumptions to easily calculate the value of \( \alpha \). The first one is that there is a curve of containing a straight–line

\[ F_1(v) = v, \quad F_{2,3,4} = 0 \quad (0 < v < 1), \tag{2.22} \]

while the second one is that the fluctuations of the profile only have one component

\[ \delta F_2(v) \equiv a(v) \quad (0 < v < 1). \tag{2.23} \]

These two assumptions have greatly simplified the calculation of symplectic currents. Finally the explicit expression of symplectic form has been obtained as

\[ \Omega = \frac{1}{2\pi \mu^2} \int a'(v) \wedge a(v), \quad \mu = \frac{g_s}{R_s \sqrt{V_4}}. \tag{2.24} \]
This results indicates that $\alpha = \pi \mu^2$. Here $R_s$ is the radius of $S^1$ and $V_4$ is the volume of $T^4$ or $K3$. Thus one can work out the commutation relation

$$[F_i(v), F'_j(\tilde{v})] = i\pi \mu^2 \delta_{ij} \delta(v - \tilde{v}).$$  \hspace{1cm} (2.25)

This commutator should coincide with the dual field theory according to AdS/CFT.

It is known that the D1–D5 system is U–dualized to the FP system which is the multiply wound fundamental string with momentum along a compactified direction. Consider quadratic action of fundamental string

$$S_{FP} = \frac{1}{4\pi} \int (\dot{X}_i^2 - X_i'^2) d\tau d\sigma.$$ \hspace{1cm} (2.26)

The commutation relation extracted from this action is

$$[X_i(\sigma), \dot{X}_j(\tilde{\sigma})] = i2\pi \delta(\sigma - \tilde{\sigma}) \delta_{ij}$$ \hspace{1cm} (2.27)

The commutator of its chiral component is 1/2 of that

$$[F_{FPi}(v), F'_{FPj}(\tilde{v})] = i\pi \delta(v - \tilde{v}) \delta_{ij}.$$ \hspace{1cm} (2.28)

This coincides with the commutator (2.25) as we identify $F(v) = \mu F_{FP}(v)$, i.e. $F(v) = \mu F_{FP}(v)$. Thus we see the canonical structure found from gravity side by the direct quantization of supergravity solutions is consistent with the results obtained from dual description. From this consistency, we may say this result has enhanced the conjecture of gravity/gauge duality or we may say the method employed in this section works well which has been checked by dual description.

2.3.2 Quantization of Bubbling AdS

In this part we will give another example, the regular horizonless supergravity solutions describing the excitations of $\text{AdS}_5 \times S^5$, which have been presented in appendix A.2. We will play the same trick to quantize these solutions and then compare results to the known dual description.
Let us start with the action of Type IIB SUGRA from which the LLM solutions [28] can be derived:

\[ S = \frac{1}{2\kappa^2_{10}} \int d^{10}x \sqrt{-g} \left( R - 4|F_5|^2 \right). \]  

(2.29)

This action only contains the metric and flux $F_5$ because LLM solutions are supported only by this five–form flux (see appendix A.2). In current case the symplectic form does not change as the selfduality constraint $F_5 = \ast F_5$ imposed. Thus the symplectic form reads

\[ \Omega = \frac{1}{2\kappa^2_{10}} \int d\Sigma_4 (J^t_G + J^t_F), \]  

(2.30)

where $J^t_G$ and $J^t_F$ are the gravity and five–form currents respectively that can be computed via Eq. (2.2).

The symplectic current of gravity $J^t_G$ is same as the one for D1–D5 system in previous section which was found by Crnković and Witten

\[ J^t_G = -\delta \Gamma^t_{mn} \wedge \delta (\sqrt{-g}g^{mn}) + \delta \Gamma^n_{mn} \wedge \delta (\sqrt{-g}g^l m). \]  

(2.31)

What we would like to point out here is that when the metric perturbations take a gauge transformation as

\[ \delta g_{mn} \to \delta g_{mn} + \nabla_{(m}\xi_{n)}, \]  

(2.32)

$J^t_G$ will change by a total derivative. This fact keeps the symplectic form (2.30) gauge invariant. Taking the potentials $A_{|k_1...k_4|}$, which satisfy the relation $F_5 = dA$, as our basic fields, and applying (2.2), we obtain $J^t_F$ directly

\[ J^t_F = -8\delta (\sqrt{-g}F^l|k_1...k_4|) \wedge \delta A_{|k_1...k_4|}, \]  

(2.33)

where $|i_1i_2i_3i_4|$ means $i_1 < i_2 < i_3 < i_4$. To figure out the symplectic form, what we need to do is to work out these symplectic currents from the fluctuations of fields involved in this system.

Recall in LLM solutions, given by appendix A.2, the ground state corresponding to $\text{AdS}_5 \times S^5$ is parametrized by circular droplet in $(x_1, x_2)$ plane which gives $z$, $V_r$, $V_\phi$ by
evaluating the integrals over it

\[ z = \frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)} - 4r_0^2}, \]
\[ V_r = 0, \]
\[ V_\phi = -\frac{1}{2} \left( \frac{r^2 + y^2 + r_0^2}{\sqrt{(r^2 + r_0^2 + y^2)^2} - 4r_0^2} - 1 \right) \]  
(2.34)

Inserting these quantities into (A.14) and making coordinates transformation (A.20), we then can get $\text{AdS}_5 \times \text{S}^5$ in global coordinates and the five form

\[ F_5 = \cosh \rho \sinh^3 \rho dt \wedge d\rho \wedge d\Omega_3 + \cos \theta \sin^3 d\theta \wedge d\phi \wedge d\tilde{\Omega}_3. \]  
(2.35)

Consider that all the fluctuations of the fields are originated from the perturbation of the background (A.21), which are mapped to the fluctuations of the circular droplet, that is the small ripples of the droplet. Thus in the polar coordinates, the boundary of the perturbed circular droplet can be presented as $r(\phi) = 1 + \delta r(\phi)$, where we have set the radius of the circular droplet $r_0$ as 1, which is also the square of the radius of AdS and spherical space. The perturbation $\delta r(\phi)$ can be expanded into Fourier series

\[ \delta r(\phi) = \sum_{n \neq 0} a_n e^{in\phi}, \quad a^* = a_{-n}. \]  
(2.36)

The presence of these ripples leads to the first order shifts of metric $h_{MN}$ which are given by

\[ h_{\mu\nu} = \sum_{n \neq 0} \left( -\frac{6}{5}|n|s_n Y_n g_{\mu\nu} + \frac{4}{|n| + 1} Y_n \nabla_{(\mu} \nabla_{\nu)} s_n \right), \]
\[ h_{\alpha\beta} = \sum_{n \neq 0} 2|n|s_n Y_n g_{\alpha\beta}, \]  
(2.37)

where $\mu, \nu$ run over the $\text{AdS}_5$ directions, while $\alpha, \beta$ run over the $\text{S}^5$ directions. Here $g_{\mu\nu}$ and $g_{\alpha\beta}$ are the unperturbed $\text{AdS}_5$ and $\text{S}^5$ metrics, respectively. The two functions in the (2.37) are spherical harmonics, which obey the following equations

\[ \nabla_{\text{S}^5}^2 Y_n = -n(n + 4)Y_n, \quad \nabla_{\text{AdS}_5}^2 s_n = n(n - 4)s_n, \]  
(2.38)
and the explicit expression for them are

\[ s_n = \frac{|n| + 1}{2|n| \cosh |n|} a_n e^{int}, \quad Y_n = e^{in\phi} \cos |n| \theta. \] (2.39)

Expressing the perturbations of metric in terms of functions \( s_n \) and \( Y_n \) enable one to identify it with a subclass of modes studied in [43, 44]. According to [43], the perturbations of the four–form which is viewed as the potential of five–form flux, can be derived directly from the shifts of metric as

\[ \delta A_{\alpha\beta\gamma\delta} = -\epsilon'_{\alpha\beta\gamma\delta} s_n \nabla^\alpha Y_n, \]
\[ \delta A_{\mu\nu\rho\lambda} = \epsilon'_{\mu\nu\rho\lambda} Y_n \nabla^\mu s_n, \] (2.40)

where \( \epsilon'_{\alpha\beta\gamma\delta} (\epsilon'_{\mu\nu\rho\lambda}) \) is the curved–space \( \epsilon \) symbol on \( S^5 \) (AdS\(_5\)).

By these preparations, one can directly compute the symplectic currents \( J_G \) and \( J_F \). The fact that the LLM solutions is independent on \( t \) allows us to make the natural choice for the hypersurface as \( \Sigma = \{ t = \text{constant} \} \). Thus one can obtain

\[ \int_{t=\text{constant}} J_G^t = 8\pi^5 i \sum_{n \neq 0} \frac{n^2 - 3|n| - 8}{(|n| - 1)(|n| + 2)n} a_n \wedge a_{-n}, \]
\[ \int_{t=\text{constant}} J_F^t = 8\pi^5 i \sum_{n \neq 0} \frac{n^2 + 5|n| + 4}{(|n| - 1)(|n| + 2)n} a_n \wedge a_{-n}. \] (2.41)

Here the coefficients of Fourier series \( a_n \) is regarded as one–forms. Adding these two integrals up, we then get the symplectic form as

\[ \omega = \frac{8\pi^5 i}{\kappa^2_{10}} \sum_{n \neq 0} \frac{1}{n} a_n \wedge a_{-n}. \] (2.42)

Promoting the Poisson bracket corresponding to this symplectic form to Dirac bracket, we can acquire the commutation relation

\[ [a_m, a_n] = \frac{\kappa^2_{10}}{16\pi^5} n \delta_{m+n}. \] (2.43)

Noting this result is derived from LLM solutions, thus it must coincide with the results obtained from the dual super Yang–Mills states in the large \( N \) limit according to AdS/CFT.
On the SYM side, there is a system of $N$ fermions in harmonic potential. The states of this system can be well described by droplets in the one-particle phase space in the large $N$ limit. The Hamiltonian of the harmonic potential one-fermion is

$$H = \frac{p^2 + q^2}{2}$$  \hspace{1cm} (2.44)

The boundary of a droplet described in polar coordinates are given by

$$p = r(\phi) \sin \phi, \quad q = r(\phi) \cos \phi.$$  \hspace{1cm} (2.45)

Thus the total energy of the droplet state then can be written as the integral of the one-particle Hamiltonian

$$H_{\text{tot}} = \int \int \frac{dp \, dq}{2\pi \hbar} \frac{p^2 + q^2}{2} = \frac{1}{16\pi \hbar} \int d\phi r^4(\phi) \equiv \frac{1}{16\pi \hbar} \int d\phi f^2(\phi)$$  \hspace{1cm} (2.46)

where the constant $\hbar$ here is defined by (A.22). By the definition $f(\phi) \equiv r^2(\phi)$, there is the following Poisson bracket [45]:

$$\{f(\phi), f(\tilde{\phi})\} = 8\pi \hbar \delta(\phi - \tilde{\phi}).$$  \hspace{1cm} (2.47)

By standard relation between $\kappa_{10}$ and the radius of AdS$_5$ (see e.g. [46])

$$R_{\text{AdS}}^4 = \frac{\kappa_{10} N}{2\pi^{5/2}},$$  \hspace{1cm} (2.48)

and noting (A.22), the constant $\hbar$ can be expressed as $\kappa_{10}$, which leads to commutation relation promoted from Poisson bracket (2.47) be

$$\left[f(\phi), f(\tilde{\phi})\right] = i\frac{\kappa_{10}^2}{2\pi^4} \delta(\phi - \tilde{\phi}).$$  \hspace{1cm} (2.49)

Expanding $f(\phi)$ as

$$f(\phi) = \sum_{n \neq 0} f_n e^{i n \phi}, \quad f_{-n} = f_{n}^*, \quad f_n = f_{n}^*,$$  \hspace{1cm} (2.50)

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the commutator then can be rewritten as

\[ [f_m, f_n] = \frac{\kappa_4^2}{4\pi^5} n\delta_{m+n}. \] (2.51)

This is just a U(1) Kac–Moody algebra. We see that this commutation relation is consistent with (2.43) as one identify \( a_n = 2f_n \). Thus we can say that by the direct quantization of supergravity solutions, i.e. LLM solutions, one can find the same canonical structures with the ones derived from SYM theory.

### 2.4 Minisuperspace quantization of bubbling geometries of \( \text{AdS}_2 \times S^2 \)

For the cases that the dual descriptions are known the quantization from gravity side may be only a check of Maldacena’s conjecture, while for the situations that the dual quantum mechanics are unknown, the approach to gravity quantization may play a important role in counting the microstates of black holes and shedding the light into the corresponding conformal field theory. As we have mentioned above, the dual description of \( \text{AdS}_2 \) is not clear so far. Thus in this section we will perform the trick employed in previous section, which has been well–checked, to quantize the supergravity solutions with \( \text{AdS}_2 \times S^2 \) asymptotics which are presented in appendix A.3.

For a method to quantize lagrangian theory, the starting point is always the action. Similarly to the case of bubbling \( \text{AdS}_5 \), the solutions (A.25) that we would like to study are also only supported by the five–form flux \( F_5 \). Therefore, the action we are interested in here is same as (2.29). Because of this, the symplectic currents \( J_G \) and \( J_F \) that we are looking for in this section have the same expression as (2.31) and (2.33).

To evaluate symplectic currents \( J_G \) and \( J_F \), we have to find the fluctuations of the fields involved in (2.31) and (2.33), that is the perturbations of metric and the four–form potential, which is produced by the shifts of two harmonic functions \( H_1 \) and \( H_2 \) in (A.25). We will start the computation with the ground state which is parametrized by circular profile, and then consider the the perturbations of ground state which corresponds to the ripples on the circular profile since that parameterize the excitations.

Recall in the parameterization (A.25), the \( \text{AdS}_2 \times S^2 \) background is specified by a cir-
cular closed curve

\[ f_1(v) = L \cos \frac{v}{L}, \quad f_2(v) = L \sin \frac{v}{L}, \quad f_3 = 0, \quad 0 \leq v \leq 2\pi L, \quad (2.52) \]

and the related vector \( \mathbf{b} \) is given by \( \mathbf{b} = 2L(\cos \frac{v}{L}, \sin \frac{v}{L}, i) \). After plugging (2.52) and \( \mathbf{b} \) into (A.26) and (A.27), it is straightforward to get

\[
H = \frac{L}{\sqrt{r^2 + (y - iL)^2}}, \\
V = -\frac{L}{2} \left[ \frac{r^2 + y^2 + L^2}{\sqrt{4L^2y^2 + (r^2 + y^2 - L^2)^2}} - 1 \right] \, d\phi. \quad (2.53)
\]

Substituting the above expressions for \( H \) and \( V \) into (A.25), and making the coordinate transformation as

\[
t = \frac{\tilde{t}}{L}, \quad r = L\sqrt{\rho^2 + 1} \sin \theta, \quad y = L\rho \cos \theta, \quad \phi = \tilde{\phi} - t, \quad (2.54)
\]

one obtains immediately the geometry of the global AdS\(_2\)\( \times \)S\(_2\):

\[
ds^2 = L^2 \left[ -(\rho^2 + 1)dt^2 + \frac{d\rho^2}{\rho^2 + 1} + d\theta^2 + \sin^2 \theta d\tilde{\phi}^2 \right] + dz^a dz_a, \\
F_5 = \frac{L}{4} d\rho d\tilde{t} \wedge \text{Re}(dz_{123}) + \text{dual}. \quad (2.55)
\]

The circular profile (2.52), parameterizing AdS\(_2\)\( \times \)S\(_2\) solution, is corresponding to the ground state of the system with a given amount of flux. To find the symplectic form, we focus on the perturbations of this solution, which is equivalent to considering the small perturbations of the two harmonic functions \( H_1 \) and \( H_2 \). The profile can vibrate in \((x_1, x_2, y)\), but in this paper we only consider its fluctuations in the \((x_1, x_2)\) plane and restrict vector \( \mathbf{b} \) to be orthogonal to this plane as (A.27). We can express \( \delta f \) (\( f = \sqrt{f_1^2 + f_2^2} \)) in Fourier series:

\[
\delta f = \sum_{|n|>1} L a_n e^{in\phi}, \quad a_n^* = a_{-n}. \quad (2.56)
\]

From now on we will set \( L = 1 \), and this scale will be restored in the end (see equations
In this computation we need only the first order fluctuations of the profile since only such perturbations enter the expression of symplectic current (2.2). In the first order the fluctuations with \( n = \pm 1 \) describe the circular profile with shifted origin, so they still correspond to the ground state. Thus we focus on \(|n| > 1\).

The first order perturbations of the two harmonic functions \( H_1, H_2 \), and vector field \( V \) caused by the fluctuations (2.56) are given by

\[
\delta H_1 = \sum_{|n|>1} \frac{a_n e^{in(\tilde{\phi} - \tilde{t})}}{2R^6} \left[ \frac{s_\theta}{\sqrt{\rho^2 + 1}} \right] |n| \rho \left[ \rho^2 - 3c_\theta^2 \right] \left[ |n|R^2 + \rho^2 - c_\theta^2 + 2 \right],
\]

\[
\delta H_2 = \sum_{|n|>1} \frac{a_n e^{in(\tilde{\phi} - \tilde{t})}}{2R^6} \left[ \frac{s_\theta}{\sqrt{\rho^2 + 1}} \right] |n| c_\theta \left[ 3\rho^2 - c_\theta^2 \right] \left[ |n|R^2 + \rho^2 - c_\theta^2 + 2 \right],
\]

\[
\delta V_r = -\sum_{|n|>1} \frac{ina_n e^{in(\tilde{\phi} - \tilde{t})}}{(\rho^2 + 1)R^2} \left[ \frac{s_\theta}{\sqrt{\rho^2 + 1}} \right] |n|-1,
\]

\[
\delta V_\phi = \sum_{|n|>1} \frac{a_n e^{in(\tilde{\phi} - \tilde{t})}}{R^6} \left[ \frac{s_\theta}{\sqrt{\rho^2 + 1}} \right] |n| |n|R^2 \left( c_\theta^2 - \rho^2 + 2\rho^2 c_\theta^2 + 2(\rho^2 + 1)s_\theta^2(c_\theta^2 - \rho^2) \right),
\]

where \( R^2 = (\rho^2 + \cos^2 \theta) \), and \( s_\theta \equiv \sin \theta \), \( c_\theta \equiv \cos \theta \). The details of the calculation can be seen in appendix A.4.1. Although perturbations (2.57) appear singular at the location of the profile \((R = 0)\), we will now show that they give rise to regular metric perturbations after an appropriate gauge transformation.

The metric perturbations are obtained by plugging the expressions (2.57) into (2.58). In order to make the field perturbations regular, we perform a linear gauge transformation (3.51) with

\[
\xi = \sum_{|n|>1} a_n e^{in(\tilde{\phi} - \tilde{t})} \left[ \frac{s_\theta}{\sqrt{\rho^2 + 1}} \right] |n| \left[ -\frac{i|n|}{n} d\tilde{t} + \frac{\rho s_\theta^2}{(\rho^2 + 1)R^2} d\rho + \frac{s_\theta c_\theta}{R^2} d\theta \right],
\]

This leads to the final form of the metric perturbations:

\[
\begin{align*}
\delta h_{\mu\nu} &= \sum_{|n|>1} \left( -\frac{2|n|(|n| - 1)}{|n|+1} s_n Y_n g_{\mu\nu} + \frac{4}{|n|+1} Y_n \nabla(\mu \nabla_\nu) s_n \right), \\
\delta h_{\alpha\beta} &= \sum_{|n|>1} 2|n| s_n Y_n g_{\alpha\beta},
\end{align*}
\]
where $\mu, \nu$ run over the AdS$_2$ directions, $\alpha, \beta$ run over the S$^2$ directions, and $s_n$ and $Y_n$ are

$$s_n = a_n e^{-in} \frac{|n| + 1}{2|n|(1 + \rho^2)|n|/2},$$
$$Y_n = e^{in\tilde{\phi}} \sin^{n/2} \theta. \quad (2.60)$$

They are the spherical harmonics of AdS$_2$ and S$^2$, respectively,

$$\nabla_{S^2}^2 Y_n = -|n|(|n| + 1) Y_n, \quad \nabla_{AdS_2}^2 s_n = |n|(|n| - 1) s_n. \quad (2.61)$$

The symplectic current of gravity $J^t_G$ is obtained by substituting (2.59) into (2.31).

The second part of the symplectic current, $J^l_F$, is constructed from the 4–form potential according to (2.33). For the solutions (A.25), we need the fluctuations of the 2–forms $F$ and $\tilde{F}$ in (A.25) and their potentials. These fluctuations can be calculated by substituting (2.56) or (2.57) into (A.25), (A.26) and (A.27), but this path involves tedious algebraic manipulations.

Fortunately, there is an alternative option given by [31, 43, 44], which relates fluctuations of $F_5$ with the metric perturbations (2.59). Following the procedure outlined in [31, 43, 44], we can compute the one–form $B$ and $\tilde{B}$ through the relations

$$\delta a_\alpha = \frac{1}{4} \epsilon_{\beta\alpha} s_n \nabla^\beta Y_n,$$
$$\delta a_\mu = -\frac{1}{4} \epsilon_{\nu\mu} Y_n \nabla^\nu s_n. \quad (2.62)$$

Here $\mu, \nu$ run over the AdS$_2$ directions, $\alpha, \beta$ run over the S$^2$ directions, and $\epsilon_{\beta\alpha}, \epsilon_{\mu\nu}$ are the volume forms on S$^2$ and AdS$_2$, respectively. Plugging (2.60) into above relations (2.62), we immediately get

$$\delta B = -\sum_{|n|>1} \frac{1}{8} a_n e^{in(\tilde{\phi}-\tilde{t})} \left[ \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right] |n| |n| + 1 \left[ \rho d\tilde{t} + \frac{in}{n|1 + \rho^2|} d\rho \right],$$
$$\delta \tilde{B} = -\sum_{|n|>1} \frac{1}{8} a_n e^{in(\tilde{\phi}-\tilde{t})} \left[ \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right] |n| |n| + 1 \left[ \frac{in}{n|\sin \theta - \cos \theta d\tilde{\phi} \right] \right], \quad (2.63)$$
\[
\delta F = - \sum_{|n|>1} \frac{1}{8} a_n e^{i n (\delta - \tilde{\iota})} \left[ \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right] |n| (|n| + 1) \left[ (|n| - 1) d\tilde{\iota} \wedge d\rho - |n| \cot \theta d\tilde{\iota} \wedge d\theta 
- in \rho d\tilde{\iota} \wedge d\tilde{\phi} - \frac{in \cot \theta}{\rho^2 + 1} d\rho \wedge d\theta + \frac{|n|}{\rho^2 + 1} d\rho \wedge d\tilde{\phi} \right],
\]
\[
\delta \tilde{F} = - \sum_{|n|>1} \frac{1}{8} a_n e^{i n (\delta - \tilde{\iota})} \left[ \frac{\sin \theta}{\sqrt{\rho^2 + 1}} \right] |n| (|n| + 1) \left[ (|n| + 1) \sin \theta d\theta \wedge d\phi + |n| \csc \theta d\tilde{\iota} \wedge d\theta 
+ in \cos \theta d\tilde{\iota} \wedge d\tilde{\phi} - \frac{in \rho \csc \theta}{\rho^2 + 1} d\rho \wedge d\theta + \frac{|n| \rho \cos \theta}{\rho^2 + 1} d\rho \wedge d\tilde{\phi} \right].
\]

It is convenient to pick the hypersurface \( \Sigma = \{ \tilde{\iota} = \text{const} \} \) in the symplectic form (2.30), since the solutions (A.25) are independent of \( \tilde{\iota} \). This leads to the gravity and to 5–form contributions to the symplectic form:

\[
\int_{\tilde{\iota} = \text{const}} J_{\tilde{\iota}}^G = - \sum_{|n|>1} \frac{2 \pi^2 i V_6 |n| (|n| + 1)(n^2 - 3|n| - 2)}{n(2|n| + 1)} a_n \wedge a_{-n}, \tag{2.65}
\]
\[
\int_{\tilde{\iota} = \text{const}} J_{\tilde{\iota}}^F = - \sum_{|n|>1} \frac{2 \pi^2 i V_6 |n|(|n| + 1)^3}{n(2|n| + 1)} a_n \wedge a_{-n}, \tag{2.66}
\]

where \( V_6 \) is the volume of \( T^6 \). In equations (2.65) and (2.66) the Fourier coefficients \( a_n \) should be interpreted as one–forms on the phase space of solutions (A.25). The derivation of (2.65) and (2.66) can be found in appendix A.4.2. Adding the individual contributions, we get the symplectic form as

\[
\Omega = - \frac{2 \pi^2 i V_6}{\kappa_{10}^2} \sum_{|n|>1} \text{sign } n (n^2 - 1) a_n \wedge a_{-n}. \tag{2.67}
\]

As we mentioned in section 3.1, once the symplectic form is found, the Poisson brackets can be extracted from it following the procedure summarized by (2.3) to (2.4). Restoring the radius \( L \), we obtain

\[
\{ a_n, a_m \} = \frac{L^2 \kappa_{10}^2}{4 \pi^2 V_6} \frac{i \text{sign } n}{n^2 - 1} \delta_{m+n}. \tag{2.68}
\]

The commutators are immediately obtained in the usual way by using the relation that
\[ [a_m, a_n] = \frac{L^2 \kappa_1^2}{4 \pi^2 V_6} \frac{\text{sign } n}{n^2 - 1} \delta_{m+n}. \]  

(2.69)

The microstate geometries in supergravity correspond to different profiles \( f(v) \). Although classically there is a continuum of such curves, the commutators (2.69) lead to expansions into discrete quantum oscillators:

\[
\begin{aligned}
  f(\phi) &= f_0(\phi) + \lambda \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 1}} \left( c_n e^{-i \phi} + c_n^\dagger e^{i \phi} \right), \\
  [c_n, c_m^\dagger] &= \delta_{mn},
\end{aligned}
\]

(2.70)

where

\[
\lambda = \frac{L \kappa_1}{2 \pi \sqrt{V_6}}, \quad c_n = \left[ \frac{L^2 \kappa_1^2}{4 \pi^2 V_6} \frac{1}{n^2 - 1} \right]^{-1/2} a_{-n}.
\]

The Hilbert space of this theory is a bosonic Fock space equipped with annihilation operators \( c_n \) and creation operators \( c_n^\dagger \).

The counterparts of the commutators (B.44) have been encountered in [31, 32, 33], and these quantization conditions derived in supergravity turned out to be in a perfect agreement with properties of supersymmetric states in \( N = 4 \) super Yang–Mills on \( S^3 \times \mathbb{R} \) [38, 47] and in the two–dimensional orbifold CFT [12, 13]. We expect that a better understanding for the boundary theory in the AdS_2 case would lead to a similar agreement for our relations (B.44).

### 2.5 Discussion

In this chapter, we focused on regular horizon–free supersymmetric solutions describing normalizable excitations of AdS_2 \( \times S^2 \), which are parameterized by a complex harmonic function \( H \) in \( \mathbb{R}^3 \) of possessing the sources located in closed curves. From the gravity side, we have generalized the method used in [31, 32, 33] to quantize the moduli space of these solutions directly. The basic idea of this approach is that commutation relations can be encoded from the symplectic form. Starting with the profile fluctuations, we obtain the perturbations of fields, graviton and five–form, involved in Type IIB SUGRA action corresponding to solutions (A.25). Finally these perturbations are used to find the symplectic
form and to quantize the profiles describing microscopic states. As in the AdS$_3$ and AdS$_5$
cases discussed in [31, 32, 33], this quantization is expected to agree with results in the dual
theory on the boundary. However, in the present situation where the boundary theory is
poorly understood, our results can be viewed as predictions rather than consistency checks
as in [31, 32, 33].
CHAPTER 3

Supersymmetric D branes on AdS$_2 \times $S$^2$ background

3.1 Introduction

Supersymmetric states play a very important role in string theory since they allow one to answer a wide range of dynamical questions using analytical techniques. The action of the type II superstring is invariant under 32 supersymmetry transformations, so the string vacuum, such as the ten–dimensional flat space, preserves 32 supercharges. The excited states may break some supersymmetries or all of them, and a great progress has been achieved in understanding so–called 1/2–BPS states, the configurations preserving 16 supercharges. In this chapter we will analyze two types of such states, D–branes and regular geometries, which are related to excitations of AdS$_2 \times $S$^2$ discussed in chapter 2.

The supersymmetric D–branes in AdS$_p \times $S$^q$ backgrounds are known as giant gravitons [35, 48]. For example, a giant graviton in AdS$_5 \times $S$^5$ is a classical D3 brane wrapping an S$^3 \subset $S$^5$ and rotating along one of the transverse directions to it within S$^5$[35]. A dual giant graviton is a 1/2 BPS D3 brane wrapping S$^3 \subset $AdS$^5$ and rotating along a maximal circle of S$^5$ [48]. All 1/2 BPS states in AdS$_5 \times $S$^5$ have been counted by enumerating the (dual) giant gravitons [49]. Similar giant and dual–giant gravitons can be constructed for AdS$_p \times $S$^q$.

These BPS branes can either expand on contractible cycles on AdS or on the sphere and it is their angular momentum that prevents these objects from collapsing. If $p = 3$, the giant gravitons can be expanded to an arbitrary size, while for $p \geq 3$ angular momentum will determine the size of giant gravitons. The giant gravitons to branes wrapping cycles on both AdS and the sphere on AdS$_3 \times $S$^3$ are generalized in [19]. The extension to the AdS$_2 \times $S$^2$ background is similar to the case of AdS$_3 \times $S$^3$ but the ‘giant gravitons’ are point–like.

In the previous chapter, we applied quantization only to light excitations. This procedure gives us the canonical structure of the dual quantum field theory, but it is not sufficient for counting all microscopic states. One should also include the high energy excitations, which are described by the supersymmetric branes on the AdS$_2 \times $S$^2$ background.
In section 3.2, we first give a review of the Dirac bracket of a single giant graviton in $\text{AdS}_5 \times S^5$ in the reduced phase space, and then we show that the result derived from LLM solutions is consistent with the one derived from DBI action in section 3.3. In the section 3.4, we pay our attention on supersymmetric D branes on $\text{AdS}_2 \times S^2$ background. A discussion is given in the last section.

3.2 Review of giant graviton

In this section we would like to review the Dirac bracket of a single giant graviton on $\text{AdS}_5 \times S^5$ background from DBI action [50]. The DBI action is described by a constrained Hamilton system and thus in appendix B.1 we give a short review of Dirac’s method for tackling such systems.

A giant graviton on $\text{AdS}_5 \times S^5$ is a D3 brane extending in $S^5$ which corresponds to a hole in the circular droplet in LLM solutions (see appendix A.2). We consider a D3 brane with the embedding in static gauge as

$$t = \tau, \quad \theta = \theta(\tau), \quad \phi = \phi(\tau), \quad \tilde{\Omega}_i = \sigma_i, \quad \rho = 0.$$  

(3.1)

We would like to utilize the global coordinates which enable us to express metric of $\text{AdS}_5 \times S^5$ as

$$ds^2 = r_0 \left[ -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\tilde{\Omega}_3^2 \right]$$  

(3.2)

We then can write down the DBI action for this D3 brane as

$$S = N \int d\tau \left( -\sin^3 \theta \sqrt{1 - \cos^2 \theta \dot{\phi}^2 - \dot{\theta}^2 - \sin^4 \theta \dot{\phi}} \right)$$  

(3.3)

Here the prefactor $N$ is the same $N$ in (A.22). Note that in appendix A.2 $N$ is given by

$$N = \frac{\pi r_0^2}{4\pi^2 l_p^2} = \frac{r_0^2}{4\pi l_p^2}.$$  

(3.4)

From the definition of DBI action $N$ should be

$$N = T_3 \omega_3 r_0^2$$  

(3.5)
where $T_3 = 1/(8\pi^3 l_p^4)$ is the tension of D3 brane and $\omega_3 = 2\pi^2$ is the volume of the unit $S^3$. Thus it is easy to check that the $N$ in (A.22) is the same $N$ in (3.3).

The configuration space of the giant graviton is parametrized by $\theta(\tau)$ and $\phi(\tau)$, which leads to a four-dimensional phase space parametrized by $\theta(\tau)$, $p_\theta(\tau)$ and $\phi(\tau)$, $p_\phi(\tau)$. The conjugate momenta derived from the DBI action (3.3) is given by

$$p_\theta = 0, \quad p_\phi = -N \sin^2 \theta$$  \hspace{1cm} (3.6)

or, alternatively we have [51]

$$\dot{\theta} = 0, \quad \dot{\phi} = -1.$$  \hspace{1cm} (3.7)

Here we should point out that (3.6) is the result of the BPS condition. To see this, we can write the Hamiltonian from action (3.3)

$$H = \left[ p_\phi^2 + p_\theta^2 + \tan^2 \theta(p_\phi + N \sin^2 \theta)^2 \right]^{1/2}$$  \hspace{1cm} (3.8)

From this expression we see that for fixed $p_\phi$ we must have $H \geq |p_\phi|$ (this is the BPS bound), and equality is achieved only when the conditions $p_\theta = 0$ and $p_\phi + N \sin^2 \theta = 0$ are imposed. On the other hand, we have $p_\theta = \partial L/\partial \dot{\theta}$ and $p_\phi = \partial L/\partial \dot{\phi}$, and from these equations we can show that our BPS conditions then imply (3.7).

We see that (3.6) do not give the relation between momenta and velocities, we thus should consider them as two primary constraints according to appendix B.1

$$f_1 = p_\theta, \quad f_2 = p_\phi + N \sin^2 \theta.$$  \hspace{1cm} (3.9)

Imposing these two constraints can reduce the four-dimensional phase space to two-dimensional one which is coordinatized by $\theta(\tau)$, $\phi(\tau)$. It easy to check that there are no secondary constraints and these two constraints are second class constraints. As a consequence, we can find the Dirac bracket as

$$\{ \theta, \phi \}_{DB} = \frac{1}{N \sin 2\theta}.$$  \hspace{1cm} (3.10)

Promoting this to commutator, we see that the two coordinates in this reduced phase space are not commutative. The Hamiltonian of the giant graviton in the reduced phase space is
given by
\[ H' = p_{\phi} \dot{\phi} = -p_{\phi} = N \sin^2 \theta. \] (3.11)

We see this is exact BPS condition, and thus we say this D3 brane is supersymmetric.

### 3.3 Consistency to microscopic description

A D3 brane in LLM description can be identified as a hole in the droplet. In appendix A.2, the relation between LLM coordinates and \( \rho, \theta \) is given by
\[ r = r_0 \cosh \rho \cos \theta, \quad y = r_0 \sinh \rho \sin \theta \] (3.12)

Noting in LLM description the location of D3 brane (3.1) is given in the \( y = 0 \) plane and
\[ r_D = r_0 \cos \theta. \] (3.13)

The Energy of this D brane can be computed from (A.23). Let us consider, at \( y = 0 \) plane, a small circular hole with radius \( r_h \) \( (r_h \ll r_0) \) centered at \( (r_D, \theta_D) \) and then the points in this hole can be expressed as
\[ (x_1, x_2) = (r_D \cos \theta_D + a(\alpha) \cos \alpha, r_D \sin \theta_D + a(\alpha) \sin \alpha) \] (3.14)

where \( 0 \leq a(\alpha) \leq r_h \) and \( 0 \leq \alpha \leq 2\pi \). Since this is a hole inside the circular droplet, we should change the sign when apply (A.23) to it. Then the total energy of this system (a supersymmetric D3 brane on \( \text{AdS}_5 \times S^5 \)) is given by
\[ \Delta = \Delta_{\text{circular}} - \Delta_{\text{hole}} = 0 - \Delta_{\text{hole}} = -\int_{\text{hole}} \frac{d^2 x}{2\pi \hbar} \frac{1}{\hbar} \left( \int_{\text{hole}} \frac{d^2 x}{2\pi \hbar} \right)^2 \]
\[ = -\frac{1}{8h^2} r_h^2 (r_h^2 - 2r_0^2 + 2r_D^2) \]
\[ \approx \frac{1}{4h^2} r_h^2 (r_0^2 - r_D^2) = \frac{1}{4h^2} r_h^2 r_0^2 \sin^2 \theta \] (3.15)
The area of the small hole is $2\pi \hbar$ thus give $r_h^2 = 2\hbar$. In (A.22), we read $r_0^2 = 2\hbar N$. Substituting $r_h$ and $r_0$ into the above equation we find

$$\nabla = \frac{1}{4\hbar^2} r_h^2 r_0^2 \sin^2 \theta = \frac{1}{4\hbar^2} (2\hbar)(2\hbar N) \sin^2 \theta = N \sin^2 \theta$$

(3.16)

which perfectly coincides with the results derived from DBI action (3.11).

### 3.4 Supersymmetric D branes on AdS$_2 \times$S$^2$ background

In this section, we will first study supersymmetric D3 branes on AdS$_2 \times$S$^2$ background from DBI action, deriving its angular momentum and energy. As we will see, the phase space of the D branes is reduced to the 2–dimensional from the 6–dimensional. We also consider the grand canonical partition function obtained from a one–dimensional simple harmonic oscillator Hamiltonian. We then turn to the bubbling geometries and compute the angular momentum from gravity side.

#### 3.4.1 Counting BPS states in reduced phase space

We start this part with AdS$_2 \times$S$^2$ space in the global coordinates

$$ds^2 = L^2 \left[ -(\rho^2 + 1) dt^2 + \frac{d\rho^2}{\rho^2 + 1} + d\theta^2 + \sin^2 \theta d\phi^2 \right] + dz_k d\bar{z}_k$$

$$F_5 = \frac{L}{4} d\rho dt \wedge \text{Re}(dz_{123}) + \text{dual},$$

(3.17)

where

$$z_k = X_k + iY_k, \quad k = 1, 2, 3$$

(3.18)

are the coordinates on T$^6$.

We are interested in putting D3 branes on above background (A.34), embedding static gauge for the worldvolume:

$$t = \tau, \quad X_1 = \xi_1 \cos \beta, \quad Y_1 = \xi_1 \sin \beta, \quad X_2 = \xi_2, \quad X_3 = \xi_3, \quad Y_2 = Y_3 = 0.$$  

(3.19)
Then one can find that the DBI action

\[ S = -T \int d\xi \sqrt{-\det \left[ g^{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^n}{\partial \xi^b} \right]} + 4T \int P[C_4], \]  

(3.20)

for these D branes on AdS$_2 \times$S$^2$ is

\[ S = -TL \int d^4\xi \sqrt{(\rho^2 + 1) - \frac{\dot{\rho}^2}{\rho^2 + 1} - \sin^2 \theta \dot{\phi}^2 + TL \int d^4\xi \left[ \cos \beta \rho + \sin \beta \cos \theta \dot{\phi} \right]}. \]  

(3.21)

Equations of motion for fields ($\rho, \theta, \phi$) can be written as

\[ \dot{\rho} = 0, \quad \dot{\theta} = 0, \quad \dot{\phi} = 1, \]  

(3.22)

under the conditions

\[ \frac{\rho}{\sqrt{\rho^2 + 1 - \sin^2 \theta \dot{\phi}^2}} = \cos \beta, \quad \frac{\cos \theta \dot{\phi}}{\sqrt{\rho^2 + 1 - \sin^2 \theta \dot{\phi}^2}} = \sin \beta. \]  

(3.23)

Again, similarly to (3.7), (3.22) is the requirement of supersymmetry, which can be seen clear from the Hamiltonian as (3.8). It is easy to check, the Eqs. (3.22) is equivalent to

\[ p_\rho = p_\theta = 0, \quad p_\phi = \frac{TLV}{\sqrt{\rho^2 + \cos^2 \theta}} = \frac{TLV \cos \beta}{\rho}, \]  

(3.24)

Where \( V = \int d\xi_1 d\xi_2 d\xi_3 \). When \( \rho = 0 \), the D3 brane become the AdS$_2 \times$S$^2$ counterpart of the giant graviton presented in previous section 3.2, while \( \theta = \pi/2 \), it is the counterpart of dual giant graviton. In this part, we focus on the counterpart of dual giant graviton, which require \( \beta = 0 \), and thus (3.24) changes to

\[ p_\rho = p_\theta = 0, \quad p_\phi = \frac{TLV}{\rho}. \]  

(3.25)

27
The energy of D branes is calculated as

\[ E = p_\phi \dot{\phi} - L \]
\[ = VTL \left( \frac{\sin^2 \theta}{\sqrt{\rho^2 + \cos^2 \theta}} + \sin \beta \cos \theta \right) \dot{\phi} + TL \left( \sqrt{\rho^2 + \cos^2 \theta} - \cos \beta \rho - \sin \beta \cos \theta \right) \]
\[ = \frac{TLV}{\sqrt{\rho^2 + \cos^2 \theta}} \]  

(3.26)

We see \( E = p_\phi = J \), which is exactly the BPS condition. Therefore, the D branes are supersymmetric.

The configuration space of the brane probes is labeled by \( \rho(\tau), \theta(\tau), \phi(\tau) \), which correspond to a 6–dimensional phase space parametrized by \( \rho(\tau), p_\rho(\tau), \theta(\tau), p_\theta(\tau), \phi(\tau), p_\phi(\tau) \). As can be seen from (3.25), the angular momenta are written as constant or function of coordinates but the velocities. Thus to quantize this system, one needs to consider Dirac bracket in reduced phase space spanned by \( \rho, \phi \).

There are three primary constraints given by (3.25)

\[ f_1 = p_\rho, \quad f_2 = p_\theta, \quad f_3 = p_\phi - \frac{TLV}{\rho}. \]  

(3.27)

It can be checked according to appendix B.1, there are no secondary constraints and only two of them is second class constraints, \( f_1, f_3 \). It is straightforward to compute the Poisson bracket of the these two constraints:

\[ \{f_1, f_3\}_{PB} = -\frac{LTV}{\rho^2}. \]  

(3.28)

Define matrix \( M_{ab} \) as \( M_{ab} = \{f_a, f_b\}_{PB} \), where \( f_a, f_b \) are second class constraints. One can easily find this matrix as follow

\[ M_{ab} = \begin{bmatrix} 0 & -\frac{LTV}{\rho^2} \\ \frac{LTV}{\rho^2} & 0 \end{bmatrix} \]  

(3.29)

Recall that one can obtain Dirac bracket from Poisson bracket and matrix \( M_{ab} \) through the relation

\[ \{q_i, q_j\}_{DB} = \{q_i, q_j\}_{PB} - \{q_i, f_a\}_{PB} M_{ab}^{-1} \{f_b, q_j\}_{PB}. \]  

(3.30)
Thus it is straightforward to find the Dirac bracket of $\rho$ and $\phi$ as

$$\{\rho, \phi\}_{DB} = \frac{\rho^2}{LTV}. \quad (3.31)$$

Defining the new coordinates as

$$\eta = \frac{1}{\sqrt{\rho}} e^{-i\phi}, \quad \bar{\eta} = \frac{1}{\sqrt{\rho}} e^{i\phi} \quad (3.32)$$

one then can rewrite the Dirac bracket (3.31) as the bracket for new variables $\eta$ and $\bar{\eta}$

$$\{\eta, \bar{\eta}\}_{DB} = -\frac{i}{TL}. \quad (3.33)$$

Therefore, the reduced two-dimensional phase space is simply complex space equipped with the symplectic form as

$$\Omega = iTLd\eta \wedge d\bar{\eta}. \quad (3.34)$$

With the help of the new coordinates (3.32), one can rewrite Hamiltonian of the system as

$$H = J = TLV \eta \bar{\eta} \quad (3.35)$$

This is the Hamiltonian of a one-dimensional simple harmonic oscillator. Thus the Hilbert space of it is given by eigenvectors of harmonic oscillator which are produced from the rising operator,

$$|n\rangle = \frac{a^\dagger}{\sqrt{n!}} |0\rangle \quad (3.36)$$

where we have define creation and annihilation operators as

$$a = \sqrt{TLV} \eta, \quad a^\dagger = \sqrt{TLV} \bar{\eta}. \quad (3.37)$$

Setting $\hbar = 1$, the commutation relation can be achieved by multiply $i$ to Dirac bracket. Thus noting bracket (3.33), we find the commutation relation for $a$ and $a^\dagger$

$$[a, a^\dagger] = 1, \quad (3.38)$$
and the Hamiltonian can be rewritten as

$$H = a a^\dagger \equiv n.$$ \hspace{1cm} (3.39)

If we consider \( N \) supersymmetric D3 branes, then the total angular momentum should be written as the sum of angular momentum of the individual D brane

$$J = \sum_{k=1}^{N} J_k.$$ \hspace{1cm} (3.40)

Therefore, according to the standard definition, the grand canonical partition function is expressed as

$$Z(\zeta, q) \equiv \text{Tr} \exp[-\mu N - \beta H] = \prod_{n=0}^{\infty} \frac{1}{1 - \zeta q^n},$$ \hspace{1cm} (3.41)

where \( q = e^{-\beta}, \zeta = e^{-\mu} \). On the other hand, the grand canonical partition function also can be written as

$$Z(\zeta, q) = \sum_{N=0}^{\infty} \zeta^N Z_N(q)$$ \hspace{1cm} (3.42)

with

$$Z_N(q) = \sum_{J=0}^{\infty} \omega_N(J) q^J.$$ \hspace{1cm} (3.43)

where \( \omega_N(J) \) is the degeneracy.

### 3.4.2 Angular Momentum from supergravity solutions

A family of supersymmetric solutions (A.25) of ten–dimensional supergravity, which are asymptotic to \( \text{AdS}_2 \times S^2 \times T^6 \), is found in [19]. The regularity of these geometries requires that all the sources of two harmonic functions \( H_1 \) and \( H_2 \) are distributed along closed curves in \( \mathbf{R}^3 \). These harmonic functions are calculated from the integral (A.27)

$$H = H_1 + iH_2 = \frac{1}{2\pi} \int \frac{\sqrt{(d - F)(d - F + A)}}{(d - F)^2} dl$$ \hspace{1cm} (3.44)
where

\[
A(l) = S(l) - i\tilde{F}(l) \times S(l)
\]

\[
= \left( S_1 - iS_3 \tilde{F}_2, S_2 + iS_3 \tilde{F}_1, S_3 + i(S_1 \tilde{F}_2 - S_2 \tilde{F}_1) \right),
\]

\[
d = (r \cos \phi, r \sin \phi, y).
\]  

(3.45)

Under the conditions that \(|d| \gg |F|\) and \(|d| \gg |S|\), we can expand these harmonic functions \(H_1\), and \(H_2\) as

\[
H_1 = \frac{Q}{\sqrt{r^2 + y^2}} + \frac{I_{S_3} y + I_1 \cos \phi + I_2 \sin \phi}{2(r^2 + y^2)^{3/2}}
+ \frac{W_{s_1}(r^2 - 2y^2) + I_3 \cos \phi + I_4 \sin \phi + I_5 \cos 2\phi + I_6 \sin 2\phi}{16(r^2 + y^2)^{5/2}},
\]

\[
H_2 = \frac{Wy + I_7 \cos \phi + I_8 \sin \phi}{(r^2 + y^2)^{3/2}}
+ \frac{W_{s_2}(r^2 - 2y^2) + I_9 \cos \phi + I_{10} \sin \phi + I_{11} \cos 2\phi + I_{12} \sin 2\phi}{8(r^2 + y^2)^{5/2}},
\]  

(3.46)

where

\[
Q = \frac{1}{2\pi} \int dl, \quad I_{S_3} = \frac{1}{2\pi} \int S_3 dl,
\]

\[
W_{s_1} = \frac{1}{2\pi} \int \left[ S_3^2 - S_1^2 - S_2^2 + 4(F_1^2 + F_2^2) \right] dl,
\]

\[
W = \frac{1}{2\pi} \int (S_1 \dot{F}_2 - S_2 \dot{F}_1) dl,
\]

\[
W_{s_2} = \frac{1}{2\pi} \int S_3 (S_1 \dot{F}_2 - S_2 \dot{F}_1) dl,
\]  

(3.47)

and \(I_1 \cdots I_{12}\) are some integrals along the profiles.

Starting with harmonic functions \(H_1\) and \(H_2\) and using (A.26), we have

\[
h^2 = H_1^2 + H_2^2,
\]

\[
\frac{\partial V_\phi}{\partial y} = -2r \left( H_1 \frac{\partial H_2}{\partial r} - H_2 \frac{\partial H_1}{\partial r} \right),
\]

\[
\frac{\partial V_r}{\partial y} = \frac{2}{r} \left( H_1 \frac{\partial H_2}{\partial \phi} - H_2 \frac{\partial H_1}{\partial \phi} \right).
\]  

(3.48)
Here we have chosen gauge $V_y = 0$. Substituting $h, V_r, V_\phi$ into metric (A.25), we then can express the metric in terms of the integrals $Q, W$ and so on. Since the region $y > 0$ is not geodesically complete, an analytic continuation is required. This can start with a harmonic function with $y > 0$, and then introduce other three sheets [19]:

$$
H_A(x_1, x_2, y) = H(x_1, x_2, y), \quad H_B(x_1, x_2, y) = \overline{H(x_1, x_2, -y)}
$$

$$
H_C(x_1, x_2, y) = -\overline{H(x_1, x_2, -y)}, \quad H_D(x_1, x_2, y) = -H(x_1, x_2, y)
$$

(3.49)

Harmonic functions $H_A, H_B$ give the same metric and $H_C, H_D$ give the same metric due to taking conjugate of $H(x_1, x_2, y)$ and minus sign for $y$ change both the sign of $V_r$ and $V_\phi$. Taking a minus sign of harmonic function $H(x_1, x_2, y)$ changes nothing, and thus the four sheets will give the same metric.

In order to analyze the asymptotic form of the metric, it is convenient to use coordinates $u, v$, which are defined as $y = Quv$, then

$$
r = Qv\sqrt{1 - u^2} + \frac{C_1(u, \phi)}{v} + \frac{C_2(u, \phi)}{v^3}
$$

(3.50)

The reason for using $u, v$ coordinates is that when $v$ goes to infinity the metric will reduce to $\text{AdS}_2 \times S^2$. Making a change of variables as

$$
u \to u + \frac{f_1}{v^2} + \frac{f_2}{v^4}, \quad v \to v + \frac{g_1}{v} + \frac{g_2}{v^3}, \quad \phi \to \phi + \frac{p_1}{v^2} + \frac{p_2}{v^4},
$$

(3.51)

Picking up a appropriate form of $p_1$, we will get

$$
\frac{g_\phi}{g_{\phi\phi}} = \frac{W}{2Q^3} + o(\frac{1}{v^2}),
$$

(3.52)

Then we can define

$$
D\phi^2 = (d\phi + \frac{1}{Q}dt - \frac{2Q^2 - W}{2Q^3}dt)^2
$$

$$
= (d\phi - \frac{2Q^2 - W}{2Q^2}dt)^2,
$$

(3.53)
where

\[ d\tilde{\phi} = d\phi + \frac{1}{Q} dt. \] (3.54)

What we can obtain from these formulae is the expression for angular momentum, which is also the energy due to the BPS condition,

\[ E = J = \frac{Q^6}{16\pi^2 l_p^8}(Q^2 - \frac{W}{2}). \] (3.55)

To compare this result with the one found from DBI action. We consider a circular profile with a radius L and a small circular profile with a radius \( r_0 \). The big one gives AdS_2 \times S^2, while the small profile is equivalent to a supersymmetric D brane on AdS_2 \times S^2. We can write down the vectors \( \mathbf{F} \) and \( \mathbf{S} \) for the big profile as

\[ \mathbf{F}_L = \left( L \cos \left( \frac{q}{L} \right), L \sin \left( \frac{q}{L} \right), 0 \right), \]
\[ \mathbf{S}_L = 2L \left( \cos \left( \frac{q}{L} \right), \sin \left( \frac{q}{L} \right), 0 \right), \] (3.56)

and for the small profile as

\[ \mathbf{F}_{r_0} = \left( R_0 \cos w + r_0 \cos \left( \frac{q}{r_0} \right), R_0 \sin w + r_0 \sin \left( \frac{q}{r_0} \right), 0 \right), \]
\[ \mathbf{S}_{r_0} = 2r_0 \left( \cos \left( \frac{q}{r_0} \right), \sin \left( \frac{q}{r_0} \right), 0 \right). \] (3.57)

Here we consider the case \( R_0 > L \), which implies that is the AdS_2 \times S^2 counterpart of dual–giant graviton in higher dimensions [48]. Applying these profiles into (3.47), we find

\[ Q = L + r_0, \quad W = 2L^2 + 2r_0^2. \] (3.58)

When (3.58) are substituted into (3.55), we find

\[ J \propto 2LR_0(L + r_0)^6. \] (3.59)

This is a constant value which is different from (3.25) or (3.26) that depends on \( 1/\rho \):

\[ J = \frac{TLV}{\rho} = \frac{TL^2V}{\sqrt{R_0^2 - L^2}}. \] (3.60)
We see that the $J$ in (3.59) does not depends on $R_0$ but the $J$ in (3.60) does. This difference may indicate problems with the expression obtained from supergravity side. In appendix B.2 we also study the contributions at higher orders, but unfortunately the discrepancy between brane picture and supergravity persists.

3.5 Discussion

In this chapter we first derived the angular momentum and energy for D branes on the background of AdS$_2 \times S^2$ from DBI action. The phase space can be reduced to two dimensions, which has a symplectic form (3.34). On the the geometric side, we derived the expression for angular momentum from asymptotic metric. Unfortunately, this result is not consistent with the one obtained from DBI action. This discrepancy indicates that our method of extracting charges, that has been originally developed for gauged supergravities in higher dimensions, fails for AdS$_2$. It would be interesting to find a proper way of dealing with this low-dimensional space.
CHAPTER 4

Hidden symmetries of superstrata

4.1 introduction

The significance of symmetries in the development of physics has been presented by countless examples. In this chapter we are interested in the symmetries of spacetimes, which can help us not only to understand the properties of spacetimes but also to study the dynamics of particles in these spacetimes. According to Noether’s theorem, these symmetries must correspond to some conserved quantities. Symmetries generated by the Killing vectors (KV) usually are called explicit symmetries that give the isometries of spacetimes. In Hamiltonian formalism, the motion of a free particle in curved space governed by the Hamiltonian that is quadratic in momenta \( p_M \). If there is a Killing vector \( \xi^M \), we then have a quantity \( p_M \xi^M \) which is a constant along geodesics of the particle. These conserved quantities are integrals of motion which are observables of phase space whose Poisson brackets with Hamiltonian vanish.

There are some conserved quantities that are not linear in momenta as given by Killing vectors, which are integrals of motion that is the higher-order polynomials in particle momenta. We call the symmetries correspond to these conserved quantities as hidden symmetries which are represented by Killing tensors (KT) [52, 53] and Killing–Yano tensors (KYT) [54]. These symmetries indicate the separability of differential equations which leads to the integrability of equation of motions. Specifically, the existence of symmetric tensors, that is Killing tensors, implies that the Hamilton–Jacobi, Klein–Gordon equations are separable, while the separability of Dirac equations are guaranteed by Killing–Yano tensors which is antisymmetric. For example, the geodesic equations in Kerr spacetime are fully integrated thanks to a nontrivial Killing tensor of rank two, which allows us to separate Hamilton–Jacobi, KleinGordon equations [20]. Equations for the gravitational and electromagnetic perturbations are also separable [21]. It had been proven by [55] that this Killing tensor is the square of a Killing–Yano tensor of rank 2 which implies the integrability of Dirac equation [22] in the Kerr background. Killing and Killing–Yano tensors later were found in
other geometries not only in general relativity [56] but also in string theory [57, 58].

In this chapter we would like to consider the hidden symmetries of superstrata. A
review of superstrata is given in appendix C.1. Superstrata are microstate geometries which
are regular, horizonless and asymptotically to AdS\(_3\) \(\times\) S\(_3\). It had been shown in [59] that for a
special families of superstratum metrics there are a set of conserved quantities and the null
geodesic is completely integrable since the existence of a nontrivial conformal Killing tensor.
This tensor indicates the separability of the massless scalar wave equation. However we do
not know if the separability of massless Dirac equation in these geometries is allowed or not,
which is the question we want to answer in this chapter. Other similar studies on hidden
symmetries or integrability on two–charge system and geometries stacked by D branes can
be found in [58, 60].

In section 4.2, we will give basic definitions of (conformal) Killing vectors and (con-
formal) Killing and Killing–Yano tensors. In section 4.3, we review the nontrivial conformal
Killing tensor for a special families of superstratum metrics found in [59]. Our work is
presented in section 4.4, where separability of the massless Dirac equations is investigated.

### 4.2 (Conformal) Killing(–Yano) tensors

If there exits a transformation which keeps the metric of the spacetime and all other
quantities obtained from metric invariant, then we can say there is a symmetry in this
spacetime. The symmetries in phase space are easily reduced to configuration space are called
explicit symmetries, while for the symmetries in phase space have no direct counterpart in
configuration space are called hidden symmetries.

Let us start with a explicit continuous transformation along a vector \(V\) in a spacetime

\[
x^M \to x^M + \epsilon V^M(x).
\]

The variation of a tensor caused by this transformation is given by Lie derivative. For a
(0, 2) tensor, its Lie derivative along vector \(V\) is

\[
L_V T_{MN} = T_{MN,L}V^L + T_{MP}V^P_N + T_{PN}V^P_M
\]

The variation of a tensor caused by this transformation is given by Lie derivative. For a
(0, 2) tensor, its Lie derivative along vector \(V\) is

\[
L_V T_{MN} = T_{MN,L}V^L + T_{MP}V^P_N + T_{PN}V^P_M
\]
Thus if we require this transformation is a symmetry of this geometry, it has to keep the metric invariant, which indicates that the Lie derivative of the metric should vanish

\[ L_V g_{MN} = 0. \]  

(4.3)

Applying Lie derivative of a tensor (4.2) to metric, one can obtain an equation for vector \( V \)

\[ \nabla_M V_N + \nabla_N V_M = 0. \]  

(4.4)

In other words, the transformation along a vector \( V \) satisfying (4.4) render the geometry invariant. We call such a vector as Killing vector. Symmetries corresponding to Killing vectors are called explicit symmetries, including translational, rotational symmetries and so on. Since any combination of these symmetries is again an symmetry of metric, they form a Lie group called isometry group. The corresponding Lie algebra is formed by the generators of these symmetries, i.e. the Killing vectors. The conserved quantities corresponding to these symmetries are given by an integral of motion

\[ I = V_M \frac{dx^M}{ds}, \]  

(4.5)

which is a constant along all geodesics.

The conserved quantities (4.5) in phase space can be written as \( p^M V_M \) which are linear in momenta. There indeed exits an integral of motion that is not linear in momentum but as

\[ I = p^M p^N K_{MN} \]  

(4.6)

where the tensor \( K_{MN} \) is symmetric. The condition for \( I \) is an integral of motion is given by the vanishing Poisson bracket

\[ \{ I, H \} = 0. \]  

(4.7)

Substituting Hamiltonian \( H = 1/2g_{MNP}p^M p^N \) in to above bracket, we have

\[ \{ I, H \} = \nabla_{(L} K_{MN)} p^L p^M p^N = 0. \]  

(4.8)
Thus we define that a symmetric tensor obey the equation

\[ \nabla_M K_{NL} + \nabla_N K_{LM} + \nabla_L K_{MN} = 0 \] (4.9)

is called a Killing tensor [52, 53]. Since the covariant derivative of metric tensor is zero, there is at least one Killing tensor for all spacetime, that is the spacetime metric itself. The corresponding conserved quantity given by metric tensor is the Hamiltonian for the relativistic particle which is the square of particle’s mass. It easy to prove that the Killing tensor can be constructed from Killing vectors as

\[ K_{MN} = \sum_{i,j} a_{ij} V^i_M V^j_N \] (4.10)

where \(a_{ij} = a_{ji}\) are constant coefficients. The corresponding conserved quantities to Killing tensor depend on particle’s momenta are given by (4.6). The phase space symmetries generated by Killing tensors have no a simple corresponding description in the spacetime, which is the reason for us calling them as hidden symmetries. However, studying the particles’ dynamics in spacetime, one can detect the presence of these symmetries.

All the quantities we discussed above are conserved along any geodesic. Now we would like to do the generalization to quantities which are only conserved along null geodesic and we call the corresponding vectors and tensors as conformal Killing vectors and conformal Killing tensors. A vector is called conformal Killing vector (CKV) if it satisfies the equation

\[ \nabla_M V_N + \nabla_N V_M = \alpha g_{MN}, \] (4.11)

where \(\alpha\) is an arbitrary function. It is clear that Killing vectors are special conformal Killing vectors with conformal factor \(\alpha = 0\). The integral of motion generated by conformal Killing vector is given by

\[ I = l_M V^M, \] (4.12)

where \(l_M\) is the momentum of massless particles whose geodesic equation is given by

\[ l^M \nabla_M l^N = 0. \] (4.13)
It is straightforward to prove that (4.12) is indeed a constant along null geodesic

\[
\dot{I} = l^M \nabla_M I = l^M \nabla_M (l^N \nabla_N) = l^M l^N \nabla_M \nabla_N = l^M l^N \nabla_{(M} \nabla_{N)} = \alpha l^M l^N g_{MN} = 0. \tag{4.14}
\]

A symmetric tensor is a conformal Killing tensor (CKT) if it obeys the equation \([53, 61]\)

\[
\nabla (L K_{MN}) = W(L g_{MN}), \tag{4.15}
\]

where \(Z_L\) is usually called associated vector of conformal Killing tensor \(K_{MN}\). The conserved quantity corresponding to conformal Killing tensor is given by integral of motion

\[
I = l_M l_N K^{MN}, \tag{4.16}
\]

which is a constant along null geodesic. It is easy to prove that metric multiplied by an arbitrary function is a conformal Killing tensor:

\[
\nabla (P [F g_{MN}]) = \nabla (P F) g_{MN} + F \nabla (P g_{MN}) = \nabla (P F) g_{MN}, \tag{4.17}
\]

where \(F\) is arbitrary function. The associated vector is given by \(\nabla_P F\).

Killing tensors and conformal Killing tensors can be generalized to complete symmetric tensors of rank \(k\) as

\[
\nabla (M_0 K_{M_1 M_2 \cdots M_k}) = 0
\]

\[
\nabla (M_0 K_{M_2 \cdots M_k}) = W(M_0 \cdots M_{k-2} g_{M_{k-1} M_k}), \tag{4.18}
\]

where \(W_{M_0 \cdots M_{k-2}}\) is symmetric tensor when \(k \geq 2\). Conformal Killing tensors can be extended to standard Killing tensors under certain condition: if there is a conformal Killing tensor \(K_{MN}\) with \(W_M = -\nabla_M \phi\), one then can construct a Killing tensor as

\[
K_{MN} = K_{MN} + \phi g_{MN}. \tag{4.19}
\]

This can be shown by taking a covariant derivative of the Killing tensor

\[
\nabla (P K_{MN}) = \nabla (P K_{MN}) + \nabla (P \phi g_{MN}) = 0. \tag{4.20}
\]
Here the second equality using the definition of conformal Killing tensor (4.15).

There are antisymmetric tensors that are closely related to Killing tensors. Different from Killing tensors that indicate the separability of Hamilton–Jacobi and Klein–Gordon equation, these antisymmetric tensors enable Dirac equation to be separable. We call these antisymmetric tensors as Killing–Yano tensors which are defined by [54]

\[ \nabla_M Y_{NP} + \nabla_N Y_{MP} = 0, \quad Y_{MN} = -Y_{NM}. \] (4.21)

For the generalized Killing–Yano tensors with rank \( k \), the equation satisfied is [62]

\[ \nabla_{(M} Y_{N)P_1\ldots P_k} = 0, \quad Y_{P_1\ldots P_k} = Y_{[P_1\ldots P_k]}. \] (4.22)

Similarly to the generalization of conformal Killing tensor, there are conformal Killing–Yano tensor (CKYT) [62]

\[ \nabla_{(M} Y_{M_1\ldots M_k N)} = g_{M_1 M_2} Z_{M_3\ldots M_{k+1}} + \sum_{i=3}^{k+1} (-1)^i g_{M_i (M_1 Z_{M_2}\ldots M_{i-1} M_{i+1}\ldots M_{k+1})} \] (4.23)

where \( Z \) is an antisymmetric tensor.

If we have a Killing–Yano tensor, there always have a Killing tensor given by

\[ K_{MN} = Y_{M A_1\ldots A_{k-1}} Y_{N}^{A_1\ldots A_{k-1}}. \] (4.24)

Similarly, any conformal Killing–Yano tensor can give rise to a conformal Killing tensor of rank two as

\[ \mathcal{K}_{MN} = Y_{M A_1\ldots A_{k-1}} Y_{N}^{A_1\ldots A_{k-1}}, \quad W_M = 2 Y_{M A_1\ldots A_{k-1}} Z^{A_1\ldots A_{k-1}}. \] (4.25)

These two relations imply that there must exit a (conformal) Killing tensor if there is (conformal) Killing–Yano tensor. However, the existence of a (conformal) Killing tensor cannot guarantee the existence of a (conformal) Killing–Yano tensor. In other words, the separability of Dirac equation always implies that the Klein–Gordon equation is separable with the same coordinates.
4.3 Conformal Killing tensor of superstrata

Dynamics of particles and fields in curved backgrounds is described by partial differential equations, which often turn out to be quite complicated. Nevertheless, for large classes of black hole geometries, the PDEs turn out to be integrable due to some hidden symmetries of the spacetimes. Such symmetries are often encoded in Killing vectors and tensors. In this section we review the conformal Killing tensor of a special family of supergravity solutions presented in appendix C.1.2. As shown in [59] these geometries admit full separation of variables, and in this section we will derive the relevant conformal Killing tensor from Hamilton–Jacobi equation studied in [59], and our result will be used in the next section.

Killing tensor implying the separability of Hamilton–Jacobi equation indicates that we can extract Killing tensor from separability of the equation

\[ g^{MN} \partial_M S \partial_N S + \mu^2 = 0. \]  

(4.26)

In this part we focus on one notion of separability for equation (4.26) and assume that

\[ S = S(x_1 \cdots x_k) + S(x_{k+1} \cdots x_n) + S_0(x_1 \cdots x_k) \]  

(4.27)

where \( S_0(x_1 \cdots x_k) \) is a known function of its arguments [63], which has no effects on our discussion here.

There three conditions that have to be satisfied such that equation (4.26) can be separated [58]:

1. There are three groups coordinates \( x^M \): one is cyclic coordinates \( z \) which need metric without dependence on \( z \); other two groups denoted as \( x \) and \( y \).

2. A certain function \( f \) exists and it enables the following equations to be obeyed

\[ g^{MN} = \frac{1}{f} (X^{MN} + Y^{MN}), \quad \partial_x Y^{MN} = \partial_y X^{MN} = 0, \]

\[ X^{x^M} = 0, \quad Y^{x^M} = 0. \]  

(4.28)
3. The decomposition of the function $f$ such that

$$f = f_x - f_y, \quad \partial_y f_x = 0, \quad \partial_x f_y = 0, \quad \partial_z f_x = \partial_z f_y = 0. \quad (4.29)$$

Under these three conditions, one can express Hamilton–Jacobi equation (4.26) as

$$X^{MN} \partial_M S \partial_N S + \mu^2 f_x = -Y^{MN} \partial_M S \partial_N S + \mu^2 f_y. \quad (4.30)$$

The left–hand side of this equality only depends on $x$, while the right–hand side only depends on $y$, which implies there is an integral of motion as

$$I \equiv [X^{MN} - f_x g^{MN}] \partial_M S \partial_N S. \quad (4.31)$$

As we know that an integral of motion generated by a Killing tensor, there must exist a Killing tensor as

$$I = K^{MN} \partial_M S \partial_N S. \quad (4.32)$$

Thus the Killing tensor related to the separability of the Hamilton–Jacobi equation is given by

$$K^{MN} = X^{MN} - \frac{f_x}{f} (X^{MN} + Y^{MN}) = -\frac{f_y X^{MN} + f_x Y^{MN}}{f}. \quad (4.33)$$

For the massless Hamilton–Jacobi equation, that is $\mu = 0$, the third condition is unnecessary. The conformal Killing tensor associated to the separability of the massless Hamilton–Jacobi equation can be expressed as

$$\mathcal{K}^{MN} = c_1 X^{MN} + c_2 Y^{MN} \quad (4.34)$$

where $c_1$ and $c_2$ are two constants. In this section we will consider the case that the third condition is not satisfied.

Now we focus on a special family superstratum solution (C.14) in appendix C.1.2. There are four cyclic coordinates in (C.14), i.e. $(u, v, \phi_1, \phi_2)$, which immediately give us
four Killing vectors

\[ V^{(1)} = \frac{\partial}{\partial \phi_1}, \quad V^{(2)} = \frac{\partial}{\partial \phi_2}, \quad V^{(3)} = \frac{\partial}{\partial v}, \quad V^{(4)} = \frac{\partial}{\partial u}. \]  

(4.35)

The corresponding conserved quantities are momenta

\[ L_1 = V^{(1)} \frac{dx^M}{ds}, \quad L_2 = V^{(2)} \frac{dx^M}{ds}, \quad P = V^{(3)} \frac{dx^M}{ds}, \quad E = V^{(4)} \frac{dx^M}{ds}. \]  

(4.36)

As a Killing tensor, metric gives the standard quadratic conserved quantity

\[ \varepsilon \equiv g_{MN} \frac{dx^M}{ds} \frac{dx^N}{ds}. \]  

(4.37)

There is one more quadratic conserved quantity by separating variables in the massless Hamilton–Jacobi equation \[59\]

\[ \Xi \equiv K_{MN} \frac{dx^M}{ds} \frac{dx^N}{ds} \equiv Q_1 Q_5 \Lambda^2 \frac{d\theta}{\sin^2 \theta} + \frac{L_1^2}{\cos^2 \theta} + \frac{L_2^2}{\cos^2 \theta}. \]  

(4.38)

This is a constant along null geodesic which can be checked easily by calculate

\[ \frac{d}{d\lambda} \Xi = R_y \frac{d\theta}{ds} \frac{\partial \Lambda}{\partial \theta} \left( g_{MN} \frac{dx^M}{ds} \frac{dx^N}{ds} \right) = 0. \]  

(4.39)

This quadratic conserved quantity must be associated with a nontrivial conformal Killing tensor.

We now follow the procedures presented at the beginning of this section. First of all, the three group coordinates are: cyclic coordinates \((u, v, \phi_1, \phi_2)\); other two groups are \(r\) and \(\theta\). The next step is to look for function \(f\) such that the metric \(g^{MN}\) can be divided into two parts as \(X(r)\) and \(Y(\theta)\). We find one can take function \(f\) as

\[ f = Q_1 Q_5 \Lambda, \]  

(4.40)

where recall

\[ \Lambda = \sqrt{1 - \frac{a^2 b^2}{(2a^2 + b^2) (r^2 + a^2)^{n+1} \sin^2 \theta}}. \]  

(4.41)

Thus we can write metric \(g^{MN}\) by the function only depends \(r\) and the function only depends
on $\theta$

\[ g^{MN} = \frac{1}{f} \left[ X(r)^{MN} + Y(\theta)^{MN} \right]. \quad (4.42) \]

Obviously, the function $f$ cannot be decomposed as (4.29), which implies there only exits conformal Killing tensor. According to (4.34), this conformal Killing tensor can write as

\[ \mathcal{K}^{MN} = c_1 X^{MN} + c_2 Y^{MN} \quad (4.43) \]

where the components of $X^{MN}$ are given by

\[
\begin{align*}
X^{rr} &= r^2 + a^2, & X^{\theta\theta} &= 1, & x^{uv} &= \frac{a^4 \tilde{R}^2}{a^2 r^2 + r^4}, \\
X^{uu} &= \frac{\tilde{R}^2 [b^2 (2a^2 + b^2) r^{2n+2} + (a^6 - b^2 r^2 (2a^2 + b^2)) (a^2 + r^2)^n]}{a^2 r^2 (a^2 + r^2)^{n+1}}, \\
X^{\phi_1\phi_1} &= -\frac{a^2}{a^2 + r^2}, & X^{\phi_2\phi_2} &= \frac{a^2}{r^2}, & X^{uv} &= -\frac{\tilde{R}^2 [a^4 + r^2 (2a^2 + b^2)]}{r^2 (a^2 + r^2)}, \\
X^{u\phi_1} &= \frac{\tilde{R} [b^2 r^{2n} - (a^2 + b^2) (a^2 + r^2)^n]}{(a^2 + r^2)^{n+1}}, \\
X^{u\phi_2} &= -\frac{a^2 \tilde{R}}{r^2}, & X^{v\phi_1} &= -\frac{a^2 \tilde{R}}{(a^2 + r^2)}, & X^{v\phi_2} &= \frac{a^2 \tilde{R}}{r^2} \quad (4.44)
\end{align*}
\]

and components of $Y^{MN}$ are given by

\[ Y^{\phi_1\phi_1} = \frac{1}{\sin^2 \theta}, \quad Y^{\phi_2\phi_2} = \frac{1}{\cos^2 \theta}, \quad (4.45) \]

where $\tilde{R} = R_y/\sqrt{2}$. Noting that $X^{MN}$ and $Y^{MN}$ are two independent conformal Killing tensors, thus their linear combination is still a conformal Killing tensor. Here the two constant $c_1$ and $c_2$ should not be equal otherwise the conformal Killing tensor is just metric multiplied by a function because of (4.42), which is trivial. The existence of such a nontrivial conformal Killing tensor implies that the massless scalar wave equation is separable, which has been showed in [59]. As we discussed above, the existence of conformal Killing tensor does not guarantee the separability of the equation of motion for particle with half spin. Thus more exploration on hidden symmetries needs to be done.
4.4 Towards conformal Killing–Yano tensor of superstrata

To know about the trajectories of fermionic particles in spacetime give by appendix C.1.2, Dirac equation need to be solved. The usual way to solve these partial differential equations is by separating the variables for these equations. In the previous section we see that there exist a nontrivial conformal Killing tensor which implies the separable massless Klein–Gordon equation. To separate massless Dirac equation, we need the nontrivial conformal Killing–Yano tensor.

4.4.1 Constraints on eigenvalues of CKT

In this section we will look for the nontrivial conformal Killing–Yano tensor by its relation to the conformal Killing tensor. As can be seen from (4.25), a conformal Killing tensor maybe the square of a conformal Killing–Yano tenor. However the conformal Killing tensor given in previous section does not guarantee the existence of a conformal Killing–Yano tensor but it is indeed suggestive. On the contrary, if there exist a conformal Killing–Yano tensor, its square must give a conformal Killing tensor which should be a combination of the two nontrivial conformal Killing tensors and other possible conformal Killing tensors constructed from Killing vectors and metric tensor.

Due to the relation between conformal Killing tensor and conformal Killing–Yano tensor (4.25), there should be some constraints on the eigenvalues on such conformal Killing tensors. To see this, let us assume there is a conformal Killing–Yano tensor of rank 2 in six dimensions. Noting conformal Killing–Yano tensor is antisymmetric, we can write it as the canonical form

\[ \mathcal{Y} = q_1 e_1 \wedge e_2 + q_2 e_3 \wedge e_4 + q_3 e_5 \wedge e_6 \]  

(4.46)
in the frame

\[ g = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2. \]  

(4.47)
Performing relation (4.25), one can find the corresponding conformal Killing tensor as

\[ \mathcal{K} = q_1^2 (e_1^2 + e_2^2) + q_2^2 (e_3^2 + e_4^2) + q_3^2 (e_5^2 + e_6^2) \]  

(4.48)
The six eigenvalues of this conformal Killing tensor are given by

\[ \lambda_1 = \lambda_1 = q_1^2, \quad \lambda_3 = \lambda_4 = q_2^2, \quad \lambda_5 = \lambda_6 = q_3^2. \quad (4.49) \]

As we can see from this, these eigenvalues are three pairs. In other words, for any conformal Killing tensor constructed from two–rank conformal Killing–Yano tensor, its eigenvalues must be allowed to divide into three pairs.

Noting relation (4.25) does not confine the conformal Killing–Yano tensor to tensor with rank two. It is possible that the two–rank conformal Killing tensor is obtained from three–rank conformal Killing–Yano tenor. Thus it is necessary to consider the constraints on the eigenvalues of conformal Killing tensor constructed from three–rank conformal Killing–Yano tensor.

First of all, we would like to determine the canonical form of the three–form \( \mathcal{Y}_{mnp} \) in frames. We will again be focusing on six dimensions, where naively \( \mathcal{Y} \) has

\[ \frac{6 \cdot 5 \cdot 4}{3!} = 20 \quad (4.50) \]

components. We begin with looking at an antisymmetric \( 5 \times 5 \) matrix \( \mathcal{Y}_{1ij} \) and perform a rotation in 5d–space to bring it to the form

\[ \frac{1}{2} \mathcal{Y}_{1ij} e_i \wedge e_j = \mathcal{Y}_{123} e_2 \wedge e_3 + \mathcal{Y}_{145} e_4 \wedge e_5. \quad (4.51) \]

Note that this form is still invariant under rotations in (23) and (45) planes. The first rotation can be used to set \( \mathcal{Y}_{345} = 0 \) and the second one is fixed by requiring \( \mathcal{Y}_{235} = 0 \). This leads to the following non–zero components of the rank–three tensor

\[ \{ \mathcal{Y}_{123}, \mathcal{Y}_{145}, \mathcal{Y}_{245}, \mathcal{Y}_{234}, \mathcal{Y}_{ij6} \}, \quad (i, j) = \{ 2, 3, 4, 5 \}. \quad (4.52) \]

The last equation gives ten independent components. Rotation in (16) plane can be used to set \( \mathcal{Y}_{236} = 0 \), so we conclude that *nine is an upper bound on the number of independent components of \( \mathcal{Y} \).* Perhaps some clever parameterization can reduce this number further.

Coming back to the system (4.52), we now impose an additional requirement on the
direction 1, which was chosen in the very beginning of the entire construction. Specifically, we assume that $e_1$ is an eigenvector of a symmetric tensor

$$K_{MN} = \mathcal{Y}_{MAB} \mathcal{Y}^{AB}_{N}$$  \hspace{1cm} (4.53)

This implies that $K_{14} = K_{12} = 0$, which leads to additional constraints:

$$\mathcal{Y}_{245} = \mathcal{Y}_{234} = 0$$  \hspace{1cm} (4.54)

Furthermore, the relation $K_{16} = 0$ leads to equation

$$\mathcal{Y}_{123} \mathcal{Y}_{236} + \mathcal{Y}_{145} \mathcal{Y}_{456} = 0 : \quad \mathcal{Y}_{623} = \lambda \mathcal{Y}_{145}, \quad \mathcal{Y}_{645} = -\lambda \mathcal{Y}_{123}$$  \hspace{1cm} (4.55)

At this point the nontrivial elements of $\mathcal{Y}$ are

$$\{\mathcal{Y}_{123}, \mathcal{Y}_{145}, \mathcal{Y}_{ij6}\}, \quad (i, j) = \{2, 3, 4, 5\},$$  \hspace{1cm} (4.56)

and rotations in the (23) and (45) planes again became symmetries. These two rotations, $A, B$, can be used to diagonalize a non–symmetric matrix

$$\begin{pmatrix}
\mathcal{Y}_{246} & \mathcal{Y}_{256} \\
\mathcal{Y}_{346} & \mathcal{Y}_{356}
\end{pmatrix} \rightarrow A \begin{pmatrix}
\mathcal{Y}_{246} & \mathcal{Y}_{256} \\
\mathcal{Y}_{346} & \mathcal{Y}_{356}
\end{pmatrix} B^T$$  \hspace{1cm} (4.57)

We conclude that the possible non–zero elements are

$$\{\mathcal{Y}_{123}, \mathcal{Y}_{145}, \mathcal{Y}_{246}, \mathcal{Y}_{356}, \mathcal{Y}_{623}, \mathcal{Y}_{645}\},$$  \hspace{1cm} (4.58)

and they are subject to a constraint (4.55). Let us now compute the corresponding matrix
\( \mathcal{K} : \)

\[
\mathcal{K} = \begin{bmatrix}
\mathcal{K}_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathcal{K}_{22} & 0 & 0 & \mathcal{K}_{25} & 0 \\
0 & 0 & \mathcal{K}_{33} & \mathcal{K}_{34} & 0 & 0 \\
0 & 0 & \mathcal{K}_{34} & \mathcal{K}_{44} & 0 & 0 \\
0 & \mathcal{K}_{25} & 0 & 0 & \mathcal{K}_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathcal{K}_{66}
\end{bmatrix}
\]

\( \mathcal{K}_{11} = [\gamma_{123}]^2 + [\gamma_{145}]^2, \quad \mathcal{K}_{66} = \lambda^2 \mathcal{K}_{11} + [\gamma_{246}]^2 + [\gamma_{356}]^2, \)

\( \mathcal{K}_{25} = -\gamma_{246}\gamma_{156} - \gamma_{236}\gamma_{356}, \quad \mathcal{K}_{34} = \gamma_{356}\gamma_{145} + \gamma_{236}\gamma_{246}, \quad \mathcal{K}_{55} = [\gamma_{145}]^2 + [\gamma_{356}]^2 + [\gamma_{456}]^2, \ldots \)

The sub–matrix in the (25) sector is

\[
\begin{bmatrix}
[\gamma_{123}]^2 + [\gamma_{246}]^2 + [\lambda \gamma_{145}]^2 & \lambda \gamma_{123}\gamma_{246} - \lambda \gamma_{145}\gamma_{356} \\
\lambda \gamma_{123}\gamma_{246} - \lambda \gamma_{145}\gamma_{356} & [\gamma_{145}]^2 + [\gamma_{356}]^2 + [\lambda \gamma_{123}]^2
\end{bmatrix}
\]

(4.60)

The eigenvalues (\( \Lambda_1, \Lambda_2 \)) of this matrix satisfy an interesting relation that follows from the Vieta theorem:

\[
\Lambda_1 + \Lambda_2 = [\gamma_{123}]^2 + [\gamma_{246}]^2 + [\lambda \gamma_{145}]^2 + [\gamma_{145}]^2 + [\gamma_{356}]^2 + [\lambda \gamma_{123}]^2
\]

\[
= \mathcal{K}_{11} + \mathcal{K}_{66},
\]

(4.61)

and the same relation hold for the eigenvalues of the (34) block. For example, for \( \lambda = 0 \) the off–diagonal terms disappear, and the last relation becomes

\[
\mathcal{K}_{22}\mathcal{K}_{55} = \mathcal{K}_{11} + \mathcal{K}_{66} = \mathcal{K}_{33} + \mathcal{K}_{44}
\]

(4.62)

To conclude, we proved that eigenvalues of the matrix \( \mathcal{K} \) appear in pairs \( (\lambda, \mu) \), and they are subject to constraints

\[
\lambda_1 + \mu_1 = \lambda_2 + \mu_2 = \lambda_3 + \mu_3.
\]

(4.63)

This generalizes the notion of triple degeneracy. Thus if the conformal Killing–Yano ten-
tensor that we are looking for exists and it is a tensor of rank three, the eigenvalues of the corresponding conformal Killing tensor must satisfy the relation (4.63).

4.4.2 Perturbation method

Solving the conformal Killing–Yano tensor equations directly in spacetime with such complicated metric (C.14) looks hopeless. In this section we will show that the separability of Dirac equation in this spacetime is not allowed since we there does not exist conformal Killing–Yano tensor. The idea is simple: we will show that the constraints (4.63) on the eigenvalues of conformal Killing tensor with the most general form cannot be satisfied.

Let us first focus on the nontrivial conformal Killing tensor and assume \( Q_1 = Q_5 = Q \). As \( b \) in metric (C.14) goes to zero, the \((1, 0, n)\) family should coincide with the near horizon region of F1–NS5 system [58]. It is known that in F1–NS5 system, there is a conformal Killing tensor of rank two and its “square root”, the conformal Killing–Yano tensor of rank three. Therefore, as \( b \) goes to zero, the nontrivial part of conformal Killing tensor should coincide with the one of F1–NS5 system in near horizon limit. It is easy to find the six eigenvalues of conformal Killing tensor of F1–NS5 system in near horizon limit

\[
\lambda_1 = \lambda_2 = \lambda_3 = -Q, \quad \lambda_4 = \lambda_5 = \lambda_6 = Q,
\]

which obviously satisfy the constraints (4.63). On the other hand when \( b = 0 \), the six eigenvalues of \( K^{MN} \) (4.34) are given by

\[
\lambda_1 = \lambda_2 = \lambda_3 = c_1 Q, \quad \lambda_4 = \lambda_5 = \lambda_6 = c_2 Q.
\]

Thus, we find that \( c_1 = -1 \), \( c_2 = 1 \) by comparing (4.64) and (4.65), which leads to the nontrivial conformal Killing tensor as

\[
K^{MN} = -X^{MN} + Y^{MN}.
\]

What F1–NS5 system and eigenvalues (4.65) tell us is that a conformal Killing–Yano tensor of rank 2 is not allowed. Recall the prerequisite of the existence of the conformal Killing–Yano tenor of rank three is (4.63). The eigenvalues of the nontrivial conformal Killing tensor
(4.66) are given by

\[
\begin{align*}
\lambda_1 &= \lambda_2 = \lambda_3 = -\lambda_4 = -\frac{Q}{\tilde{\Lambda}} \left[ 2a^2 + b^2 - \frac{a^2 b^2 r^{2n} \sin^2 \theta}{(a^2 + r^2)^{n+1}} \right], \\
\lambda_5 &= \frac{Q}{\tilde{\Lambda}} \left[ 2a^2 + b^2 - \frac{a^2 b^2 r^{2n} \sin^2 \theta}{(a^2 + r^2)^{n+1}} \right], \\
\lambda_6 &= \frac{Q}{\tilde{\Lambda}} \left[ 2a^2 + b^2 - \frac{a^2 b^2 r^{2n} (1 + \cos^2 \theta)}{(a^2 + r^2)^{n+1}} \right],
\end{align*}
\]  

(4.67)

where

\[
\tilde{\Lambda} = \sqrt{(2a^2 + b^2) \left( 2a^2 + b^2 - \frac{a^2 b^2 r^{2n} \sin^2 \theta}{(a^2 + r^2)^{n+1}} \right)}
\]  

(4.68)

Obviously, these eigenvalues do not satisfy constraint (4.63), which implies that there is no conformal Killing–Yano tensor that can give this nontrivial Killing tensor by squaring. Therefore, to know if there exists a conformal Killing–Yano tensor, we have to investigate the most general form of conformal Killing tensor.

Noting the discussion in section 4.2, the most general form of conformal Killing tensor is given by the nontrivial one (4.34) plus the trivial one which is consistent of metric multiplied by an arbitrary function and construction (4.10):

\[
\mathcal{K}^{MN} = \mathcal{K}_\text{nontriv}^{MN} + \mathcal{K}_\text{triv}^{MN}
\]

\[
\mathcal{K}_\text{nontriv}^{MN} = -X^{MN} + Y^{MN}, \quad \mathcal{K}_\text{triv}^{MN} = K^{MN} + f_n g^{MN},
\]  

(4.69)

where \(K^{MN}\) is Killing tensor constructed from Killing vectors and \(f_n\) is an arbitrary function. Noting Killing vectors (4.35), we infer that \(K^{MN}\) is a symmetric tensor with constant components along directions \((u, v, \phi_1, \phi_2)\):

\[
K^{MN} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & K^{uu} & K^{uv} & K^{u\phi_1} & K^{u\phi_2} \\
0 & K^{uv} & K^{vv} & K^{v\phi_1} & K^{v\phi_2} \\
0 & K^{u\phi_1} & K^{v\phi_1} & K^{\phi_1\phi_1} & K^{\phi_1\phi_2} \\
0 & K^{u\phi_2} & K^{v\phi_2} & K^{\phi_1\phi_2} & K^{\phi_2\phi_2}
\end{bmatrix},
\]  

(4.70)

where all components are are independent of \(r, \theta\). Unfortunately, direct computation to
the eigenvalues of the conformal Killing tensors with general form \((4.69)\) is too complicated to work out. Thus to investigate if the conditions \((4.63)\) are satisfied or not, one of the best option is perturbation method for the current case. This is because the eigenvalues of conformal Killing tensor and the corresponding conformal Killing–Yano tensor are known at \(b = 0\). Thus we can pick \(b\) as the perturbation parameter and the perturbative form of general conformal Killing tensor can be expressed as

\[
K^{MN} = -X^{MN} + Y^{MN} + bC_1^{MN} + b^2C_2^{MN} + bf_1g^{MN} + b^2f_2g^{MN},
\]

where \(C_1^{MN}\) and \(C_2^{MN}\) are two constant matrices which have the same form as \((4.70)\) and \(f_1, f_2\) are two arbitrary functions. Now the question is that if there exists certain constant matrices \(C_1^{MN}, C_2^{MN}\) and function \(f_1, f_2\) which enable the eigenvalues of conformal Killing tensor \((4.71)\) to obey relation \((4.63)\). In appendix C.2, we checked this possibility, but unfortunately, this is impossible. Thus we conclude that there is no conformal Killing–Yano tensor in the solutions of \((1, 0, n)\) family. In other words, Dirac equation is not separable in these geometries.

### 4.5 Discussion

To study the dynamics of particles, finding the hidden symmetries of the background spacetime is a powerful approach, especially for the particles in highly curved spacetime, for instance the particles near even horizon of black hole. Hawking radiation incurs the information paradox. To resolve this puzzle, the study of particles near the horizon of black hole become more essential. Therefore, in this chapter we first introduced the representation of hidden symmetries that are the (conformal) Killing and Killing–Yano tensors. These symmetric and antisymmetric tensors implies the separability of Klein–Gordon and Dirac equations, respectively. In this chapter we mainly focus on a very complicated infinite family of superstratum solutions whose conformal Killing tensor can be obtained by separating Hamilton–Jacobi equation. Thanks to the existence of this conformal Killing tensor \([59]\) showed that the massless scalar wave function is separable.

The studies on the particle with half spin need to solve Dirac equation. It is extremely hard to do that in such complicated spacetime unless we can separate the variables in the
PDEs, which results from the hidden symmetry of the background spacetime represented by conformal Killing–Yano tensor. The main goal of this chapter is to check if this kind hidden symmetry exists or not in these solutions. In order to investigate this possibility we focus on the relation between conformal Killing tensor and conformal Killing–Yano tensor. The relation between them gives constraints on the eigenvalues of conformal Killing tensor which provide us with a direct check of the existence of conformal Killing–Yano tensor. For the technical difficulties, we cannot solve the eigenvalues of the conformal Killing tensor with general form. We thus employ the perturbation method which has proved that there does not allow a conformal Killing–Yano tensor. As we have mentioned above, the separability of Dirac equation must be accompanied by the separability of Klein–Gordon equation. However sometimes, Klein–Gordon equation is separable, while Dirac equation is not. Other known example for such case can be found in [58]: there is a Killing tensor but Killing–Yano tensor in F1–NS5 system when $Q_1 = Q_5$. 
5.1 Introduction

A $n$–dimensional manifold is maximally symmetric if it possesses the maximal number of isometries, which is equivalent to that there are $n(n + 1)/2$ linearly independent KVs (Killing vectors). For instance, Euclidean, Minkowski, (anti–)de Sitter and spherical space are maximally symmetric spaces. The maximally symmetric space also admits the maximum number of KYTs (Killing–Yano tensors) and closed CKYTs whose special properties have been studied in [64]. The translational and rotational Killing–Yano $p$–forms in $n$–dimensional flat space can be found in [65], while all solutions to equation of CKYT in anti–de Sitter spacetime were constructed by [66].

As mass put into the maximally symmetric space, the symmetries will decrease and the introduction of rotation, the symmetries may reduce more. The hidden symmetries of the Kerr black hole in four dimensions were studies in [22, 52, 53]. The study of higher dimensional black holes started with [67], which is the higher dimensional generalization of Schwarzschild black hole, while the higher–dimensional generalization of Kerr spacetime was done by [68]. Adding cosmological constant into Myers–Perry metric [68] can be found in [69] and NUT parameters were considered by [70, 71]. It was demonstrated by [72] that both Myers–Perry black hole [68] and the most general Kerr–NUT–(A)dS spacetime [71] in arbitrary dimensions possess a nondegenerate closed conformal Killing–Yano 2–form, which implies complete integrability of geodesic motion. The separability of the Maxwell’s equations and the dynamical equations for all $p$–form fluxes in the Myers–Perry–(A)dS geometry was demonstrated in [73, 74]. The review of hidden symmetries and black holes can be seen in [58, 75, 76].

The nondegenerate closed CKYT of rank 2 mentioned above is usually called principle tensor due to its role in constructing all CKYTs and CKTs in Kerr–NUT–(A)dS spacetime. Since this two–rank CKYT is closed, its Hodge dual is a KYT and its wedge product is again a closed CKYT [75, 77]. These two properties enable the principle tensor to be the
seed of “Killing tower”: The wedge products of it give all CKYTs which all are closed and whose square are CKTs; the Hodge dual of these closed CKYTs give all KYTs whose square are KTs; the covariant derivative of it gives primary KV which can generate other KVs by product with KTs. It was proved by [78] that the most general geometry admitting a principle tensor is locally the off–shell Kerr–NUT–(A)dS spacetime.

As we have seen in previous chapters, there are microstate geometries which are asymptotically to AdS$_p$×S$^p$ where $p = 2, 3, 5$. These asymptotical spacetimes are not maximally symmetric space but they are conformal flat and their subspace are maximally symmetric. As we will show that under certain condition, there are lots of CKYTs in AdS$_p$×S$^p$. Different from CKYTs in maximally symmetric which are all closed, most CKYTs in AdS$_p$×S$^p$ are not closed, but the total number of CKYTs in AdS$_p$×S$^p$ is same as the ones in maximally symmetric space due to they are conformal flat. As one more sphere added for $p = 3$, we have AdS$_3$×S$^3$×S$^3$, which is a very interesting background in string theory. AdS$_3$×S$^3$×S$^3$×S$^1$ is another class of AdS$_3$ background with 16 superchargers beside AdS$_3$×S$^3$×T$^4$ which we have been encountered in previous chapters. Thus it is also interested to study the hidden symmetries of AdS$_3$×S$^3$×S$^3$, and we will show that there are no nontrivial CKTs in this background.

In the next section, we give a short review of “Killing tower” and the uniqueness theorem. In the following section we derive the CKYTs of flat space and point out which of them will survive as mass and rotation introduced. All hidden symmetries of AdS$_p$×S$^p$ are considered in section 5.4. We present the solutions to CKT equations in AdS$_2$×S×S$^2$ in section 5.5, while the CKTs in AdS$_3$×S$^3$×S$^3$ are given by section 5.6. A short discussion is in the last section.

5.2 Review of “Killing tower”

Before proceeding this section, we list four useful properties for CKYT:

1. The Hodge dual of a CKYT is a CKYT;
2. The Hodge dual of a closed CKYT is a KYT and vice versa;
3. Under conformal transformation $\tilde{g} = \Omega^2 g$, the $p$–rank CKYT transform as $\tilde{Y} =$
4. The wedge product of a $p$–rank closed CKYT with a $q$–rank closed CKYT is a $(p + q)$–rank closed CKYT [77].

As we know that the closed form is the form with the vanished exterior derivative, the equation for a closed rank–two CKYT $h$ is thus given by

$$\nabla_P h_{MN} = g_{PM} \xi_N - g_{PN} \xi_M, \quad \xi_M = \frac{1}{D - 1} \nabla^N h_{NM} \quad (5.1)$$

where $D = 2n + \varepsilon$ ($\varepsilon = 0$ for even dimension and $\varepsilon = 1$ for odd dimensions) is the dimensions. The closed CKYT $h$ is called principle tensor when it is nondegenerate. Starting with two–rank tensor, one can generate closed CKYTs, CKTs, KYTs, KTs and KVs as followings.

**Closed CKYTs** Using property 4, one can found $n + 1$ closed CKYTs of rank $2j$

$$Y^{(j)} = \frac{1}{j!} h^{\wedge j} \quad (5.2)$$

where $j = 0, \cdots, n$. For $j = 0$, we obtain a constant, which can be regarded as a trivial 0–form. Note that all the tensors found by this construction are tensors of even ranks and the largest rank for these tensors is $2n$.

**CKTs** Employing the relation (4.25) between CKYT and CKT , one can obtain CKT of rank two as

$$K^{(j)}_{MN} = \frac{1}{(2j - 1)!} Y^{(j)}_{M A_1 \cdots A_{2j - 1}} Y^{(j) A_1 \cdots A_{2j - 1}}_{N} \quad (5.3)$$

From this construction we see that the number of CKTs of rank two is same as the number of CKYTs of all possible ranks.

**KYTs** Noting the property 2, one can find the standard KYTs as

$$Y^{(j)} = * Y^{(j)} \quad (5.4)$$

Recall how to evaluate Hodge star and we thus know the rank of these KYTs obtained by this relation is $D - 2j$, where $2j$ is the rank of CKYTs.
KTs Similarly to finding CKYTs, one can construct the KTs from the relation (4.24)

\[ K^{(j)}_{MN} = \frac{1}{(D - 2j - 1)!} Y^{(j)}_{MA_1 \cdots A_{2j-1}} Y^{(j)A_1 \cdots A_{2j-1}}. \]  

(5.5)

When \( j = 0 \), the KT reduces to metric \( g_{MN} \). This is because when \( j = 0 \), \( Y^{(0)} \) is Levi–Civita tensor.

KV The associated vector \( \xi_M \) in the definition of closed CKYT (5.1) is a Killing vector, and it is usually called the primary KV. Other KVs are given by

\[ V^{(j)}_M = K^{(j)}_{MN} \xi^N. \]  

(5.6)

Since these KVs are generated from \( \xi_M \) which is called primary KV, we call them as secondary KVs.

By the operations presented above, starting with the principal tensor, one can figure out all other representation of symmetries in Kerr–NUT–(A)dS spacetime. It seems that this rank–two tensor can determine certain class of geometries and it indeed does. As shown by [78], if \((M, g)\) is a \((2n + \varepsilon)\)-dimensional spacetime admitting a nondegenerate closed CKYT of rank 2, i.e. the principle tensor, the metric \( g \) is then can be written locally as off–shell Kerr–NUT–(A)dS metric. This theorem is called the uniqueness theorem. As the Einstein equations imposed, the on–shell Kerr–NUT–(A)dS metric is obtained. The Hamilton–Jacobi, Klein–Gordon and Dirac equation are separable in this spacetime due to the geodesic motion is complete integrable.

5.3 CKYT in maximally symmetric space

All the CKYT in maximally symmetric space can be constructed as “Killing tower”. However the seed is not a nondegenerate closed CKYT of rank two but are several conformal Killing one–forms.

In this section, we will take AdS space as example to see how to build this construction.
In the \((p + 3)\)-dimensional flat space

\[
ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2 \tag{5.7}
\]

the hyperboloid

\[
- X_0^2 - X_{p+2}^2 + \sum_{i=1}^{p+1} X_i^2 = -R^2, \tag{5.8}
\]

can be utilized to represent \((p + 2)\)-dimensional anti–de Sitter space which has isometry \(SO(2, p + 1)\), and it is homogenous and isotropic. The solutions to (5.8) can be set as

\[
X_0 = R \cosh \rho \cos \tau, \quad X_{p+2} = R \cosh \rho \sin \tau, \\
X_i = R \sinh \rho \Omega_i \quad (i = 1, \cdots, p + 1; \sum_i \Omega_i^2 = 1), \tag{5.9}
\]

where \(0 \leq \rho\) and \(0 \leq \tau \leq 2\pi\) which can cover the entire hyperboloid once. Plugging this solutions into (5.7), we obtain the AdS\(_{p+2}\) in global coordinates as

\[
ds^2 = R^2 \left( -\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega^2 \right). \tag{5.10}
\]

We will use global coordinates in the rest sections of this chapter. However in this section it is convenient to consider the so called Stereographic coordinates \((t, x_i), \ i = 1, \cdots, p + 1:\)

\[
X_0 = -At, \quad X_i = -Ax_i, \quad X_{p+2} = R(1 + 2A) \tag{5.11}
\]

where

\[
A \equiv \frac{4R^2}{-4R^2 - t^2 + \sum x_i^2}. \tag{5.12}
\]

The metric of AdS\(_{p+2}\) in these coordinates can be written as

\[
ds^2 = 4R^2 A(-dt^2 + dx_i^2). \tag{5.13}
\]

There are \(p + 3\) closed conformal Killing one–forms in AdS\(_{p+2}\)

\[
dX_0, \quad dX_i, \quad dX_{p+2}. \tag{5.14}
\]

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The prove of this was given by [66]. These conformal Killing one–forms can be regarded as the seeds of “Killing tower” as (5.2) to (5.6): the wedge products of them produce closed CKYTs of arbitrary possible ranks and the Hodge dual of these CKYTs give KYTs; All the CKTs and KTs are constructed from CKYTs, KYTs and KVs. Noting property 3 and the AdS metric (5.13) and making conformal transformation \( \tilde{g} = \Omega^2 g \), \( \Omega = (4R^2 A)^{-1} \), one can work out the conformal Killing–Yano \( p \)–forms in flat space as \( \mathcal{Y} = \Omega^{p+1} \mathcal{Y} \).

According to the construction above, it is easy to count that the number of CKVs in \( n \)–dimensional AdS space is \( n + 1 \), while the number of KVs is

\[
\binom{n + 1}{n - 1} = \frac{1}{2}(n + 1)n. \tag{5.15}
\]

The numbers of CKYTs and KYTs of rank \( p \) are given by

\[
\begin{align*}
\binom{n + 1}{p}_{\text{CKYT}} & , \\
\binom{n + 1}{n - p}_{\text{KYT}}.
\end{align*} \tag{5.16}
\]

Flat space is maximally symmetric space which has the largest number of Killing vectors. In the realistic situation, the spacetime is not flat due to the existence of mass or charges, which can kill some symmetries in flat space. As rotating mass considered, symmetries may be killed more.

We have known from previous section that the closed CKYT of rank two can determine all the symmetries in four–dimensional Kerr spacetime. This closed CKYT is known as

\[
\mathcal{Y}_{MN} = \begin{bmatrix}
0 & -r & -a^2 c_\theta s_\theta & 0 \\
-2 \phi & 0 & 0 & -a r s_\phi^2 \\
a^2 c_\theta s_\theta & 0 & 0 & -a (a^2 + r^2) c_\phi s_\theta \\
0 & a r s_\phi^2 & a (a^2 + r^2) c_\phi s_\theta & 0
\end{bmatrix}, \tag{5.17}
\]

where \( c_\theta \equiv \cos \theta \), \( s_\theta \equiv \sin \theta \) and \( a \) is the rotation parameter. Obviously, this tensor does not depend on mass, which indicates that this symmetry can be identified with one symmetry of flat space. In other words, this symmetry is the one survived from adding mass and rotation.
to flat space. In four–dimensional flat space
\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \] (5.18)
there is a closed CKYT does not depend on \( t \),
\[ \mathcal{Y}_{MN} = xdt \wedge dx + ydt \wedge dy + zdt \wedge dz. \] (5.19)
Adding a KYT \( Y = adx \wedge dy \) and making coordinates transformation as
\[ t \rightarrow t, \quad z \rightarrow r \cos \theta \]
\[ x \rightarrow \sqrt{r^2 + a^2 \sin \theta \cos \phi}, \quad y \rightarrow \sqrt{r^2 + a^2 \sin \theta \sin \phi} \] (5.20)
one then can find (5.17). This operation in looking for closed CKYT of Kerr spacetime
from flat space can be not only generalized to higher dimensional general Kerr–NUT–(A)dS
spacetime but also higher–rank closed CKYTs or KYTs. The basic procedures are simple:
we first work out the CKYTs (including KYTs) in flat space and pick out the ones does not
depends on \( t \), these CKYTs can survive as mass introduced; If these picked CKYTs do not
depend on azimuthal angle, they can survived as rotations added and even the cosmological
constant, by adding a KYT only depending on rotating parameter. The specific form of this
constant KYT depends on how the rotation is added. For example, as (5.17), the rotation is
in \((x, y)\) plane, then \( Y = adx \wedge dy \). We have investigated the CKYTs of rank two and three
in five– and six–dimensional Kerr black hole from flat space, which perfectly agree with the
results from the “Killing tower”.

### 5.4 (Conformal) Killing–Yano tensors in \( \text{AdS}_p \times S^p \)

In this section we will take \( \text{AdS}_3 \times S^3 \) as example to demonstrate how to build (Confor-
mal) Killing–Yano tensors in \( \text{AdS}_p \times S^p \) from (Conformal) Killing–Yano tensors in AdS and
sphere. The discussion in this section requires the AdS and spherical space have same radii.

To find the Killing vectors and Killing–Yano tensors of \( S^3 \) whose metric, using coordi-
nates $\theta$, $\phi$, $\psi$, can be written as

$$ds^2 = L^2 \left(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2\right),$$

(5.21)

Let’s first put it into a four-dimensional flat space and its four coordinates in the flat space are

$$(S^{(1)}, S^{(2)}, S^{(3)}, S^{(4)}) = L(\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta \cos \psi, \sin \theta \sin \psi),$$

(5.22)

where $L$ is the radius of the 3-sphere. According to the discussion in the previous section, all the (conformal) Killing vectors and Killing–Yano tensors of 3-sphere can be constructed from $S^{(a)}$, $a = 1, 2, 3, 4$ as follows.

1. One–form
   - 4 CKVs
     $$\gamma^{(a)} = dS^{(a)}, \quad Z \propto S^{(a)}$$
     (5.23)
     Here $Z$ is the conformal factor. These four closed conformal Killing vectors can be thought as the seeds of “Killing tower”. The wedge products of them are conformal Killing–Yano forms according to property 4.
   - 6 KVs
     $$V^{(i)} = \ast \left[\gamma^{(a)} \wedge \gamma^{(b)}\right], \quad i = 1, \cdots, 6, \quad a, b = 1, 2, 3, 4,$$
     (5.24)
     The Hodge dual is evaluated in (5.21). There are 6 ways for the combination of $a$ and $b$. Thus $i$, labeling the number of Killing vectors, takes 1 to 6. The wedge products in the square bracket are closed conformal Killing–Yano tensor of rank 2. Due to the property 2, the Hodge dual of them give Killing one–forms.

2. Two–form
   - 4 KYTs
     $$Y_2^{(a)} = \ast \gamma^{(a)}$$
     (5.25)
     Again, noting property 2, one can obtain Killing–Yano forms from the seeds of “Killing tower” (5.23). Recall that we are dealing with three-dimensional sphere,
and thus the Killing–Yano tensors obtained are two–rank.

• 6 CKYTs

\[ \mathcal{Y}_2^{(i)} = \mathcal{Y}^{(a)} \wedge \mathcal{Y}^{(b)}, \quad Z \propto V^{(i)} \]  

(5.26)

Employing property 4, it is straightforward to write down the conformal Killing–Yano two–forms, which all are closed. This is consistent with the description in [66]: the conformal Killing–Yano tensors of rank two are given by the pullbacks of constant two–forms in four–dimensional flat space. There is another option

\[ \mathcal{Y}_2^{(i)} = LdV^{(i)}, \quad Z \propto V^{(i)}, \]  

(5.27)

which is exterior derivative acting on Killing one form. This not only increases the rank but also give us conformal Killing–Yano two forms.

3. Three–form

• 1 KYT

\[ Y_3 = \sqrt{|\det g|} \; d\theta \wedge d\phi \wedge d\psi \]  

(5.28)

This can be understood as the Hodge dual of “zero–form” obtained from \( \mathcal{Y}^{\wedge 0} \).

• 4 CKYTs

\[ \mathcal{Y}_3^{(a)} = LdY_2^{(a)} \propto \mathcal{Y}^{(b)} \wedge \mathcal{Y}^{(c)} \wedge \mathcal{Y}^{(d)}, \quad Z \propto Y_2^{(a)} \]  

(5.29)

Again these conformal Killing–Yano forms are closed. Although any three–rank tensors are conformal CKYTs in current case which is 3–dimensional, CKYTs found in (5.29) will be helpful to construct three–rank CKYTs in \( \text{AdS}_3 \times \text{S}^3 \).

Similarly, in four–dimensional flat space, the 4 coordinates of \( \text{AdS}_3 \) space are

\[ (A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}) = L(\cosh \rho \cos \tau, \sinh \rho \cos \alpha, \sinh \rho \sin \alpha, \cosh \rho \sin \tau). \]  

(5.30)

Following the same procedures (5.24)–(5.29), one can find all (conformal) Killing vectors and two and three–rank (conformal) Killing–Yano tensors in \( \text{AdS}_3 \). In the rest of this chapter, we will using \( Y_A \) to denote differential forms in \( \text{AdS}_3 \), \( Y_S \) to denote differential forms in \( \text{S}^3 \) and \( Y \) without \( A, S \) indices to denote the differential forms in \( \text{AdS}_3 \times \text{S}^3 \).
After obtaining all (conformal) Killing vectors and Killing–Yano tensors in AdS and spherical space, we can construct all the Killing vectors and (conformal) Killing–Yano tensors in $\text{AdS}_3 \times \text{S}^3$ as follows.

1. One–form
   
   - 12 KVs
     \[ V^{(i)} = (V_A^{(i)}, 0, 0, 0), \quad V^{(j)} = (0, 0, 0, V_S^{(j)}), \quad i, j = 1, \cdots, 6 \]  

   It is easy to check that adding three zero components to Killing vectors of $\text{AdS}_3$ along $\theta$, $\phi$, $\psi$ directions, one will obtain Killing vectors in $\text{AdS}_3 \times \text{S}^3$. Similarly, adding three zero components to Killing vectors of $\text{S}^3$ along $\rho$, $\tau$, $\alpha$ directions, one will get the rest Killing vectors in $\text{AdS}_3 \times \text{S}^3$.

   - 16 CKVs
     \[ Y^{(ab)} = S^{(a)} V_A^{(b)} - A_A^{(b)} V_S^{(a)}, \quad Z^{(ab)} \propto S^{(a)} A^{(b)} \]  

   Since $a$, $b$ can take values from 1 to 4, there are 16 conformal Killing vectors obtained by this construction. Noting 16 is the number of conformal Killing vectors when 12 Killing vectors (5.31) are not counted, thus there are 28 conformal Killing vectors in total. We see from the expression (5.32), all conformal Killing vectors constructed by two parts: one originated from AdS, another originated from 3–sphere, which is quite reasonable.

2. Two–form
   
   - 8 KYTs
     \[ Y_A^{(a)}, \quad Y_S^{(a)} \]  

   Similarly to Killing vectors, directly expanding Killing–Yano tensors in AdS and 3–sphere to 6–dimensional space, we will find 8 Killing–Yano tensors of rank two in $\text{AdS}_3 \times \text{S}^3$. 

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Recall \(a, b = 1, \cdots, 4\), \(i, j = 1, \cdots, 6\). Thus each equation gives \(4 \times 6 = 24\) conformal Killing–Yano tensors. Being different from the case of conformal Killing vectors, to find all conformal Killing–Yano tensors, we need two constructions: one is by conformal Killing–Yano tensors of 3–sphere and conformal Killing vectors of AdS; another one is by conformal Killing–Yano tensors of AdS and conformal Killing vectors of 3–sphere.

3. Three–form

• 2 KYTs

\[
Y_A^3 = \sqrt{|\det g_A|} \, d\tau \wedge d\alpha \wedge d\rho, \quad Y_S^3 = \sqrt{|\det g_S|} \, d\theta \wedge d\phi \wedge d\psi, \quad (5.35)
\]

Again, similarly, expanding three–rank Killing–Yano tensors of AdS and 3–sphere to six–dimensions, it is straightforward to obtain three–rank Killing–Yano tensors for \(\text{AdS}_3 \times S^3\).

• 68 CKYTs

\[
\begin{align*}
\mathcal{Y}^{(ab)} &= A^{(a)} \mathcal{Y}^{(b)}_{S_3} + L^2 Z^{(b)}_{S_3} dA^{(a)}, \quad Z^{(ab)} \propto A^{(a)} Z^{(b)}_{S_3} \\
\mathcal{Y}^{(cd)} &= S^{(c)} \mathcal{Y}^{(d)}_{A_3} - L^2 Z^{(d)}_{A_3} dS^{(c)}, \quad Z^{(cd)} \propto S^{(c)} Z^{(d)}_{A_3} \\
\mathcal{Y}^{(ij)} &= Ld \left[ V^{(i)}_{A} \wedge V^{(j)}_{S} \right], \quad Z^{(ij)} \propto K^{(i)} \wedge K^{(j)} \quad (5.36)
\end{align*}
\]

Compared two–rank conformal Killing–Yano tensors, one more construction is needed to find all three–rank conformal Killing–Yano tensors. The first two equations are similar to the constructions in two–rank conformal Killing–Yano tensors, while the third one is never appeared in lower rank which is constructed by Killing vectors of AdS and sphere. The first two equation give us \(4 \times 4 \times 2 = 32\) conformal Killing–Yano tensors, and the third one gives \(6 \times 6 = 36\) three–rank conformal Killing–Yano tensors. What we should point out is that the \(\mathcal{Y}_{A_3}\) and \(\mathcal{Y}_{S_3}\) in
(5.36) are established from (5.29) but arbitrary three–rank conformal Killing–Yano tensors in AdS and 3–sphere because, as we known, all 3–rank tensors in three dimensions are conformal Killing–Yano tensors.

As we mentioned at the beginning of this section, our discussion requires that the radii of the AdS and sphere are same, which enables AdS\(_p \times S^p\) to be conformal flat. This tells us that starting with a 6–dimensional flat space

\[
d s^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2
\]

\[
= -dt^2 + dx_1^2 + dr^2 + r^2 (d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2),
\]

and then making a conformal transformation as

\[
g_{MN} \rightarrow \frac{g_{MN}}{r^2},
\]

we will get AdS\(_3 \times S^3\) as

\[
d s^2 = \frac{1}{r^2} (-dt^2 + dx_1^2 + dr^2) + d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2.
\]

Under this conformal transformation, according to property 3, the conformal Killing–Yano k–forms are obtained as

\[
\mathcal{V} \rightarrow \left(\frac{1}{r^2}\right)^{k+1} \mathcal{V}.
\]

Therefore, one can find the conformal Killing–Yano tensors in AdS\(_p \times S^p\) from the conformal Killing–Yano tensors in 2p–dimensional flat space, or even AdS\(_2p\), S\(_{2p}\). All the constructions above can be directly generalized to AdS\(_2 \times S^2\) and AdS\(_5 \times S^5\).

### 5.5 (Conformal) Killing tensors in AdS\(_2 \times S^2 \times S^2\)

The two methods to obtain (conformal) Killing–Yano tensors introduced above require the spacetime is conformal flat. In string theory, what is interesting is AdS\(_3 \times S^3 \times S^3\) background, which has one more sphere than we discussed above and is not conformal flat. Therefore, the constructions above are failed to obtain (conformal) Killing–Yano tensors and thus (conformal) Killing tensors. It is known that one can obtain (conformal) Killing tensors
from Killing vectors and Killing–Yano tensors. However it is not clear if they give all (conformal) Killing–Yano tensors. Therefore, to find all (conformal) Killing tensors in AdS$_3 \times$S$^3 \times$S$^3$, the first principle maybe the best choice. However calculation in nine dimensions is so tedious and instead of it, we will apply first principle into AdS$_2 \times$S$^2 \times$S$^2$. This six–dimensional spacetime has the same structure of AdS$_3 \times$S$^3 \times$S$^3$ but lower dimensions which will simplify the computation a lot.

The metric of AdS$_2 \times$S$^2 \times$S$^2$ in global coordinates is given by

$$ds^2 = L^2 (d\rho^2 - \cosh^2 \rho dt^2) + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \tilde{R}^2 \left(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2\right). \quad (5.41)$$

From the discussion in previous section, we know that the Killing one–forms and Killing–Yano forms in AdS space and spherical spaces are also the Killing one forms and Killing–Yano forms in AdS$_2 \times$S$^2 \times$S$^2$ by expanding to higher dimensions. Therefore, the one–forms

$$V^{(1)} = -L ch_\rho^2 dt, \quad V^{(2)} = L s_t d\rho - L c_t s_\rho c_\rho d\rho, \quad V^{(3)} = -L c_t d\rho - L s_t s_\rho c_\rho d\rho,$$

$$V^{(4)} = -R s_\phi^2 d\phi, \quad V^{(5)} = R c_\phi d\theta + R c_\phi s_\phi c_\theta d\phi, \quad V^{(6)} = -R c_\phi d\theta + R s_\phi s_\phi c_\theta d\phi,$$

$$V^{(7)} = -\tilde{R} s_{\tilde{\phi}}^2 d\tilde{\phi}, \quad V^{(8)} = \tilde{R} s_{\tilde{\phi}} d\tilde{\theta} + \tilde{R} c_{\tilde{\phi}} s_{\tilde{\phi}} c_{\tilde{\theta}} d\tilde{\phi}, \quad V^{(9)} = -\tilde{R} c_{\tilde{\phi}} d\tilde{\theta} + \tilde{R} s_{\tilde{\phi}} s_{\tilde{\phi}} c_{\tilde{\theta}} d\tilde{\phi}, \quad (5.42)$$

are the Killing one–forms in AdS$_2 \times$S$^2 \times$S$^2$. $V^{(1)}, V^{(2)}, V^{(3)}$ are found from AdS space and $V^{(4)}$ to $V^{(9)}$ are found from two spherical spaces. As we mentioned above that using these Killing vectors and Killing–Yano tensors, one can obtain Killing tensors and conformal Killing tensors by different approaches. To find Killing tensors, one can use the relation

$$K_{MN} = V_M V_N + V_N V_M, \quad (5.43)$$

to build Killing tensors from Killing vectors. By Killing–Yano tensors, we can apply the known relation of Killing tensors and Killing–Yano tensors

$$K_{MN} = Y_{M, K_1 K_2 \ldots} Y_{N, K_1' K_2' \ldots}. \quad (5.44)$$

Noting the fact that the hodge dual of Killing one–forms and Killing–Yano forms in AdS$_2 \times$S$^2 \times$S$^2$
will give closed conformal Killing–Yano forms, i.e.

\[ \mathcal{Y}_5 = \ast V, \quad \mathcal{Y}_4 = \ast Y_2. \]  

Similarly to relation (5.44), there is a relation between conformal Killing tensors and conformal Killing–Yano tensors

\[ \mathcal{K}_{MN} = \mathcal{Y}_{MK_1K_2...}, \quad W_M = 2\mathcal{Y}_{MK_1K_2...}Z^{K_1K_2...}, \]  

where \( W_M \) and \( Z^{K_1K_2...} \) are the conformal factors corresponding to conformal Killing tensors and conformal Killing–Yano tensors, respectively. However, there is no any evidence to show that (conformal) Killing tensors found by the (5.44) and (5.46) are all (conformal) Killing tensors in \( \text{AdS}_2 \times S^2 \times S^2 \). Unfortunately, all the (conformal) Killing tensors found by the (5.44) and (5.46) are trivial. Thus we expect by first principle, we can find something interesting, i.e. nontrivial (conformal) Killing tensors.

### 5.5.1 Results from first principles

To make calculation easier, we can write components of conformal Killing tensors as

\[ \mathcal{K}_{MN}(\rho, t, \theta, \phi, \tilde{\theta}, \tilde{\phi}) = \mathcal{K}_{MN}'(\rho, \theta, \tilde{\theta})e^{int+im\phi+i\tilde{m}\tilde{\phi}}, \]  

where \( n, m, \tilde{m} \) are three integers. We will classify all the conformal Killing tensors found by first principle via these three integers. In solving partial differential equations, we have considered all the possible constraints on radii of AdS and spherical spaces that can affect computation, which does not lead to new conformal Killing tensors, compared the case for arbitrary radii.

We will directly give our results obtained from first principle as different \((n, m, \tilde{m})\). The outline of the calculation can be found in appendix D. For each \((n, m, \tilde{m})\), we will compare the (conformal) Killing tensors obtained from Killing vectors, Killing–Yano tensors and from first principle.
For \( (n, m, \tilde{m}) = (0, 0, 0) \), by first principle we find

\[
K'_{11} = f, \quad K'_{22} = \cosh^4 \rho \left( c_1 - f \text{sech}^2 \rho \right), \quad K'_{33} = \frac{f R^2}{L^2} + c_4
\]

\[
K'_{44} = s_\theta^2 \left( \frac{f R^2}{L^2} + c_5 s_\theta^2 + c_4 \right), \quad K'_{55} = \frac{f \tilde{R}^2}{L^2} + c_7, \quad K'_{66} = s_\tilde{\theta}^2 \left( \frac{f \tilde{R}^2}{L^2} + c_8 s_\tilde{\theta}^2 + c_7 \right),
\]

\[
K'_{24} = c_2 \cosh^2 s_\theta^2, \quad K'_{26} = c_3 \cosh^2 s_\tilde{\theta}^2, \quad K'_{46} = c_6 \cosh^2 s_\theta^2.
\]

(5.48)

where \( f \) is an arbitrary function of \( \rho, \theta, \tilde{\theta} \), and \( c_1 \ldots c_8 \) are 8 constants which give 8 independent conformal Killing tensors and metric multiplied by function \( f \). If picking \( f \) as a constant, the results give 8 independent Killing tensors and metric multiplied by a constant.

Due to these two spheres can be regarded equivalently, the (conformal) Killing–tensors are symmetric about these two spheres, which means if we know components related to one sphere, we then can write the components related to another sphere, for example, if we know \( K'_{24} \), then we can write down \( K'_{26} \) directly by changing \( \theta \) to \( \tilde{\theta} \). As it is expected, we see that our results from first principle satisfy this symmetry.

Next we will see how many (conformal) Killing tensors we can find from Killing vectors and Killing–Yano tensors. As we have shown in previous part, there are 9 Killing vectors in \( \text{AdS}_2 \times \mathbb{S}^2 \times \mathbb{S}^2 \). To find Killing tensors belonging to \((0, 0, 0)\) from Killing vectors, we require there are no \( t, \phi, \tilde{\phi} \) in the expression of Killing vectors. From (5.42), we see only \( V^{(1)}, V^{(4)}, V^{(7)} \) do not depend on \( t, \phi, \tilde{\phi} \). Therefore, the Killing tensors independent on \( t, \phi, \tilde{\phi} \) can be obtain by the these three Killing vectors. There are 6 ways to combine them

\[
V^{(1)}V^{(1)}, \quad V^{(4)}V^{(4)}, \quad V^{(7)}V^{(7)}, \quad V^{(1)}V^{(4)}, \quad V^{(1)}V^{(7)}, \quad V^{(4)}V^{(7)}.
\]

(5.49)

where \( V^{(a)}V^{(b)} = V^{(a)}V^{(b)} + V^{(b)}V^{(a)} \) with \( a, b = 1, 2 \ldots 9 \). Thus they give 6 independent Killing tensors. The Hodge dual of these three Killing vectors

\[
\star V^{(1)}, \quad \star V^{(4)}, \quad \star V^{(7)}
\]

(5.50)

are three closed conformal Killing–Yano tensors. Using (5.46), one can find 3 conformal Killing tensors by these three closed conformal Killing–Yano tensors and their conformal factors. However, it can be proved directly that the linear combination of the Killing tensors found from \( V^{(1)}V^{(1)} \) and conformal Killing tensors obtained from \( \star V^{(1)} \) is metric multiplied
by a function, so does the combinations of conformal Killing tensors obtained from $V^{(4)}V^{(4)}$ and $\ast V^{(4)}$, and the combinations of conformal Killing tensors obtained from $V^{(7)}V^{(7)}$, $\ast V^{(7)}$. Therefore, so far we find 6 independent Killing tensors by Killing vectors.

On the other hand, one can construct 3 Killing tensors from Killing–Yano tensors:

$$Y^{(1)} = L^2 \cosh \rho d \rho \wedge dt,$$

$$Y^{(2)} = R^2 \sin \theta d \theta \wedge d \phi,$$

$$Y^{(3)} = \tilde{R}^2 \sin \tilde{\theta} d \tilde{\rho} \wedge d \tilde{\phi}. \quad (5.51)$$

It is easy to check that the Killing tensors found by (5.49) and (5.51) are same with the Killing tensors given by (5.48). Thus what we found from the first principle coincides with the Killing tensors obtained from Killing vectors and Killing–Yano tensors, i.e. there are no nontrivial (conformal) Killing tensors found in the case of $(n,m,\tilde{m})=(0,0,0)$.

For $(n,m,\tilde{m})=(1,0,0)$, by first principle we found\(^1\):

$$K'_{12} = c_1 ch^2_{\rho}, \quad K'_{22} = -2ic_1 ch^3_{\rho} sh_{\rho}, \quad K'_{14} = c_2 s_\theta^2 $$

$$K'_{16} = c_3 s_{\tilde{\theta}}^2, \quad K'_{24} = -ic_2 s_{\theta}^2 ch_{\rho} sh_{\rho}, \quad K'_{26} = -ic_3 s_{\tilde{\theta}}^2 ch_{\rho} sh_{\rho}. \quad (5.52)$$

where $c_1$, $c_2$, $c_3$ are 3 constants. This results, as we expected again, are symmetric about two sphere, for example, replacing $\theta$ to $\tilde{\theta}$, $K'_{14}$ then changes to $K'_{16}$. Three constants in (5.52) means there 3 independent $K'_{\rho, \theta, \tilde{\theta}}$. Recall that $(1,0,0)$ indicates the factor $e^{it}$. After adding this factor, there are 6 independent Killing tensors found by first principle.

On the other hand, to obtain Killing tensors for $(1,0,0)$ from Killing vectors, we need the combinations between a Killing vector only depending on $\rho$, $\theta$, $\tilde{\theta}$ and a Killing vector only depending on $\rho$, $\theta$, $\tilde{\theta}$, $t$. From (5.42), we see that $V^{(2)}$ and $V^{(3)}$ are only the function of $\rho$, $t$ and $V^{(1)}$, $V^{(4)}$ and $V^{(7)}$ only depends on $\rho$, $\theta$, $\tilde{\theta}$. Therefore, there are 6 combinations

$$V^{(2)}V^{(1)}, \quad V^{(2)}V^{(4)}, \quad V^{(2)}V^{(7)}, \quad V^{(3)}V^{(1)}, \quad V^{(3)}V^{(4)}, \quad V^{(3)}V^{(7)}. \quad (5.53)$$

Thus from Killing vectors, the total number of Killing tensors we found is 6, which is the

---

\(^1\)Here we have picked up the arbitrary function in the results found by first principle such that all tensors become Killing tensors.
same number of Killing tensors found from first principle. After a little calculation, by direct
comparison, it is to see that the 6 Killing tensors found from Killing vectors are same as the
6 Killing tensors obtained from first principle, which tells us that in this case, i.e. (1, 0, 0),
again, same as (0, 0, 0), there are no nontrivial Killing tensors.

For \((n, m, \tilde{m})=(0, 1, 0)\), by first principle we found

\[
K'_{23} = c_1 c h^2_\rho, \quad K'_{24} = i c_1 c_\theta s_\theta c h^2_\rho, \quad K'_{34} = c_2 s^2_\theta \\
K'_{44} = 2i c_2 c_\theta s^3_\theta, \quad K'_{36} = c_3 s^2_\theta, \quad K'_{46} = i c_3 c_\theta s_\theta s^2_\theta.
\] (5.54)

Same as the previous case, (5.54) only give Killing tensors after choosing proper conformal
factor. The three constant in (5.54) indicates there are 3 independent \(K'(\rho, \theta, \tilde{\theta})\). There
is another factor \(e^{i\phi}\) in the full expression of Killing tensors. Therefore, in current case, the
total number of independent Killing tensors found by first principle is 6.

In (5.42), only \(V^{(5)}\) and \(V^{(6)}\) are the function of \(\theta, \phi\). Therefore, in order to find Killing
tensors belonging to (0, 1, 0) from Killing vectors, we need 6 combinations as:

\[
V^{((5)V^{(1)}}), \quad V^{((5)V^{(4)}}, \quad V^{((5)V^{(7)}}, \quad V^{((6)V^{(1)}}, \quad V^{((6)V^{(4)}}, \quad V^{((6)V^{(7)}}. \quad (5.55)
\]

It is straightforward to prove that the 6 Killing tensors found from the combinations (5.55)
are same as the 6 Killing tensors found from first principle. Thus, in this case, we also do
not find nontrivial conformal Killing tensors.

Due to the symmetry about the two sphere, which we have seen from the results
above, Replacing \(\theta, \phi\) with \(\tilde{\theta}, \tilde{\phi}\) in (0, 1, 0), we immediately obtain Killing tensors for
\((n, m, \tilde{m})=(0, 0, 1)\). In this case, of course, same as (0, 1, 0), there are no nontrivial con-
formal Killing tensors.

For \((n, m, \tilde{m})=(1, 1, 0)\), by first principle we found

\[
K'_{13} = -1, \quad K'_{14} = -i c_\theta s_\theta, \quad K'_{23} = i c_\rho s_\rho s_\theta, \quad K'_{24} = -c_\theta c_\rho s_\rho s_\theta.
\] (5.56)

In this case, there is one \(K'(\rho, \theta, \tilde{\theta})\), and there are 4 independent conformal Killing tensors
after we considering the factor \(e^{i(t+\phi)}\).
To find conformal Killing tensors depending on $t$, $\phi$ from Killing vectors, we need the combinations of Killing vectors depending on $t$ and Killing vectors depending on $\phi$. From (5.42), we know that only $V^{(2)}$, $V^{(3)}$ are function of $t$ and only $V^{(5)}$, $V^{(6)}$ are function of $\phi$. Therefore, the Killing tensors, we are interested in for this case can be found from

$$V^{((2)V^{(5)}}, \quad V^{((2)V^{(6)}}, \quad V^{((3)V^{(5)}}, \quad V^{((3)V^{(6)}}. \tag{5.57}$$

As we mentioned below (5.42), $V^{(1)}$ to $V^{(6)}$ are obtained from AdS and one sphere. As can been seen from (5.56), all these non–vanished components are only involved in AdS and one sphere. Therefore, it is not surprise that the 4 Killing tensors obtained from Killing vectors are same as the Killing tensors found by first principle, which tell us the unfortunate fact that we still have not found nontrivial conformal Killing tensors.

Similarly from $(0, 1, 0)$ to $(0, 0, 1)$, replacing $\theta$ to $\tilde{\theta}$, we immediately obtain all Killing tensors for $(n, m, \tilde{m})=(1, 0, 1)$ from $(1, 1, 0)$. In this case, all non–vanished components are only involved in AdS and another spherical space directions, and they can be found by combinations of Killing vectors in AdS and another 2–sphere.

For $(n, m, \tilde{m})=(0, 1, 1)$, by first principle we found

$$K_{35}'=1, \quad K_{36}'=ic\tilde{\theta}s\tilde{\theta}, \quad K_{45}'=ic\theta s\theta, \quad K_{46}'=-c\theta c\tilde{\theta}s\theta s\tilde{\theta}. \tag{5.58}$$

As expected, all non–vanished components, similarly to the cases $(1, 1, 0)$ and $(1, 0, 1)$, are only involved two spheres. Note that, in (5.42), only $V^{(5)}$, $V^{(6)}$ are function of $\phi$ and only $V^{(8)}$, $V^{(9)}$ are function of $\tilde{\phi}$. Thus from Killing vectors, 4 Killing tensors are obtained from the combinations

$$V^{((5)V^{(8)}}, \quad V^{((5)V^{(9)}}, \quad V^{((6)V^{(8)}}, \quad V^{((6)V^{(9)}}. \tag{5.59}$$

These four Killing tensors and the Killing tensors found by first principle (5.58) are same. Thus no nontrivial Killing tensors are found.

For $(n, m, \tilde{m})=(2, 0, 0)$, by first principle we find the non–vanished components $K''(\rho, \theta, \tilde{\theta})$
for Killing tensors

\[ K'_{11} = i, \quad K'_{12} = ch_\rho sh_\rho, \quad K'_{22} = -ich_\rho^2 sh_\rho^2. \]  

(5.60)

These three components are only related to AdS space. Adding factor \( e^{2it} \) for this case, we can write the full expression for the components of Killing tensors as

\[ K_{MN} = K'_{MN}(\rho, \theta, \tilde{\theta}) \left( A_{MN1} \cos^2 t + 2iA_{MN2} \cos t \sin t - A_{MN3} \sin^2 t \right) \]  

(5.61)

where \( A_{MN1}, A_{MN2}, A_{MN3} \) are constants. Substituting (5.60) into (5.61), we get the specific expression for three non–vanished components of Killing tensors with 9 constants. By solving Killing tensor equations, we can reduce 9 constants to 3

\[

table

K_{11} = -4(A_3 c_1^2 + 2iA_2 c_1 s_1 - A_1 s_1^2) \\
K_{12} = \frac{1}{2} sh_\rho (2A_2 c_2 t + i(A_1 + A_3)s_2 t) \\
K_{22} = 2ch_\rho^2 sh_\rho^2 (-iA_1 c_1^3 + iA_3 s_1^2 + A_2 s_2 t).
\]

(5.62)

where \( A_1, A_2, A_3 \) are three constants. Thus through first principle, we find three independent Killing tensors for \((n, m, \tilde{m})=(2, 0, 0)\) whose components are only function of \( \rho, t \) which are the coordinates of AdS space.

By observing (5.42), we see that, to find Killing tensors belonging to \((2, 0, 0)\) from Killing vectors, we need three possible ways to combine \( V^{(2)} \) and \( V^{(3)} \), which originates from the Killing vectors of AdS space

\[ V^{(2)} V^{(2)}, \quad V^{(3)} V^{(3)}, \quad V^{(2)} V^{(3)} \]  

(5.63)

It is easy to check that the three Killing tensors obtained from Killing vectors are same as the three Killing tensors (5.62) found by first principle. However, using \( V^{(2)} \) and \( V^{(3)} \) we can find two closed conformal Killing–Yano tensors, and thus two conformal Killing tensors, which seems not to be covered by our results via solving PDEs of conformal Killing tensors. In fact the two conformal Killing tensors found by the dual of Killing vectors

\[ \star V^{(2)}, \quad \star V^{(3)} \]  

(5.64)
are not independent with what we found from Killing vectors (5.63). It is easy to prove that the linear combination of conformal Killing tensors obtained from \( V^{(2)} V^{(2)} \) and \( \star V^{(2)} \) can be metric multiplied by a function, and conformal Killing tensors obtained from \( V^{(3)} V^{(3)} \) and \( \star V^{(3)} \) can be also metric multiplied by a function. Therefore, what we found from first principle is complete same as what we found from Killing vectors, which unfortunately means there are still no nontrivial conformal Killing tensors so far.

For \((n, m, \tilde{m}) = (0, 0, 2)\), by first principle we find the non–vanished components of \( K'(\rho, \theta, \tilde{\theta}) \) are

\[
\begin{align*}
K'_{55} & = -1, \quad K'_{56} = -i c_\tilde{\theta} s_\tilde{\theta}, \quad K'_{66} = c_\tilde{\theta}^2 s_\tilde{\theta}^2. \tag{5.65}
\end{align*}
\]

It is not surprising that, similarly to \((2, 0, 0)\), these components only related to second 2–sphere. Following what we have done for \((2, 0, 0)\), we can write the full expression for non–vanished components of Killing tensors as

\[
K_{MN} = K'_{MN}(\rho, \theta, \tilde{\theta}) \left( B_{MN_1} \cos^2 \tilde{\phi} + 2i B_{MN_2} \cos \tilde{\phi} \sin \tilde{\phi} - B_{MN_3} \sin^2 \tilde{\phi} \right) \tag{5.66}
\]

Again applying what we found in (5.65) into (5.66), then substituting into Killing tensors equations, only three constants are left

\[
\begin{align*}
K'_{55} &= -2B_3 c_\phi^2 - 4iB_2 c_\phi s_\phi + 2B_1 s_\phi^2, \\
K'_{56} &= \frac{1}{2} s_\phi s_{\tilde{\phi}} (-2iB_2 c_\phi s_\phi + (B_1 + B_3) s_{\tilde{\phi}}), \\
K'_{66} &= 2c_\phi^2 s_{\tilde{\phi}}^2 (B_1 c_\phi^2 + 2iB_2 c_\phi s_\phi - B_3 s_\phi^2).
\end{align*} \tag{5.67}
\]

where \(B_1, B_2, B_3\) are 3 constants. Like the case \((2, 0, 0)\), there are 3 independent Killing tensors found by first principle, which all components are only function of the coordinates of second sphere.

On the other hand, from (5.42) we know that only \(V^{(8)}\) and \(V^{(9)}\) are functions of \(\tilde{\phi}\), which are, as mentioned below (5.42), found from the second sphere. Therefore to build Killing tensors are classified to \((0, 0, 2)\) from Killing vectors, we need the combinations

\[
\begin{align*}
V^{(8)} V^{(8)}, \quad V^{(9)} V^{(9)}, \quad V^{(8)} V^{(9)}. \tag{5.68}
\end{align*}
\]
The three Killing tensors are found by these combinations are complete same with what we found from conformal Killing tensors equations, which can be checked easily after a little calculation. We expect that, according to the experience got from the case \((2, 0, 0)\), the two conformal Killing tensors obtained by the closed conformal Killing–Yano tensors which found by computing the Hodge dual of Killing vectors \(V^{(8)}\) and \(V^{(9)}\)

\[ \ast V^{(8)}, \ast V^{(9)}, \]  

(5.69)

are not independent. On can prove, same as \((2, 0, 0)\) case, that the linear combination of conformal Killing tensors obtained from \(V^{(8)}V^{(8)}\) and \(\ast V^{(8)}\) can be metric multiplied by a function, and conformal Killing tensors obtained from \(V^{(9)}V^{(9)}\) and \(\ast V^{(9)}\) can be also metric multiplied by a function. Therefore, in this case, we still do not find nontrivial Killing tensors.

Recall that \((0, 2, 0)\) and \((0, 0, 2)\) are symmetric. Replacing \(\tilde{\theta}\) with \(\theta\) and \(\tilde{\phi}\) with \(\phi\) in \((0, 0, 2)\), we immediately obtain Killing tensors for \((n, m, \tilde{m}) = (0, 2, 0)\).

In the calculation of solving PDEs for conformal Killing tensors, we meet the factors such as \((L^2 - \tilde{R}^2), (R^2 + \tilde{R}^2)\). Unfortunately, the constraints given by vanishing these factors does not give anything new.

Other cases \((1, 1, 1), (n \geq 2, 1, 0), (n \geq 2, 1, 1), (n \geq 3, 0, 0), (1, 0, \tilde{m} \geq 2), (0, 1, \tilde{m} \geq 2), (1, 1, \tilde{m} \geq 2), (0, 0, \tilde{m} \geq 3), (n \geq 2, m \geq 2, \tilde{m}), (n, m \geq 2, \tilde{m} \geq 2)\) only give the metric multiplied by an arbitrary function.

### 5.5.2 Summary of results

We have indicated that the Killing tensors found from first principle completely coincide with the ones obtained from Killing vectors and Killing–Yano tensors, which demonstrates that, in \(\text{AdS}_2 \times \mathbb{S}^2 \times \mathbb{S}^2\), all conformal Killing tensors can be constructed from Killing vectors and Killing–Yano tensors which originate in AdS and spherical spaces and all these conformal Killing tensors are trivial. Even we have tried all the possible constraints on radii that we can in the calculation, there are still no nontrivial conformal Killing tensors.

In this summary, to clearly show all Killing tensors in \(\text{AdS}_2 \times \mathbb{S}^2 \times \mathbb{S}^2\), we give a table
below. In this table we only give expression for the part depends on $\rho$, $\theta$, $\tilde{\theta}$. Adding the factor $e^{int+im\phi+i\tilde{m}\tilde{\phi}}$ will give the full expression.

<table>
<thead>
<tr>
<th>$(n, m, \bar{m})$</th>
<th>$K'(\rho, \theta, \tilde{\theta})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 0)$</td>
<td>$K'<em>{11} = f$, $K'</em>{22} = ch^4_\rho \left( c_1 - f sech^2_\rho \right)$, $K'_{33} = \frac{Fr^2}{L^2} + c_4$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{44} = s^2</em>\theta \left( \frac{Fr^2}{L^2} + c_5 s^2_\theta + c_4 \right)$, $K'<em>{55} = \frac{Fr^2}{L^2} + c_7$, $K'</em>{66} = s^2_\theta \left( \frac{Fr^2}{L^2} + c_8 s^2_\theta + c_4 \right)$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{24} = c_2 ch^2</em>\rho s^2_\theta$, $K'<em>{26} = c_3 ch^2</em>\rho s^2_\theta$, $K'<em>{46} = c_6 s^2</em>\theta s^2_\bar{\theta}$</td>
</tr>
<tr>
<td>$(1, 0, 0)$</td>
<td>$K'<em>{12} = c_1 ch^2</em>\rho$, $K'<em>{22} = -2ic_1 ch^3</em>\rho sh_\rho$, $K'<em>{14} = c_2 s^2</em>\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{16} = c_3 s^2</em>\theta$, $K'<em>{24} = -ic_2 s^2</em>\rho ch_\rho sh_\rho$, $K'<em>{26} = -ic_3 s^2</em>\rho ch_\rho sh_\rho$</td>
</tr>
<tr>
<td>$(0, 1, 0)$</td>
<td>$K'<em>{23} = c_1 ch^2</em>\rho$, $K'<em>{24} = ic_1 c</em>\theta s_\theta ch^2_\rho$, $K'<em>{34} = c_2 s^2</em>\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{44} = 2ic_2 c</em>\theta s^2_\theta$, $K'<em>{36} = c_3 s^2</em>\theta$, $K'<em>{46} = ic_3 c</em>\theta s_\theta s^2_\bar{\theta}$</td>
</tr>
<tr>
<td>$(0, 0, 1)$</td>
<td>$K'<em>{25} = c_1 ch^2</em>\rho$, $K'<em>{26} = ic_1 c</em>\bar{\theta} s_\bar{\theta} ch^2_\rho$, $K'<em>{56} = c_2 s^2</em>\bar{\theta}$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{66} = 2ic_2 c</em>\bar{\theta} s^2_\bar{\theta}$, $K'<em>{45} = c_3 s^2</em>\bar{\theta}$, $K'<em>{46} = ic_3 c</em>\bar{\theta} s_\bar{\theta} s^2_\theta$</td>
</tr>
<tr>
<td>$(1, 1, 0)$</td>
<td>$K'<em>{13} = -1$, $K'</em>{14} = -ic_\theta s_\theta$, $K'<em>{23} = ich</em>\rho sh_\rho$, $K'<em>{24} = -c</em>\theta ch_\rho s_\theta sh_\rho$</td>
</tr>
<tr>
<td>$(1, 0, 1)$</td>
<td>$K'<em>{15} = -1$, $K'</em>{16} = -ic_\bar{\theta} s_\bar{\theta}$, $K'<em>{25} = ich</em>\rho sh_\rho$, $K'<em>{26} = -c</em>\bar{\theta} ch_\rho s_\bar{\theta} sh_\rho$</td>
</tr>
<tr>
<td>$(0, 1, 1)$</td>
<td>$K'<em>{35} = 1$, $K'</em>{36} = ic_\theta s_\theta$, $K'<em>{45} = ic</em>\bar{\theta} s_\bar{\theta}$, $K'<em>{46} = -c</em>\theta c_\bar{\theta} s_\theta s_\bar{\theta}$</td>
</tr>
<tr>
<td>$(n, \ m, \ \tilde{m})$</td>
<td>$K'(\rho, \ \theta, \ \tilde{\theta})$</td>
</tr>
<tr>
<td>-----------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>$(2, \ 0, \ 0)$</td>
<td>$K'<em>{11} = 2i, \ K'</em>{12} = sh_{2\rho}, \ K'<em>{22} = -2ich</em>{\rho}^2sh_{\rho}^2$</td>
</tr>
<tr>
<td>$(0, \ 2, \ 0)$</td>
<td>$K'<em>{33} = -2, \ K'</em>{34} = -is_{2\theta}, \ K'<em>{44} = 2c</em>{\theta}^2s_{\theta}^2$</td>
</tr>
<tr>
<td>$(0, \ 0, \ 2)$</td>
<td>$K'<em>{55} = -2, \ K'</em>{56} = -is_{2\tilde{\theta}}, \ K'<em>{66} = 2c</em>{\tilde{\theta}}^2s_{\tilde{\theta}}^2$</td>
</tr>
</tbody>
</table>

5.6 (Conformal) Killing tensors in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3$

What is physically interesting is not $\text{AdS}_2 \times \text{S}^2 \times \text{S}^2$ but $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3$ in string theory. In this section we will turn to $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3$ spacetime. According the comparison of the results obtained from first principle and from Killing vectors and Killing–Yano tensors in $\text{AdS}_2 \times \text{S}^2 \times \text{S}^2$, we infer all the (conformal) Killing tensors in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3$ can be found from Killing vectors and Killing–Yano tensors.

5.6.1 Set–up

First of all, we write the metric of $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3$ as

\[
    ds^2 = L^2 (d\rho^2 - \cosh^2 \rho d\tau^2 + \sinh^2 \rho d\alpha^2) + R^2 (d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2) + \tilde{R}^2 \left( d\tilde{\theta}^2 + \cos^2 \tilde{\theta} d\tilde{\phi}^2 + \sin^2 \tilde{\theta} d\tilde{\psi}^2 \right),
\]

where $L, \ R, \ \tilde{R}$ are radii of AdS and two spheres, respectively.

In this part we will figure out (conformal) Killing vectors and (conformal) Killing–Yano tensors in AdS and two spherical spaces systematically. Following the constructions introduced in previous section, first let’s define

\[
    L(\cosh \rho \cos \tau, \ \sinh \rho \cos \alpha, \ \sinh \rho \sin \alpha, \ \cosh \rho \sin \tau) \equiv (A^{(1)}, \ A^{(2)}, \ A^{(3)}, \ A^{(4)}),
\]
\[ R(\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta \cos \psi, \sin \theta \sin \psi) \equiv (S^{(1)}, S^{(2)}, S^{(3)}, S^{(4)}), \quad (5.72) \]

\[ \tilde{R}(\cos \tilde{\theta} \cos \phi, \cos \tilde{\theta} \sin \phi, \sin \tilde{\theta} \cos \psi, \sin \tilde{\theta} \sin \psi) \equiv (\tilde{S}^{(1)}, \tilde{S}^{(2)}, \tilde{S}^{(3)}, \tilde{S}^{(4)}). \quad (5.73) \]

Recall that these \( A^{(a)}, S^{(a)}, \tilde{S}^{(a)} \) are the coordinates of AdS and two spheres in four-dimensional flat space. Then conformal Killing vectors of AdS and two spherical spaces, according to (5.23), are given by

\[ V_\mu^A = \partial_\mu A, \quad V_a^S = \partial_a S, \quad V_m^{\tilde{S}} = \partial_m \tilde{S}, \quad (5.74) \]

where \( \mu, \nu \) are indices in AdS, while \( a, b \) and \( m, n \) are indices in \( S^3 \) and \( \tilde{S}^3 \), respectively. On the other hand, according to (5.24), the Killing vectors are

\[ V^A = \star (dA^{(i)} \wedge dA^{(j)}), \quad V^S = \star (dS^{(i)} \wedge dS^{(j)}), \quad V^{\tilde{S}} = \star (d\tilde{S}^{(i)} \wedge d\tilde{S}^{(j)}), \quad (5.75) \]

where the hodge dual is evaluated in 3 dimensions, i.e. in AdS, and two \( S^3 \) respectively.

Noting the Killing vectors and Killing–Yano tensors in AdS and spherical spaces are also the Killing vectors and Killing–Yano tensors in AdS \( \times S^3 \times S^3 \), and using the expressions (5.75), we find six Killing vectors from AdS space

\[ V_A^{(1)} = -Lch^2 \rho d\tau, \quad V_A^{(2)} = -Lsh^2 \rho d\alpha, \]
\[ V_A^{(3)} = -Ls \tau s_\alpha d\rho + Lch^2 \rho (c_\tau s_\alpha d\tau - s_\tau c_\alpha d\alpha), \]
\[ V_A^{(4)} = Ls c_\alpha d\rho - Lch^2 \rho (c_\tau c_\alpha d\tau + s_\tau s_\alpha d\alpha), \]
\[ V_A^{(5)} = -Lc_\tau s_\alpha d\rho - Lch^2 \rho s_\tau s_\alpha d\tau + c_\tau c_\alpha d\alpha), \]
\[ V_A^{(6)} = -Lc_\tau c_\alpha d\rho + Lch^2 \rho (s_\tau c_\alpha d\tau - c_\tau s_\alpha d\alpha). \quad (5.76) \]

All these Killing vectors are only function of \( \rho, \tau, \alpha \). Especially, the first two Killing vectors
are only function of $\rho$. Similarly, we can find six Killing vectors from first sphere

\begin{align*}
V_S^{(1)} &= -Rc_\phi d\phi, & V_S^{(2)} &= -Rs_\psi d\psi, \\
V_S^{(3)} &= Rs_\phi s_\psi d\theta - Rc_\theta s_\phi (c_\phi s_\psi d\phi - s_\phi c_\psi d\psi), \\
V_S^{(4)} &= -Rs_\phi c_\psi d\theta + Rc_\theta s_\phi (c_\phi c_\psi d\phi + s_\phi s_\psi d\psi), \\
V_S^{(5)} &= -Rc_\phi s_\psi d\theta - Rc_\theta s_\phi (s_\phi s_\psi d\phi + c_\phi c_\psi d\psi), \\
V_S^{(6)} &= Rc_\phi c_\psi d\theta + Rc_\theta s_\phi (s_\phi c_\psi d\phi - c_\phi s_\psi d\psi). \tag{5.77}
\end{align*}

Similarly to (5.76), all the Killing vectors found here are only function of $\theta, \phi, \psi$, which are the coordinates of first sphere. The first two Killing vectors only depend on $\theta$ while the last four of them depend on all coordinates of first sphere. There are 6 more Killing vectors originating in second sphere. Due to the symmetry of the two spheres, it is straightforward to obtain the rest Killing vectors by changing $\theta, \phi, \psi$ to $\tilde{\theta}, \tilde{\phi}, \tilde{\psi}$ in (5.77). Therefore, there are two Killing vectors only depends on $\tilde{\theta}$, and other four of them depend on all coordinates of second sphere.

Using (5.74), (5.25) and (5.28), we can easily figure out four two–rank Killing–Yano tensors and one three–rank Killing–Yano tensor for AdS and two spherical spaces, respectively. Expanding these Killing–Yano tensors to 9 dimensions, we thus can find 12 two–rank Killing–Yano tensors and 3 three–rank Killing–Yano tensors in $\text{AdS}_3 \times S^3 \times S^3$. Recall the relation between Killing tensors and Killing–Yano tensors (5.44). We then can obtain 15 Killing tensors in $\text{AdS}_3 \times S^3 \times S^3$. The Killing tensors found from Killing–Yano tensors originating in AdS only depend on the coordinates of AdS, while the ones found from Killing–Yano tensor originating in sphere only depend on the coordinates of the sphere. in other words, The non–vanished components of Killing tensors originating in AdS are only along the AdS direction, while the non–vanished components of Killing tensors originating in sphere are only along spherical directions.

### 5.6.2 Constructing (Conformal) Killing tensors

As we mentioned at the beginning of this section, what the conclusion we get from previous section is in $\text{AdS}_3 \times S^3 \times S^3$ all the conformal Killing tensors are only the metric multiplied by an arbitrary function, and all the Killing tensors can be constructed from
Killing vectors and Killing–Yano tensors. Therefore, in this section, we will not solve PDEs again. Instead of it, we will obtain all Killing tensors from Killing vectors and Killing–Yano tensors which are found by expanding Killing vectors and Killing–Yano tensors in AdS and spherical spaces to 9 dimensions.

To do the parallel presentation with $\text{AdS}_2 \times S^2 \times S^2$, and to classify the Killing tensors, we can write the Killing tensors as

$$K(\rho, \tau, \alpha, \theta, \phi, \psi, \tilde{\theta}, \tilde{\phi}, \tilde{\psi}) = K'(\rho, \theta, \tilde{\theta})e^{i[n(a_n \tau + b_n \alpha) + m(a_m \phi + b_m \psi) + \tilde{m}(a_{\tilde{m}} \tilde{\theta} + b_{\tilde{m}} \tilde{\psi})]}.$$ (5.78)

We first consider the case that $a_n = b_n = a_m = b_m = a_{\tilde{m}} = b_{\tilde{m}} = 1$. We then can again classify the Killing tensors by three integers $n, m, \tilde{m}$:

<table>
<thead>
<tr>
<th>$(n, m, \tilde{m})$</th>
<th>$K'(\rho, \theta, \tilde{\theta})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 0)$</td>
<td>$K'<em>{11} = c</em>{22}$, $K'<em>{22} = ch^2</em>\rho(-c_{22} + 2c_1ch^2_\rho)$, $K'<em>{23} = c_3ch^2</em>\rho sh^2_\rho$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{25} = c</em>{10}c^2_\theta ch^2_\rho$, $K'<em>{26} = c</em>{12}s^2_\theta ch^2_\rho$, $K'<em>{28} = c</em>{14}c^2_\theta ch^2_\rho$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{29} = \tilde{R}c</em>{16}s^2_\theta ch^2_\rho$, $K'<em>{33} = sh^2</em>\rho(c_{22} + 2c_2sh^2_\rho)$, $K'<em>{35} = c</em>{13}s^2_\theta sh^2_\rho$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{36} = c</em>{11}s^2_\theta sh^2_\rho$, $K'<em>{38} = c</em>{17}s^2_\theta sh^2_\rho$, $K'<em>{39} = c</em>{15}s^2_\theta sh^2_\rho$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{44} = c</em>{23}$, $K'<em>{55} = c^2</em>\theta(c_{23} + 2c_4c^2_\theta)$, $K'<em>{56} = c_6c^2</em>\theta s^2_\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{58} = c</em>{18}c^2_\theta c^2_\theta$, $K'<em>{59} = c</em>{20}c^2_\theta s^2_\theta$, $K'<em>{66} = s^2</em>\theta(c_{23} + 2c_5s^2_\theta)$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{68} = c</em>{21}s^2_\theta c^2_\theta$, $K'<em>{69} = c</em>{19}s^2_\theta s^2_\theta$, $K'<em>{77} = c</em>{24}$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{88} = c^2</em>\theta(c_{24} + 2c_7c^2_\theta)$, $K'<em>{89} = c_9c^2</em>\theta s^2_\theta$, $K'<em>{99} = s^2</em>\theta(c_{24} + 2c_8s^2_\theta)$</td>
</tr>
<tr>
<td>$(n, m, \bar{m})$</td>
<td>$K'(\rho, \theta, \bar{\theta})$</td>
</tr>
<tr>
<td>------------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td><strong>(1, 0, 0)</strong></td>
<td>$K'<em>{12} = c_1 c^2</em>\rho, \ K'<em>{13} = c_2 s^2</em>\rho, \ K'<em>{15} = c_3 c^2</em>\theta, \ K'<em>{16} = c_4 s^2</em>\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{18} = c_5 c^2</em>\theta, \ K'<em>{19} = c_6 s^2</em>\theta, \ K'<em>{22} = -2i c_1 c^3</em>\rho s_\rho$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{23} = i c \rho s</em>\rho (c_1 c^2_\rho - c_2 s^2_\rho), \ K'<em>{25} = -i c_3 c</em>\rho s_\rho c^2_\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{26} = -i c_4 c</em>\rho s_\rho s^2_\theta, \ K'<em>{28} = -i c_5 c</em>\rho s_\rho c^2_\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{29} = -i c_6 c</em>\rho s_\rho c^2_\theta, \ K'<em>{33} = 2i c_2 c</em>\rho s^3_\rho, \ K'<em>{35} = i c_5 c</em>\rho s_\rho c^2_\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{36} = i c_4 c</em>\rho s_\rho s^2_\theta, \ K'<em>{38} = i c_5 c</em>\rho s_\rho c^2_\theta, \ K'<em>{39} = i c_6 c</em>\rho s_\rho c^2_\theta$</td>
</tr>
<tr>
<td><strong>(0, 1, 0)</strong></td>
<td>$K'<em>{24} = c_1 c^2</em>\rho, \ K'<em>{25} = -i c_1 c</em>\theta s_\theta c^2_\rho, \ K'<em>{26} = i c_1 c</em>\theta s_\theta c^2_\rho$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{34} = c_2 s^2</em>\rho, \ K'<em>{35} = -i c_2 c</em>\theta s_\theta s^2_\rho, \ K'<em>{36} = i c_2 c</em>\theta s_\theta s^2_\rho$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{45} = c_3 c^2</em>\theta, \ K'<em>{46} = c_4 s^2</em>\theta, \ K'<em>{48} = c_5 c^2</em>\theta, \ K'<em>{49} = c_6 s^2</em>\theta, \ K'<em>{55} = -2i c_3 c^2</em>\theta s_\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{56} = i c</em>\theta s_\theta (c_3 c^2_\theta - c_4 s^2_\theta), \ K'<em>{58} = -i c_5 c</em>\theta s_\theta c^2_\theta, \ K'<em>{59} = -i c_6 c</em>\theta s_\theta s^2_\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{60} = 2i c_4 c</em>\theta s^3_\theta, \ K'<em>{68} = -i c_5 c</em>\theta s_\theta c^2_\theta, \ K'<em>{69} = -i c_6 c</em>\theta s_\theta s^2_\theta$</td>
</tr>
<tr>
<td><strong>(1, 1, 0)</strong></td>
<td>$K'<em>{14} = 1, \ K'</em>{15} = -i c_\theta s_\theta, \ K'<em>{16} = i c</em>\theta s_\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{24} = i c \rho s</em>\rho, \ K'<em>{25} = -i c \rho s</em>\rho c_\theta s_\theta, \ K'<em>{26} = i c \rho s</em>\rho c_\theta s_\theta$</td>
</tr>
<tr>
<td></td>
<td>$K'<em>{34} = i c \rho s</em>\rho, \ K'<em>{35} = i c \rho s</em>\rho c_\theta s_\theta, \ K'<em>{36} = -i c \rho s</em>\rho c_\theta s_\theta$</td>
</tr>
</tbody>
</table>
The above tables present the Killing tensors obtained from Killing vectors and three–rank Killing–Yano tensors. The other cases not corresponding to \(a_n = b_n = a_m = b_m = a_{\tilde{m}} = b_{\tilde{m}} = 1\) will be given in next three tables.

For case \((0, 0, 0)\), there are 24 constants, \(c_1, c_2\cdots c_{24}\) in \(K'(\rho, \theta, \tilde{\theta})\), which indicate 24 independent Killing tensors. All these Killing tensors are only depends on \(\rho, \theta, \tilde{\theta}\), and thus they are obtained from the Killing vectors which only depend these coordinates. As can been seen from (5.76) and (5.77), there are 6 Killing vectors only depend on \(\rho, \theta, \tilde{\theta}\). Two of them come from the hidden symmetries of AdS space, and every sphere provides two such Killing vectors. Therefore, via Killing vectors, there are \(6 + 6 + \frac{5 \times 5}{2} = 21\) Killing tensors.

On the other hand, applying relation Killing tensors and Killing–Yano tensors (5.24), we can find three Killing tensors from 3 three–rank Killing–Yano tensors. Recall that there are, as we mentioned above, one three–rank Killing–Yano tensor for AdS and two spheres respectively, which can be expand to 9 dimensions thereby giving 3 three–rank Killing–Yano tensors in \(\text{AdS}_3 \times S^3 \times S^3\). One linear combination of the 3 Killing tensors constructed from these 3
three–rank Killing–Yano tensors is metric.

For case (1, 0, 0), there are 6 constants \( c_1, c_2 \cdots c_6 \) in \( K'(\rho, \theta, \tilde{\theta}) \). Adding the factor \( e^{i(\tau+\alpha)} \) corresponding to (1, 0, 0), there are 24 Killing tensors. All these Killing tensors are constructed from the combinations of 4 Killing vectors depending on \( \tau, \alpha \) and 6 Killing vectors only depending on \( \rho, \theta, \tilde{\theta} \). There are, same as (1, 0, 0), 6 constants \( c_1, c_2 \cdots c_6 \) in \( K'(\rho, \theta, \tilde{\theta}) \) for case (0, 1, 0). Therefore, there are 24 Killing tensors for this group after we consider the factor \( e^{i(\phi+\psi)} \). These 24 Killing tensors are constructed from the combinations of 4 Killing vectors depending on \( \phi, \psi \) and 6 Killing vectors only depending on \( \rho, \theta, \tilde{\theta} \). We do not present the case (0, 0, 1) in the table since it can be written down directly by changing \( \theta, \phi, \psi \) in (0, 1, 0) to \( \tilde{\theta}, \tilde{\phi}, \tilde{\psi} \). The reason for we can do this is obvious which is due to the two spheres can be treated equivalently, as we showed in previous section for \( \text{AdS}_2 \times S^2 \times S^2 \), and there are also 24 of them.

For case (1, 1, 0), there are 16 Killing tensors after adding factor \( e^{i(\tau+\alpha)+i(\phi+\psi)} \). They are obtained from the combinations of 4 Killing vectors depending on \( \tau, \alpha \) and 4 Killing vectors depending on \( \phi, \psi \). The combinations of 4 Killing vectors depending on \( \phi, \psi \) and 4 Killing vectors depending on \( \tilde{\phi}, \tilde{\psi} \) give 16 Killing tensors which corresponding to the case (0, 1, 1) in the table. For the case (1, 0, 1) which does not be shown in the table can be find again by changing \( \theta, \phi, \psi \) to \( \tilde{\theta}, \tilde{\phi}, \tilde{\psi} \) in (1, 1, 0).

For case (2, 0, 0) in the table can give 10 Killing tensors after adding \( e^{2i(\tau+\alpha)} \). To see this, one needs to expand \( e^{2i(\tau+\alpha)} \) to 9 terms and then give them by different coefficients for every terms in every components. Finally substituting them into Killing tensor equations should only leave 10 constants alive which is similar to the case (2, 0 , 0) in \( \text{AdS}_2 \times S^2 \times S^2 \) (5.62). These 10 Killing tensors are constructed by the combinations of 4 Killing vectors depending on \( \tau, \alpha \), i.e. \( 4+\frac{4 \times 3}{2} = 10 \). Similarly, there are 10 Killing tensors corresponding to case (0, 2, 0) constructed by 4 Killing vectors depending on \( \phi, \psi \) themselves. By 4 Killing vectors depending on \( \tilde{\phi}, \tilde{\psi} \), 10 more Killing tensors can be found that corresponds to case (0, 0, 2) which does not be presented in the table but can be obtained from (0, 2, 0) by simply changing \( \theta, \phi, \psi \) to \( \tilde{\theta}, \tilde{\phi}, \tilde{\psi} \).

The Killing tensors in these tables are constructed from Killing vectors and three–rank Killing–Yano tensors. Killing tensors built from 2–rank Killing–Yano tensors are not
included in the above tables, and they give the cases either $a_i = 2$ or $b_i = 2$:

<table>
<thead>
<tr>
<th>$(n, \ a_n, \ b_n)$</th>
<th>$K'(\rho, \ \theta, \ \tilde{\theta})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2, 0)$</td>
<td>$K'<em>{11} = 1, \ K'</em>{12} = -ich_{\rho}sh_{\rho}, \ K'<em>{22} = -ch</em>{\rho}^2sh_{\rho}^2, \ K'<em>{33} = ch</em>{\rho}^2sh_{\rho}^2$</td>
</tr>
<tr>
<td>$(1, 0, 2)$</td>
<td>$K'<em>{11} = 1, \ K'</em>{13} = ich_{\rho}sh_{\rho}, \ K'<em>{22} = ch</em>{\rho}^2sh_{\rho}^2, \ K'<em>{33} = -ch</em>{\rho}^2sh_{\rho}^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(m, \ a_m, \ b_m)$</th>
<th>$K'(\rho, \ \theta, \ \tilde{\theta})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2, 0)$</td>
<td>$K'<em>{44} = 1, \ K'</em>{45} = -ic_{\theta}s_{\theta}, \ K'<em>{55} = -c</em>{\theta}^2s_{\theta}^2, \ K'<em>{66} = c</em>{\theta}^2s_{\theta}^2$</td>
</tr>
<tr>
<td>$(1, 0, 2)$</td>
<td>$K'<em>{44} = 1, \ K'</em>{46} = ic_{\theta}s_{\theta}, \ K'<em>{55} = c</em>{\theta}^2s_{\theta}^2, \ K'<em>{66} = -c</em>{\theta}^2s_{\theta}^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(\tilde{m}, \ a_{\tilde{m}}, \ b_{\tilde{m}})$</th>
<th>$K'(\rho, \ \theta, \ \tilde{\theta})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2, 0)$</td>
<td>$K'<em>{77} = 1, \ K'</em>{78} = -ic_{\tilde{\theta}}s_{\tilde{\theta}}, \ K'<em>{88} = -c</em>{\tilde{\theta}}^2s_{\tilde{\theta}}^2, \ K'<em>{99} = c</em>{\tilde{\theta}}^2s_{\tilde{\theta}}^2$</td>
</tr>
<tr>
<td>$(1, 0, 2)$</td>
<td>$K'<em>{77} = 1, \ K'</em>{79} = ic_{\tilde{\theta}}s_{\tilde{\theta}}, \ K'<em>{88} = c</em>{\tilde{\theta}}^2s_{\tilde{\theta}}^2, \ K'<em>{99} = -c</em>{\tilde{\theta}}^2s_{\tilde{\theta}}^2$</td>
</tr>
</tbody>
</table>

Each case in the three tables above gives two Killing tensors after the full expression being considered. There are 12 Killing tensors in total, which correspond to 12 two–rank Killing–Yano tensors.

It is easy to check that, same as the case of AdS$_2$×S$^2$×S$^2$, the linear combinations of Killing tensors obtained from Killing vectors $V^{(i)}V^{(i)}$ and conformal Killing tensors found from $\star V^{(i)}$ are the metric multiplied by a function. Other cases that are not listed in the tables above only simply give metric multiplied by an arbitrary function according to what happened to AdS$_2$×S$^2$×S$^2$. 

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5.7 Discussion

By reviewing the so-called “Killing tower”, we present several important properties of conformal Killing form and how these properties help us to look for more hidden symmetries. Using the basic idea of the “Killing tower”, we give a way to figure out all the isometries and hidden symmetries in maximally symmetric space. We then identify the hidden symmetries in Kerr spacetime to the ones in flat spacetime, which provides us with an alternative way to finding hidden symmetries in Kerr–NUT–AdS spacetimes.

Employing this method, we constructed all symmetries in AdS space and spherical space and then we show how to construct the (conformal) Killing–Yano forms in $\text{AdS}_p \times S^p$ with same radii for AdS and sphere. One can obtain these conformal Killing forms from the symmetries of AdS and spherical space. The validity of this construction can be proved by another approach to these symmetries, which is based on a conformal transformation. Although the second method is straightforward, it is hard to proceed in higher dimensional spacetime, while constructing them from symmetries of subspace avoids the operation in very high dimensions. The other meaning of this construction is that it encodes the relation between the symmetries of $\text{AdS}_p \times S^p$ and the ones in its subspace.

By solving the conformal Killing tensor equation in $\text{AdS}_2 \times S^2 \times S^2$ directly, we find all the conformal Killing tensors in $\text{AdS}_2 \times S^2 \times S^2$. Comparing the results from the first principle, we notice that the all CKTs can be constructed from Killing vectors and Killing–Yano tensors in AdS and spherical spaces. This demonstrates that there are no nontrivial CKTs in $\text{AdS}_2 \times S^2 \times S^2$. To explore the possibilities as much as we can, we tried all the possible constraints on the radii of AdS and spheres which have potential to lead to nontrivial CKTs over the whole procedure in solving PDEs. Thus we can change our conclusion as that there are no nontrivial CKTs in $\text{AdS}_2 \times S^2 \times S^2$ no matter what relations between these three radii are.

Due to $\text{AdS}_2 \times S^2 \times S^2$ and $\text{AdS}_3 \times S^3 \times S^3$ have the very similar structure, it is unnecessary to repeat what we have done for $\text{AdS}_2 \times S^2 \times S^2$ to $\text{AdS}_3 \times S^3 \times S^3$ which would be very complicated. Thus we infer that in the space we are interested in, i.e. $\text{AdS}_3 \times S^3 \times S^3$, there are only trivial Killing tensors which can be found from Killing vectors and Killing–Yano tensors. As a consequence, we list Killing tensors obtained from Killing vectors and Killing–Yano tensors in previous sections, which are all the CKTs.
CHAPTER 6

Summary of the Results

In chapter 2, we quantized the bubbling geometries of AdS$_2 \times S^2$ by employing Crnković–Witten–Zukerman covariant method, obtaining canonical structures which can shed light into the unknown dual one-dimensional quantum mechanics. In chapter 3, we used the Dirac bracket in reduced phase space from the DBI action of supersymmetric D branes on AdS$_2 \times S^2$ to write down the grand canonical partition function for this system. We also derived the expression for the energy of bubbling solutions. In chapter 4, by inspecting the eigenvalues of conformal Killing tensor of the general form in a family of superstratum solutions, we proved that the Dirac equation on these backgrounds is not separable. In chapter 5, we showed that the symmetries of AdS$_p \times S^p$ can be constructed either from symmetries of AdS and sphere or from the symmetries of flat space. We demonstrate that there are no nontrivial conformal Killing tensors in AdS$_2 \times S^2 \times S^2$, and AdS$_3 \times S^3 \times S^3$, which implies that all symmetries of these spaces are encoded in the Killing vectors or Killing–Yano tensors.
A.1 Gravity solutions to two-charge system

By counting the excited states of D–brane configurations, one can derive the entropy of black hole in string theory. AdS/CFT indicates that each of these states is the state of conformal field theory on the boundary of the dual geometries, which can be mapped into a regular supergravity solutions. In this appendix we will give a review of supergravity solutions to D1–D5 system which are asymptotically $\text{AdS}_3 \times S^3 \times T^4$.

A.1.1 Dp–brane Supergravity solutions

If we put a D brane into a ten–dimensional flat spacetime, what will this spacetime look like? In this part, we take D2–brane as the example to show what the D–brane solution of supergravity looks like. We set the D2–brane extending along directions 0, 1, 2, where direction 0 is time direction and 1, 2 are two space directions. What we have learned from electromagnetism is that there is a three–form potential $C_{012}$ sourced by this D2–brane. The non–zero tension of D brane indicates the metric have to be coupled to it. There is one more field called the dilaton $\phi$ which is also sourced by the D brane.

Through a function $Z$ which is the function of the spacetime coordinates, these three fields are affected by this D2 brane. In other words, the three–form potential and the dilaton then can be written in terms of this function

$$C_{012} = Z^{-1}, \quad e^\phi = Z^{1/4},$$

(A.1)

while the metric is given by

$$ds^2 = g_{\mu\nu} = Z^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2) + Z^{1/2}(dx_3^2 + \cdots + dx_9^2).$$

(A.2)

The solution presented here are lorentz invariant along the directions 0, 1, 2, i.e. the D2
brane directions, while in the transverse directions, there are Euclidean symmetry.

Since the D2 brane is a point particle in the transverse space $\mathbb{R}^7$, similarly to the potential of Maxwell potential, the function $Z$ satisfies the Laplace equation on $\mathbb{R}^7$, i.e. $\nabla_7 Z = 0$, which is the results from the equation of motion of supergravity. This equation should be modified as the presence of sources $\rho_{D2}$

$$\nabla_7 Z = \rho_{D2}. \tag{A.3}$$

If there are $N_{D2}$ D2 branes, the density $\rho_{D2}$ then can be expressed as $\rho_{D2} = N_{D2}\delta(r_7)$. The solution of the Eq. (A.3) is expressed as

$$Z = 1 + \frac{N_{D2}}{r^5} \tag{A.4}$$

where $r^2 = x_3^2 + \cdots + x_5^2$, the radius of the transverse space.

Applying the rules above to the D1–D5 system, we need two $Z$ functions that we denote them as $h_1$ and $h_5$ for D1 and D5 brane, respectively. We can directly write the supergravity solution for this two–charge system as

$$ds^2 = \frac{1}{\sqrt{h_1 h_5}}(-dt^2 + dx_1^2) + \sqrt{h_1 h_5} dx^m dx^m + \sqrt{\frac{h_1}{h_5}} ds_4^2(x^a);$$

$$C = (h_1^{-1} - 1) dt \wedge dx_1 + \star_4 dh_5, \quad e^{-2\phi} = \frac{h_5}{h_1}, \tag{A.5}$$

where $m = 2, 3, 4, 5, a = 6, 7, 8, 9$ and $ds_4^2(x^a)$ represents the metric for $T^4$ or $K3$. Here the $\star_4$ denotes the Hodge dual in the four–dimensional flat space. In this solution D5 branes extend along the four wrapped direction $x^a$ and $x_1$ which is also the direction the D1 brane extended. In the table below "$\sqrt{\cdot}$" denotes the direction wrapped by the D branes. Here

<table>
<thead>
<tr>
<th>Dp–brane</th>
<th>t</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$x_3$</td>
<td>$x_4$</td>
<td>$x_5$</td>
<td>$x_6$</td>
<td>$x_7$</td>
<td>$x_8$</td>
<td>$x_9$</td>
<td></td>
</tr>
<tr>
<td>D5</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td>$\sqrt{\cdot}$</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

the coordinate $x_1$ is periodic and has radius $R_{x_1}$. There exist the single–centered solutions enable $h_i = (1 + Q_i/r^2)$, where the charges $Q_1$ can be written in terms of the integral charges.
\begin{align}
  N_i \text{ as} \quad Q_1 &= \left(\alpha'\right)^3 N_i, \quad Q_5 = \alpha' N_5, \\
  (A.6)
\end{align}

where \( \alpha' = l_s^2 \) (\( l_s \) is the string scale, the natural length that appears in string theory) and it is related to string tensor \( T \) as \( T = 1/2\pi \alpha' \). Here we have set the volume of the \( T^4 \) as \((2\pi)^4 V_4\) and string coupling as 1. In the horizon limit \( r \to 0 \), the two functions \( h_1 \), \( h_5 \) will be changed as \( h_1 \to Q_1/r^2 \) and \( h_5 \to Q_5/r^2 \). Therefore we have \( \sqrt{h_1 h_5} = \sqrt{Q_1 Q_5}/r^2 \), which leads to the metric as

\begin{align}
  ds^2 &= \frac{r^2}{Q} (-dt^2 + dx_1^2) + \frac{Q}{r^2} dr^2 + Q d\Omega_3^2 + \sqrt{Q_1 Q_5} ds_4^2(x^a) \\
  (A.7)
\end{align}

where \( Q \equiv \sqrt{Q_1 Q_5} \), and \( dx^m dx^m = r^2 dr^2 + r^2 d\Omega_3^2 \). Therefore, we see that the near horizon geometry for D1−D5 system is \( \text{AdS}_3 \times S^3 \times T^4 \). The factor \( \text{AdS}_3 \) here implies that the microstates of this system can be counted by the dual 2−dimensional CFT.

### A.1.2 Fuzzball solutions

The fuzzball solutions are families of regular solutions without horizon which are asymptotically to AdS space in the near horizon limit. For the D1−D5 system, these solutions are found by [12, 13] in Type IIB SUGRA. These solutions describe oscillating strings with arbitrary profiles \( F(v) \), where \( v \) is the light cone coordinate along the string. In these solutions, the nontrivial fields, i.e. the metric, the dilaton and the RR two–form are given by

\begin{align}
  ds^2 &= f_1^{-1/2} f_5^{-1/2} \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + f_1^{1/2} f_5^{1/2} dx \cdot dx \\
  &\quad + f_1^{1/2} f_5^{-1/2} dz \cdot dz \\
  e^{2\phi} &= f_1 f_5^{-1}, \\
  C_{ti} &= \frac{B_i}{f_1}, \quad C_{ty} = f_1^{-1} - 1, \\
  C_{iy} &= -\frac{A_i}{f_1}, \quad C_{ij} = C_{ij} + f_1^{-1} (A_i B_j - A_j B_i). \\
  (A.8)
\end{align}

Using the profile \( F(v) \), the two functions \( f_1 \), \( f_5 \) and the components of vector fields \( A \) can be written in the form of integrals as

\[
\begin{align*}
  f_5 &= 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|x - \dot{F}|^2}, \\
n_1 &= 1 + \frac{Q_5}{L} \int_0^L \frac{|\dot{F}|^2 dv}{|x - \dot{F}|^2}, \\
A_i &= -\frac{Q_5}{L} \int_0^L \frac{\dot{F}_i dv}{|x - \dot{F}|^2}.
\end{align*}
\]  

(A.9)

where \( y \) denotes the direction along \( S^1 \), while \( z \) denotes the directions along \( T^4 \). The solutions are asymptotically to \( M^5 \times S^1 \times T^4 \) at infinity, while asymptotically to \( \text{AdS}_3 \times S^3 \times T^4 \) in the near horizon limit. Here the length \( L \) depends on the number of D5 branes \( N_5 \) (\( N_5 \) is also the original number of strings) as \( L = 2\pi N_5 / R \), where the \( R \) is the radius of \( S^1 \). For convenience, in this section string coupling \( g \), the volume \( V_4 \) and \( \alpha \) have been set to 1. There exist the duality relation between \( f_5 \), \( A \) and \( C \), \( B \), which can be utilized to determine the three form \( C \) and the another vector fields \( B \)

\[
\begin{align*}
dC &= -\star dF_5, \\
\star_4 dB &= -\star_4 dA,
\end{align*}
\]  

(A.10)

where the \( \star_4 \) is evaluated in the four non–compact spatial dimensions. The charges of D1 branes and D5 branes are related by an integral along \( v \) as

\[
Q_1 = Q_5 \langle |\dot{F}|^2 \rangle = \frac{Q_5}{L} \int_0^L |\dot{F}|^2 dv.
\]  

(A.11)

As can be seen from the metric Eq. (A.8), the moduli space are parametrized by profiles \( F(v) \), closed curves living in a four–dimension space. Classically, there are infinite number of these profiles. Hence counting the microstates from these profiles directly only can be done after quantizing them. However, it is known that the degeneracy of this system is of order \( \exp(2\pi \sqrt{2Q_1Q_5}) \) which indicates that the entropy of this system is \( S = 2\pi \sqrt{2Q_1Q_5} \). Unfortunately, there is no a macroscopic horizon for two–charge black hole within supergravity, which means that the Bekenstein–Hawking entropy is not yet reproduced by this entropy. To obtain the nonzero horizon, the two–derivative Type IIB SUGRA needs higher–curvature correction as what happened in [80]. Although the two–charge system is more like a toy example, it is indeed very interesting and suggestive for the generalization for the three–charge system. However these fuzzball solutions constructed for the two–charge system from supergravity agree with CFT description.

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It is also interesting to mention that the angular momentum of the solution (A.8) can be expressed as

\[ J_{ij} = \frac{Q_5 R}{L} \int_0^L (F_i \dot{F}_j - F_j \dot{F}_i) dv. \]  

(A.12)

The two U(1) components of it are \( J_\phi = J_{12} \) and \( J_\psi = J_{34} \), which are corresponding to two rotational symmetries in the four non-compact dimensions. What should be pointed out here is that the angular momentum found here is independent on the location of the profiles which is different from the case of bubbling geometries for \( \text{AdS}_5 \times \text{S}^5 \) in chapter 3.

A.2 Bubbling AdS space and 1/2 BPS geometries

The first known and also the best known example of AdS/CFT correspondence is \( \text{AdS}_5/\text{CFT}_4 \): Type IIB string theory on the \( \text{AdS}_5 \times \text{S}^5 \) is equivalent to \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory on the four-dimensional boundary. There should exist the supergravity solutions that are asymptotically to \( \text{AdS}_5 \), which are dual to the BPS states of CFT. In this section we will give a review these supergravity solution.

A.2.1 Description in “droplets”

It is well-known that in the dual field theory there is a description in terms of free fermions [47] and other discussion can be found in [38, 81]. Configurations of ”droplets” (the region in the phase space occupied by the fermions) correspond to smooth horizonless geometries that preserve 16 of the original 32 supersymmetries.

Later, we will give the gravity solutions that are parametrized by the droplets on a particular plane which is a plane of ten (or eleven) dimensional geometry. This plane can directly correspond to the phase space of the fermions. Every droplet on this plane uniquely determines one non-singular geometry that has no horizon.

As a consequence, the topology of the droplets on the plane has fixed the topology of the gravity solutions. In other words, the shape of the droplets has determined the actual geometry. The ground state is represented by a circular droplet, which corresponds to \( \text{AdS}_5 \times \text{S}^5 \). The gravitons in the AdS are given rise by the small fluctuations on the droplet, i.e. the small ripples on the droplet, while the branes wrapping an \( \text{S}^3 \) in \( \text{S}^5 \) is dual to a hole inside the circular droplet. The small droplets with higher energy corresponds to giant
graviton branes [35]. When many branes stacked, the new geometry is encountered which corresponds to the new topology of the droplets.

A.2.2 Type IIB supergravity solutions

This part we will give the most general Type IIB geometry with $\text{SO}(4) \times \text{SO}(4) \times \mathbb{R}$ symmetry. Under the assumption that the dilaton and axion are constant and that the three–form field strengths are zero, obtaining the supergravity solutions only needs to solve the gravitino equations [82]

$$\nabla_M \eta + \frac{i}{480} \Gamma^{M_1 M_2 M_3 M_4 M_5} F^{(5)}_{M_1 M_2 M_3 M_4 M_5} \Gamma_M \eta = 0, \quad (A.13)$$

where $\eta$ is Killing spinor and $\Gamma$ is gamma matrices. Here $F$ is the five–form field strength, which obeys the self duality condition $F = \star \tilde{F}$. Applying the method of analyzing spinor bilinears [83, 84, 85], the metric and five–form are given by

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2,$$

$$h^{-2} = 2y \cosh G,$$

$$y \partial_y V_i = \varepsilon_{ij} \partial_j z, \quad y(\partial_i V_j - \partial_j V_i) = \varepsilon_{ij} \partial_y z, \quad z = \frac{1}{2} \tanh G,$$

$$F = dB_t \wedge (dt + V) + B_t dV + \hat{d} \hat{B},$$

$$\tilde{F} = d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + \hat{\tilde{d}} \hat{\tilde{B}},$$

$$B_t = -\frac{1}{4} y^2 e^{2G}, \quad \tilde{B} = -\frac{1}{4} y^2 e^{-2G},$$

$$d\hat{B} = -\frac{1}{4} y^3 \star_3 d\left(\frac{z + \frac{1}{2}}{y^2}\right), \quad \hat{d}\tilde{B} = -\frac{1}{4} y^3 \star_3 d\left(\frac{z - \frac{1}{2}}{y^2}\right), \quad (A.14)$$

where $i = 1, 2$ and $\star_3$ is evaluated in three–dimensional flat space expanding along $y, x_1, x_2$ direction. From the equations above, we see that the full solutions are parametrized by the function $z$, which satisfies the linear equation

$$\partial_t \partial_z + y \partial_y \left(\frac{\partial y z}{y}\right) = 0. \quad (A.15)$$

The regularity of the solutions is guaranteed by the requirement that $z = \pm \frac{1}{2}$ on the $y = 0$ plane spanned by $x_1, x_2$. The two signs of $z$ on $(x_1, x_2)$ plane corresponds to the fermions
and holes, and the \((x_1, x_2)\) plane corresponds to the phase space in dual description. The solution of Eq. \((A.15)\) is

\[
z(x_1, x_2, y) = \frac{y^2}{\pi} \int_D z(x_1', x_2', 0) dx_1' dx_2', \tag{A.16}
\]

\[
V_i(x_1, x_2, y) = \frac{\varepsilon_{ij}}{\pi} \int_D \frac{z(x_1', x_2', 0)(x_j - x_j') dx_1' dx_2'}{[(x - x')^2 + y^2]^2}, \tag{A.17}
\]

where the integral is over the droplets \(D\).

As we mentioned that \(\text{AdS}_5 \times \text{S}^5\) corresponds to the ground state described by a circular droplet. Therefore, by integrating \((A.16)\) over the circular droplet on the \((x_1, x_2)\) plane, we then can recover the \(\text{AdS}_5 \times \text{S}^5\) spacetime. To do so, it is convenient to introduce a new function \(\tilde{z} = z - \frac{1}{2}\). Replacing \(x_1, x_2\) with the polar coordinates \(r, \varphi\), one can find \(\tilde{z}\) function as

\[
\tilde{z}(r, y) = -\frac{y^2}{\pi} \int_{\text{Disk}} \frac{r' dr' d\phi'}{r^2 + r'^2 - 2rr' \cos \phi' + y'^2]^2} = \frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2r_0^2}} - \frac{1}{2}, \tag{A.18}
\]

where \(r_0\) is the radius of the circular droplet. Here we have set \(\phi = 0\) for the symmetric reason. The vector fields \(V\) in the polar coordinates can be written as \(V = (\cos \phi V_1 + \sin \phi V_2) dr + (-r \sin \phi V_1 + r \cos \phi V_2) d\phi\). Taking Stokes’ theorem, we change the area integral to line integral. It is easy to check that the integral to calculate the component \(V_r\) is vanished and the component \(V_\phi\) is given by

\[
V_\phi = -\frac{1}{2\pi} \int_{\partial \Omega} \frac{rr' \cos \phi' d\phi'}{r^2 + r'^2 + y^2 - 2rr' \cos \phi'} = -\frac{1}{2} \left( \frac{r^2 + y^2 + r_0^2}{\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2r_0^2}} - 1 \right). \tag{A.19}
\]

After inserting \(z\) and \(V_i\) into \((A.14)\), and changing the coordinates as

\[
y = r_0 \sinh \rho \sin \theta, \quad r = r_0 \cosh \rho \cos \theta, \quad \tilde{\phi} = \phi - t, \tag{A.20}
\]
we can obtain the global AdS$ _5$ $\times$ S$ _5$
\[
 ds^2 = r_0 \left[ - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega^2 + d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\tilde{\Omega}^2 \right] \quad (A.21)
\]

Here we see that the radius of AdS and sphere is $\sqrt{r_0}$. The standard answer for the radius of AdS is $(4\pi l_p N)^{1/4}$, where $l_p$ is Planck length and $N$ is an integer. Therefore, the exact condition for quantization on the area of the droplets $A$ in the $(r,\phi)$ plane is given by
\[
 A = 4\pi^2 l_p^4 N = 2\pi \hbar N, \quad \hbar = 2\pi l_p^4, \quad (A.22)
\]
where we have defined an effective $\hbar$ in $(r,\phi)$ or $(x_1, x_2)$ plane which is can be identified as the phase space in dual field theory.

### A.2.3 Excitation energy

In the previous part we presented 1/2 BPS excitation of AdS$ _5$ $\times$ S$ _5$, while in this section we will consider the energy of these excitations.

In the dual description [47], the energy of excitation is equal to the energy of the fermions in a harmonic oscillator potential subtract the energy of the ground state of $N$ fermions. Since the BPS condition $\Delta = J$, one can look for energy instead of angular momentum which in current case is easier. To obtain the angular momentum of the supergravity solutions, one needs to first derive the the asymptotic form of the metric. The angular momentum or the energy then can be read directly from the leading terms in the $g_{\tilde{\phi}t}$ (where $\tilde{\phi} = \phi + t$) components of the asymptotic metric
\[
 \Delta = J = \frac{1}{16\pi^3 l_p^8} \left[ \int_D d^2x (x_1^2 + x_2^2) - \frac{1}{2\pi} \left( \int_D d^2x \right)^2 \right] = \frac{1}{2\pi \hbar} \int d^2x \frac{1}{2} (x_1^2 + x_2^2) - \frac{1}{2} \left( \int_D d^2x \right)^2, \quad (A.23)
\]
where $D$ is the domain at $z = -\frac{1}{2}$, or we can say region occupied by the fermions. The ground state. Noting the definition of effective $\hbar$ in (A.22), it is obvious that this energy is exactly the quantum energy of the fermions minus the ground state energy.
A.3 Bubbling geometries for $\text{AdS}_2 \times \text{S}^2$

The study of $\text{AdS}_2 / \text{CFT}_1$ can be found in [25]. One the reason for suggesting people to study $\text{AdS}_2 \times \text{S}^2$ is that this space holds some analytic properties with the counterparts in the higher dimensions. For instance, integrability of strings that had been found for the higher dimensional cases, $\text{AdS}_5 \times \text{S}^5$ and $\text{AdS}_3 \times \text{S}^3$ [86, 87], also discovered in $\text{AdS}_2 \times \text{S}^2$ [34, 88].

A.3.1 Type IIB supergravity solutions

In [19], the normalizable BPS excitations of $\text{AdS}_2 \times \text{S}^2$ are constructed and all the corresponding geometries are regular and have no horizons. In the AdS/CFT dictionary, the ground state corresponds to $\text{AdS}_2 \times \text{S}^2$, while the low energy excitations correspond to the strings moving on $\text{AdS}_2 \times \text{S}^2$ that discussed by [34]. The heavier excitations are mapped to the probe branes on $\text{AdS}_2 \times \text{S}^2$ which is semiclassical description. As the energy increase even more, the gravitational backreaction of branes cannot be neglected and the corresponding supergravity solutions will be given in this section.

Similarly to the case in section A.2, the variations of dilatino vanish trivially under supersymmetry transformation, while the five–form is excited. Therefore, what only needs to be done is solving the gravitino equation (A.13). What is different from the previous section is that this part is looking for BPS excitation of $\text{AdS}_2 \times \text{S}^2$ which preserves the torus and the structure of $F_5$ as:

\begin{align*}
    ds^2 &= g_{mn}dx^m dx^n + dz^a d\bar{z}^a, \\
    F_5 &= \frac{1}{2} F_{mn} dx^m \wedge \text{Re}\Omega_3 - \frac{1}{2} \tilde{F}_{mn} dx^m \wedge \text{Im}\Omega_3
\end{align*}

(A.24)

where $\Omega_3 = dz_{123}$ with $z_a = X_a + iY_a, a = 1, 2, 3$ and $m = 1, 2, 3$. Here the ten coordinates used are $(t, x_1, x_2, x_3, X_1, X_2, X_3, Y_1, Y_2, Y_3)$. By the combination of the gravitino equation
and the equation of motion for flux $F_5$, one can determine the $g_{mn}$ and $F_5$ as

$$
ds^2 = -h^{-2}(dt + V)^2 + h^2 dx_\alpha dx_\alpha + dz_\dot{a} d\bar{z}_{\dot{a}},
$$

$$
F_5 = F \wedge \text{Re } \Omega_3 - \tilde{F} \wedge \text{Im } \Omega_3, \quad \Omega_3 = dz_{123},
$$

$$
F = -\partial_\alpha A_\alpha (dt + V) \wedge dx^\alpha + h^2 \star_3 d\tilde{A}_t, \quad \tilde{A}_t + iA_t = \frac{1}{4h} e^{i\beta},
$$

$$
\tilde{F} = -\partial_\alpha \tilde{A}_\alpha (dt + V) \wedge dx^\alpha - h^2 \star_3 dA_t, \quad dV = -2h^2 \star_3 d\beta,
$$

$$
\tilde{\eta} = h^{-1/2} e^{i\beta/2}, \quad \Gamma^4 \Gamma_5 \epsilon = \epsilon, \quad (A.25)
$$

Where $a = 1, 2, 3$ and $z_\dot{a} = X_\dot{a} + iY_\dot{a}$ with $\dot{a} = 1, 2, 3$ in which $X_\dot{a}$ and $Y_\dot{a}$ are the coordinates for $T^6$. Here $\star_3$ is the Hodge dual operator in three dimensions space spanned by $x_1, x_2, x_3$ and $V$ are vector fields. This geometry can be parameterized by two harmonic functions $H_1$ and $H_2$ in $\mathbb{R}^3$:

$$
H_1 = h \sin \beta, \quad H_2 = h \cos \beta, \quad d \star_3 dH_a = 0,
$$

$$
dV = -2 \star_3 [H_2 dH_1 - H_1 dH_2], \quad \tilde{A}_t + iA_t = \frac{1}{4(H_2 - iH_1)}. \quad (A.26)
$$

At the points where the two harmonic functions $H_1$ and $H_2$ have sources, the solutions $(A.25)$ may be singular. To guarantee the regularity, one has to require that all the sources are distributed along closed curves in $\mathbb{R}^3$, and the harmonic functions are obtained from:

$$
H = H_1 + iH_2 = \frac{1}{2\pi} \int \frac{\sigma \sqrt{(r - \bar{f}) \cdot (r - \bar{f} + b)}}{(r - \bar{f})^2} dv + H_{\text{reg}},
$$

$$
\mathbf{b} \cdot \bar{f} = 0, \quad \mathbf{b} \cdot \mathbf{b} = 0, \quad (A.27)
$$

where $\sigma$ is the “density charge” and $\text{f}(v)$ is the location of the profile parameterized by $v$ and $\mathbf{b}$ is a complex vector.

**A.3.2 Ground state**

In the bubbling AdS$_5$ solutions, AdS$_5 \times S^5$ is specified by a circular droplet, while in the current bubbling solutions, AdS$_2 \times S^2$ is parametrized by a circular profile $\text{f}(v)$, which should corresponds to a vacuum of the unknown quantum mechanics living on the boundary of the AdS space.
Performing the integral (A.27) along a circular profile \( f(v) \) in \( \mathbb{R}^3 \)

\[
(f_1, f_2, f_3) = \left( L \sin \frac{v}{L}, L \sin \frac{v}{L}, 0 \right)
\]  

(A.28)

we can find the complex harmonic function \( H \) as

\[
H = \frac{L}{\sqrt{\tilde{r}^2 + (y - iL)^2}},
\]

(A.29)

where we have set \( r = (x_1, x_2, x_3) \equiv (r \cos \phi, r \sin \phi, y) \). Here we have chosen the complex vector as \( b = 2L \left( \sin \frac{v}{L}, \sin \frac{v}{L}, i \right) \) such that it is satisfied with the constraints in (A.27). In order to find the geometry specified by this complex function \( H \), we first need to separate the real part and imaginary part of it. A convenient way to do so is to make a coordinates transformation as

\[
\tilde{r} = L\sqrt{\rho^2 + \sin \theta}, \quad y = L\rho \cos \theta,
\]

(A.30)

which leads to

\[
H = \frac{\rho + i \cos \theta}{\rho^2 + \cos^2 \theta}
\]

(A.31)

Reading \( H_1 \) and \( H_2 \) directly from this complex function based on the definition in (A.27) and using the equation (A.26) which relates the two harmonic functions and vector fields \( V \), one can find

\[
V_{\tilde{r}} = 0, \quad V_{\phi} = -\frac{L \sin^2 \theta d\phi}{\rho^2 + \cos^2 \theta}.
\]

(A.32)

Here we have chosen the gauge \( V_y = 0 \). On the other hand, note that the change of coordinates (A.30) results in

\[
ds_{\text{base}}^2 = dx_1^2 + dx_2^2 + dx_3^2
\]

\[
= d\tilde{r}^2 + r^2 d\phi^2 + dy^2
\]

\[
= L^2 \left[ \frac{(\rho^2 + \cos^2 \theta) d\rho^2}{\rho^2 + 1} + (\rho^2 + \cos^2 \theta) d\theta^2 + \sin^2 \theta (\rho^2 + 1) d\phi^2 \right].
\]

(A.33)

By rescaling time as \( \tilde{t} = t/L \) and taking a shift of the angular coordinate as \( \tilde{\phi} = \phi + \tilde{t} \), from
bubbling solutions \((A.25)\) we obtain the geometry of the global \(\text{AdS}_2 \times \text{S}^2\)

\[
    ds^2 = L^2 \left[ -(\rho^2 + 1)d\tilde{t}^2 + \frac{d\rho^2}{\rho^2 + 1} + d\theta^2 + \sin^2 \theta d\phi^2 \right] + dz^a d\bar{z}_a
\]

\[
    F_5 = \frac{L}{4} d\rho d\tilde{t} \wedge \text{Re}(dz_{123}) + \text{dual.} \quad (A.34)
\]

It is obvious that the expression for complex harmonic function \(H\) in \(y = 0\) plane is singular on the \(r = L\) circle. On the other hand, we notice that \(\rho\) can run from \(-\infty\) to \(\infty\). The transformations that

\[
    \rho \to -\rho, \quad \theta \to \pi - \theta \quad (A.35)
\]

have no effects on coordinates \(r, y\) except \((\tilde{r}, y) = (L, 0)\), indicates that when \(\rho\) goes from \(-\infty\) to \(\infty\), there are two pieces of base \(ds^2_{\text{base}}\). This is not surprising since the two boundaries of \(\text{AdS}_2\) are disconnected. In other words, to cover the entire \(\text{AdS}_2 \times \text{S}^2\), one need two copies of the flat base. It is pointed out by \([19]\) that these two three–dimensional flat bases are glued by a ”branch cut surface” which is a Riemann surface and the precise location of it is ambiguous but it is bounded by the singular curve \((\tilde{r}, y) = (L, 0)\). As close to the singular curve, a point moves from one copy to another, thus makes the singular curve is not really ”singular”.

In string theory, the vibrations of strings correspond to light particles such as photons, gravitons, and electrons, etc., while the collective excitations of string, the branes, can be used to construct the heavy objects, such as black holes. As we have been discuss above that using different stack of Dp branes we can build different geometries. \([89]\) supplied a way to stack D3 branes as

<table>
<thead>
<tr>
<th>Dp–brane</th>
<th>t</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(X_3)</th>
<th>(Y_1)</th>
<th>(Y_2)</th>
<th>(Y_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D3_1</td>
<td>√</td>
<td></td>
<td></td>
<td></td>
<td>√</td>
<td>√</td>
<td>√</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>D3_2</td>
<td>√</td>
<td></td>
<td></td>
<td></td>
<td>√</td>
<td>×</td>
<td>×</td>
<td>√</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>D3_3</td>
<td>√</td>
<td></td>
<td>√</td>
<td></td>
<td>×</td>
<td>√</td>
<td>×</td>
<td>√</td>
<td>×</td>
<td>√</td>
</tr>
<tr>
<td>D3_4</td>
<td>√</td>
<td></td>
<td>√</td>
<td>√</td>
<td>×</td>
<td>√</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
</tbody>
</table>

where again ”√” denotes the direction wrapped by the D3 branes while the ”×” denotes the coordinates in which branes are smeared. Putting D3 branes as the table in a ten–dimensional flat spacetime results in \(\text{AdS}_2 \times \text{S}^2 \times \text{T}^6\) in the near horizon limit.
A.4 Symplectic form evaluation

A.4.1 Shifts in harmonic functions and vector fields

The vectors \( r, f, b \) in (A.27) can be written in the cylindrical coordinates as

\[
\begin{align*}
    r &= (r \cos \phi, r \sin \phi, y), \\
    f &= (f \cos \phi_1, f \sin \phi_1, 0), \\
    b &= (b \cos \phi_2, b \sin \phi_2, \, i b_3), 
\end{align*}
\]

(A.36)

where \( r, f, b, b_3 \) are real. In this paper we only consider the planar profiles, thus taking \( f_3 = 0 \). Applying (A.36) and picking \( v = f \phi_1 \), the Eq. (A.27) becomes

\[
\begin{align*}
    H &= \frac{1}{2\pi} \int_{2\pi} \frac{f \sigma}{r^2 + f^2 + y^2 - 2fr \cos(\phi_1 - \phi)} \times \sqrt{r^2 + f^2 + y^2 - 2fr \cos(\phi_1 - \phi) + br \cos(\phi_2 - \phi) - bf \cos(\phi_2 - \phi_1) + iyb_3} \\
    &= \frac{1}{2\pi} \int C \sqrt{D + iyb_3} \ d\phi_1, 
\end{align*}
\]

(A.37)

where

\[
\begin{align*}
    C &= \frac{f \sigma}{r^2 + f^2 + y^2 - 2fr \cos(\phi_1 - \phi)}, \\
    D &= r^2 + f^2 + y^2 - 2fr \cos(\phi_1 - \phi) + br \cos(\phi_2 - \phi) - bf \cos(\phi_2 - \phi_1). 
\end{align*}
\]

(A.38)

Therefore, we can express \( H_1, H_2 \) as

\[
\begin{align*}
    H_1 &= \frac{1}{2\pi} \int_0^{2\pi} C \left[ D^2 + (yb_3)^2 \right]^{\frac{1}{4}} \cos \left[ \frac{1}{2} \tan^{-1} \left( \frac{yb_3}{D} \right) \right] \ d\phi_1, \\
    H_2 &= \frac{1}{2\pi} \int_0^{2\pi} C \left[ D^2 + (yb_3)^2 \right]^{\frac{1}{4}} \sin \left[ \frac{1}{2} \tan^{-1} \left( \frac{yb_3}{D} \right) \right] \ d\phi_1. 
\end{align*}
\]

(A.39) (A.40)

The profile corresponding to \( \text{AdS}_{2} \times \text{S}_{2} \) is given by (2.52) and the corresponding vector \( b \) is given as \( b = 2L \left( \cos \frac{v}{L}, \sin \frac{v}{L}, \, i \right) \), where \( L \) is the radius of the AdS and the sphere. We set \( L = 1 \) for convenience. Now we consider perturbations (2.56) on this profile, i.e now \( f = (1 + \delta f)(\cos \phi_1, \sin \phi_1, 0) \). Vector \( b \) has to change correspondingly due to the constrain
\( \mathbf{b} \cdot \dot{\mathbf{f}} = 0, \)

\[ \mathbf{b} = \mathbf{b}_0 + \delta \mathbf{b}, \quad (A.41) \]

where

\[ \mathbf{b}_0 = 2L \left( \cos \frac{\nu}{L}, L \sin \frac{\nu}{L}, i \right) = 2(\cos \phi_1, \sin \phi_1, i) = 2(\cos \phi_2, \sin \phi_2, i), \]

\[ \delta \mathbf{b} = 2(-\sin \phi_2, \cos \phi_2, 0) \delta \phi_2 = 2(-\sin \phi_1, \cos \phi_1, 0) \delta \phi_2. \quad (A.42) \]

Here we keep \( b=2 \) unchanged, while \( \phi_2 \to \phi_2 + \delta \phi_2 \), which can guarantee the two constrains \( \mathbf{b} \cdot \dot{\mathbf{f}} = 0, \mathbf{b} \cdot \mathbf{b} = 0 \).

Using constrain

\[ 0 = \mathbf{b} \cdot \dot{\mathbf{f}} = \delta \dot{\mathbf{f}} + (1 + \delta \mathbf{f}) \delta \phi_2 \approx \delta \dot{\mathbf{f}} + \delta \phi_2, \quad (A.43) \]

we find \( \delta \phi_2 = -\delta \dot{\mathbf{f}} = -\sum_n in a_n e^{in\phi_1} \).

Thus we can express the first order perturbations of harmonic function \( H_1 \) and \( H_2 \) as

\[ \delta H_1 = \left[ \frac{\partial H_1}{\partial f} \delta \dot{f} + \frac{\partial H_1}{\partial \phi_2} (-\delta \dot{\mathbf{f}}) \right]_{\sigma=1, f=1, b=b_3=2, \phi_1=\phi_2} , \]

\[ \delta H_2 = \left[ \frac{\partial H_2}{\partial f} \delta \dot{f} + \frac{\partial H_2}{\partial \phi_2} (-\delta \dot{\mathbf{f}}) \right]_{\sigma=1, f=1, b=b_3=2, \phi_1=\phi_2} . \quad (A.44) \]

Let \( \phi_1 - \phi = \alpha \), then \( \phi_1 = \phi + \alpha \) we obtain

\[
\begin{aligned}
\left[ \frac{\partial H_1}{\partial f} \delta \dot{f} \right]_{\sigma=1, f=1, b=b_3=2, \phi_1=\phi_2} &= \int d\alpha \frac{\sum_{|n|>1} a_n e^{in(\phi+\alpha)}}{2\pi \left[ (r^2 + y^2 - 1)^2 + 4y^2 \right]^{3/4} (r^2 - 2r \cos \alpha + y^2 + 1)^2} \\
&\times \left\{ \frac{1}{2} \tan^{-1} \left( \frac{2y}{r^2 + y^2 - 1} \right) \right\}.
\end{aligned}
\]

(A.45)
Working out this integral over $\alpha$ by residue theorem, we find first part of $\delta H_1$. Similarly, we can figure out another part, and finally we obtain, after transferring to global AdS coordinates, the perturbations of $H_1$. Repeating the same procedure, we will find the first order shifts of $\delta H_2$.

To find $\delta V$ corresponding to $\delta H$, we begin with equation

$$dV = -2 \ast_3 [H_2 dH_1 - H_1 dH_2]. \tag{A.46}$$

The exterior derivative of vector field $V$, in the gauge $V_y=0$, can be written as:

$$dV = \frac{\partial V_r}{\partial y} dy \wedge dr + \frac{\partial V_\phi}{\partial y} dy \wedge d\phi + \left( \frac{\partial V_\phi}{\partial r} - \frac{\partial V_r}{\partial \phi} \right) dr \wedge d\phi. \tag{A.47}$$

Through the comparison of the left and right hand side of (A.46), we get the differential equations

$$\frac{\partial V_r}{\partial y} = \frac{2}{r} \left( H_1 \frac{\partial H_2}{\partial \phi} - H_2 \frac{\partial H_1}{\partial \phi} \right),$$
$$\frac{\partial V_\phi}{\partial y} = -2r \left( H_1 \frac{\partial H_2}{\partial r} - H_2 \frac{\partial H_1}{\partial r} \right),$$
$$\frac{\partial V_\phi}{\partial r} - \frac{\partial V_r}{\partial \phi} = 2r \left( H_1 \frac{\partial H_2}{\partial y} - H_2 \frac{\partial H_1}{\partial y} \right). \tag{A.48}$$

Considering the first order shifts of $V_r$ and $V_\phi$, we get

$$\frac{\partial (\delta V_r)}{\partial y} = \frac{2}{r} \left[ \left( \frac{\partial H_1}{\partial f} \delta f - \frac{\partial H_1}{\partial \phi_2} \delta \phi \right) \frac{\partial H_2}{\partial \phi} + H_1 \frac{\partial}{\partial \phi} \left( \frac{\partial H_2}{\partial f} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \phi \right) \right. \right.$$  

$$\left. - \left( \frac{\partial H_2}{\partial f} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \phi \right) \frac{\partial H_1}{\partial \phi} - H_2 \frac{\partial}{\partial \phi} \left( \frac{\partial H_1}{\partial f} \delta f - \frac{\partial H_1}{\partial \phi_2} \delta \phi \right) \right]_{f=1}. \tag{A.49}$$

Similarly, we have

$$\frac{\partial (\delta V_\phi)}{\partial y} = -2r \left[ \left( \frac{\partial H_1}{\partial f} \delta f - \frac{\partial H_1}{\partial \phi_2} \delta \phi \right) \frac{\partial H_2}{\partial r} + H_1 \frac{\partial}{\partial r} \left( \frac{\partial H_2}{\partial f} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \phi \right) \right. \right.$$  

$$\left. - \left( \frac{\partial H_2}{\partial f} \delta f - \frac{\partial H_2}{\partial \phi_2} \delta \phi \right) \frac{\partial H_1}{\partial r} - H_2 \frac{\partial}{\partial r} \left( \frac{\partial H_1}{\partial f} \delta f - \frac{\partial H_1}{\partial \phi_2} \delta \phi \right) \right]_{f=1}. \tag{A.50}$$
Again, letting $\phi_1 - \phi = \alpha$, we obtain
\[
\frac{\partial (\delta V_r)}{\partial y} = \sum_{|n|>1} \frac{a_n e^{i n \phi}}{2\pi} \int_0^{2\pi} \frac{4ny(n \sin \alpha + i \cos \alpha) e^{i n \alpha} d\alpha}{\left[r^4 + 2r^2(y^2 - 1) + (y^2 + 1)^2\right] \left(r^2 - 2r \cos \alpha + y^2 + 1\right)}.
\]  
(A.51)

Completing this integral over $\alpha$, we find $\delta V_r$ in the global AdS coordinates. Proceeding (A.50), we obtain
\[
\frac{\partial (\delta V_\phi)}{\partial y} = \sum_{|n|>1} \frac{a_n e^{i n \phi}}{2\pi} \int_0^{2\pi} \frac{4ry \cos \alpha \left(r^2 - y^2 - 1\right) + 2r \left(r^2 + y^2 - 1\right) - in \sin \alpha \left(r^2 - y^2 - 1\right)}{\left[r^4 + 2r^2(y^2 - 1) + (y^2 + 1)^2\right] \left(r^2 - 2r \cos \alpha + y^2 + 1\right)^2} \times e^{i n \alpha} d\alpha.
\]  
(A.52)

Figuring out this integral and changing the coordinates as (2.54), we immediately find $\delta V_\phi$. It is easy to check $\delta V_r$ and $\delta V_\phi$ obtained from (A.51) and (A.52) are satisfied the third equation in (A.48).

### A.4.2 Symplectic currents

From metric perturbation (2.59), it is straightforward to find the components of $\delta [g^{mn} \sqrt{-g}]$:
\[
\begin{align*}
\delta [g^{tt} \sqrt{-g}] &= -\sum_{|n|>1} 2a_n e^{in(\hat{\phi} - \hat{t})} \frac{(|n| + 1)\rho^2 \sin \theta S_n}{(1 + \rho^2)^2}, \\
\delta [g^{t\rho} \sqrt{-g}] &= \sum_{|n|>1} 2a_n e^{in(\hat{\phi} - \hat{t})} \frac{i(n^2 + |n|)\rho \sin \theta S_n}{n(1 + \rho^2)}, \\
\delta [g^{\rho\rho} \sqrt{-g}] &= \sum_{|n|>1} 2a_n e^{in(\hat{\phi} - \hat{t})} (|n| + 1) \sin \theta S_n,
\end{align*}
\]  
(A.53)

where
\[
S_n = \left(\frac{\sin \theta}{\sqrt{\rho^2 + 1}}\right)^{|n|}.
\]
The shifts of connections $\delta \Gamma_{mp}^i$ that we are interested in are:

$$
\delta \Gamma_{ti}^i = \sum_{|n|>1} a_n e^{i n (\hat{t} - \hat{t})} \frac{i \left[ n^2 \left( |n| (\rho^2 - 1) + 5 \rho^2 - 1 \right) + 4 |n| \rho^2 \right] \mathcal{S}_n}{2n(1 + \rho^2)},
$$

$$
\delta \Gamma_{t\rho}^i = \sum_{|n|>1} a_n e^{i n (\hat{t} - \hat{t})} \rho \left( |n| + 1 \right) \frac{\left[ n \left( \rho^2 - 1 \right) - 4 \right] \mathcal{S}_n}{2(1 + \rho^2)^2},
$$

$$
\delta \Gamma_{\rho\rho}^i = \sum_{|n|>1} a_n e^{i n (\hat{t} - \hat{t})} \frac{i \left[ n^2 \left( 3 |n| \rho^2 + |n| - \rho^2 - 3 \right) + 4 |n| \left( |n| \rho^2 - 1 \right) \right] \mathcal{S}_n}{2n(1 + \rho^2)^3}.
$$

Some components of $\delta \Gamma_{mp}^m$ are:

$$
\delta \Gamma_{m\hat{t}}^m = - \sum_{|n|>1} a_n e^{i n (\hat{t} - \hat{t})} i n (|n| + 1) \mathcal{S}_n, \\
\delta \Gamma_{m\rho}^m = - \sum_{|n|>1} a_n e^{i n (\hat{t} - \hat{t})} \frac{|n|(|n| + 1) \rho \mathcal{S}_n}{1 + \rho^2}. 
$$

With the preparation (A.53), (A.55) and (A.55), we can easily find

$$
- \int d\vec{\phi} \delta \Gamma_{mp}^i \wedge \delta \left[ \sqrt{-g} g_{mp}^{\hat{t}} \right] = \frac{2 \pi i |n| (1 + |n|)^2 \left[ |n| \left( 3 \rho^4 - 6 \rho^2 - 1 \right) + 4 \left( \rho^2 - 1 \right)^2 \right] \sin \theta \mathcal{S}_n^2}{n(1 + \rho^2)^3} \times (a_n \wedge a_{-n}),
$$

$$
\int d\vec{\phi} \delta \Gamma_{mp}^p \wedge \delta \left[ \sqrt{-g} g_{mp}^{m\hat{t}} \right] = \frac{8 \pi i (|n| + n^2)^2 \rho^2 \sin \theta \mathcal{S}_n^2}{n(1 + \rho^2)^2} (a_n \wedge a_{-n}). \tag{A.56}
$$

Integrate the remaining integrals, and add them up we then find the symplectic form (2.65) from gravity current.

Using relations (2.62), one can easily find

$$
\delta B_{t} = \sum_{|n|>1} \frac{1}{8} a_n e^{i n (\hat{t} - \hat{t})} (1 + |n|) \rho \mathcal{S}_n, \\
\delta B_{\rho} = \sum_{|n|>1} a_n e^{i n (\hat{t} - \hat{t})} \frac{i n |n| (1 + \rho^2)}{8 |n| (1 + \rho^2)}, \\
\delta \tilde{B}_{\theta} = \sum_{|n|>1} a_n e^{i n (\hat{t} - \hat{t})} \frac{i n |n| + 1 \mathcal{S}_n}{8 |n| \sin \theta}, \\
\delta \tilde{B}_{\phi} = \sum_{|n|>1} \frac{1}{8} a_n e^{i n (\hat{t} - \hat{t})} (|n| + 1) \cos \theta \mathcal{S}_n. \tag{A.57}
$$
From these one–form perturbations we obtain

\[
\delta F_{\bar{\rho}} = - \sum_{|n|>1} \frac{1}{8} a_n e^{i(n+\bar{n})} (n^2 - 1) S_n,
\]

\[
\delta \tilde{F}_{\bar{\theta}} = - \sum_{|n|>1} \frac{1}{8} a_n e^{i(n+\bar{n})} n |n| + 1 \csc \theta S_n,
\]

\[
\delta \tilde{F}_{\bar{\phi}} = - \sum_{|n|>1} \frac{1}{8} a_n e^{i(n+\bar{n})} n \cos \theta S_n.
\]

Then we obtain

\[
\delta [\sqrt{-g} F^\alpha_\beta] = \sum_{|n|>1} \frac{1}{8} a_n e^{i(n+\bar{n})} n \sin \theta S_n,
\]

\[
\delta [\sqrt{-g} \tilde{F}^{\bar{\theta}}] = \sum_{|n|>1} \frac{1}{8} a_n e^{i(n+\bar{n})} n \frac{(n+1) S_n}{1 + \rho^2},
\]

\[
\delta [\sqrt{-g} \tilde{F}^{\bar{\phi}}] = \sum_{|n|>1} \frac{1}{8} a_n e^{i(n+\bar{n})} n \frac{\cot \theta S_n}{1 + \rho^2}.
\]

Therefore we find

\[
\int d\bar{\phi} \ 8 \ \delta A_{[i_1 i_2 i_3 i_4]} \wedge \delta [\sqrt{-g} F^\mu_{[i_1 i_2 i_3 i_4]}] = - \sum_{|n|>1} \frac{\pi |n|(n+1)^2 (2|n| + \sin^2 \theta) S_n^2}{n(1 + \rho^2) \sin \theta} \times a_n \wedge a_{-n}.
\]

Working out the remaining integrals, we then find the symplectic form (2.66) from 5-form current.
B.1 Constrained Hamiltonian system

Promoting Poisson bracket to commutator is the usual way to proceed canonical quantization. However for some constrained Hamiltonian system, Poisson brackets do not work well. Thus Dirac generalized the Hamiltonian and Poisson brackets such that more general Lagrangians can be processed and thus the canonical quantization can be well figured out [90].

To see what is the problem with the original Hamiltonian method, let us consider a simple example with Lagrangian which is linear in the velocity:

\[ L = x \dot{y} - y \dot{x} - V(x, y). \] (B.1)

In this case Euler–Lagrange equations still work very well and give us the equations of motion

\[ \dot{y} = \frac{1}{2} \frac{\partial V}{\partial x}, \quad \dot{x} = -\frac{1}{2} \frac{\partial V}{\partial y}. \] (B.2)

These equations of motion look like differently from what we usually obtain as they are the first order differential equations. Nothing can prevent this happening in the Lagrangian theory. To obtain the Hamiltonian description, we first compute the conjugate momenta

\[ p_x = \frac{\partial L}{\partial \dot{x}} = -y, \quad p_y = \frac{\partial L}{\partial \dot{y}} = x. \] (B.3)

This makes trouble with us in writing \( \dot{x}, \dot{y} \) in terms of conjugate momenta. Neglecting this problem, we proceed the Legendre transformation we still can write down the Hamiltonian

\[ H = p_x \dot{x} + p_y \dot{y} - L = V(x, y). \] (B.4)

The Hamiltonian with this form might be strange since there is no momenta dependence.
The equations of motion derived from this Hamiltonian are given by
\[ \dot{x} = \frac{\partial H}{\partial p_x} = 0, \quad \dot{y} = \frac{\partial H}{\partial p_y} = 0, \] (B.5)
while the other Hamilton’s equations and (B.3) give us another set of equations of motion
\[ \dot{x} = -\frac{\partial V}{\partial y}, \quad \dot{y} = \frac{\partial V}{\partial x}. \] (B.6)
We get two sets of equations of motion (B.5) and (B.6) from Hamiltonian formulation which are not consistent with each other and not consistent with the results derived from Euler–Lagrange equations.

Dirac suggested that we should do similar operation as what have been done for Lagrangian formulation when there are holonomic constraints to Hamiltonian formulation when there are constraints
\[ \phi_i(p, q) = 0. \] (B.7)
Before writing down the generalize Hamiltonian, let us first define weak and strong equality as what Dirac did. For two functions \( f, g \) on the phase space, if they are equal only when the condition (B.7) are satisfied, we call \( g \approx f \) as weak equality while if they are equal throughout all phase space, we call \( g = f \) as strong equality.

The generalized Hamiltonian can be written as
\[ H' = H + c_i \phi_i, \] (B.8)
where \( H \) is the “naive” Hamiltonian and \( c_i \) are functions of coordinates and momenta. Thus the variation in this new Hamiltonian is
\[
dH' = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \phi_i dc_i + c_j \left( \frac{\partial \phi_j}{\partial q_i} dq_i + \frac{\partial \phi_j}{\partial p_i} dp_i \right) \\
= \left( \frac{\partial H}{\partial q_i} + c_j \frac{\partial \phi_j}{\partial q_i} \right) dq_i + \left( \frac{\partial H}{\partial p_i} + c_j \frac{\partial \phi_j}{\partial p_i} \right) dp_i + \phi_i dc_i. \] (B.9)
Recall how to read Hamilton’s equations from the variation of “naive” Hamiltonian \( H \), and
then we can write the new Hamilton’s equations as

\[
\dot{q}_i = \frac{\partial H}{\partial p_i} + c_j \frac{\partial \phi_j}{\partial p_i}, \quad \dot{p}_i = -\left( \frac{\partial H}{\partial q_i} + c_j \frac{\partial \phi_j}{\partial q_i} \right)
\]  
(B.10)

These equations can be rewritten by usual Poisson bracket as

\[
\dot{q}_i = \{ q_i, H \} + c_j \{ q_i, \phi_j \}, \quad \dot{p}_i = \{ p_i, H \} + c_j \{ p_i, \phi_j \}
\]  
(B.11)

Noting the weak equation \( \dot{\phi}_i \approx 0 \) is always correct, when we apply the new equation of motion as (B.11) to \( \phi_i \), we can obtain the weak equation as

\[
\{ \phi_i, H \} + c_j \{ \phi_i, \phi_j \} \approx 0.
\]  
(B.12)

There are four possible cases for this equation. The first one is that this equation is inherently unsatisfied which implies the Lagrangian considered is not allowed. The second case is that the equation is true without introducing new constraints, which give us nothing. The third case is that this equation is satisfied unless we introduce new constraints on coordinates and momenta. The last possible case is that this equation is satisfied by certain values of \( c_j \).

We define the constraints derived from the definition of conjugate momenta as primary constraints, while define the constraints derived from the third case as secondary constraints. When secondary constraints are found, one should add them to Hamiltonian as we add the primary constraints. The vanishing time derivative of these secondary constraint as (B.12) may give us more secondary constraints. One needs to repeat this procedure until no new constraints produced. For the last possible case we allow to write \( c_j \) in terms of a particular solution plus a linear combination

\[
c_j = U_j + v_k V_{kj}
\]  
(B.13)

where \( k \) runs over all the independent solutions \( V_{kj} \).

After finishing the generalization of Hamiltonian, we now turn to the generalization of Poisson bracket, that is Dirac bracket. The reason for us doing the this generalization can
be seen from a simple example again: in a $N$ particles system including two constraints

$$
q_1 \approx 0, \quad p_1 \approx 0, \quad (B.14)
$$

the Poisson bracket tell us $\{q_1, p_1\} = 1$ which gives the commutator $[\hat{q}_1, \hat{p}_1] = i\hbar$, while plugging the constraints we have $[\hat{q}_1, \hat{p}_1] = 0$. This example shows that the particle without dynamics is not interesting, but the Poisson bracket do not rule them out.

For defining Dirac bracket, we need to know what is first and second class constraints. A constraint $f$ is called first class constraint unless its Poisson bracket with all of the constraints is weakly vanished,

$$\{f, \phi_i\} \approx 0, \quad \forall i. \quad (B.15)$$

Note that since the above bracket is weakly zero the Poisson bracket of first class constraint with other constraints must be the linear combination of these constraints, i.e. $\{f, \phi_i\} = c_j\phi_j$. Another thing interesting to mention is that the number of the first class constraints is the number of the unphysical degrees of freedom.

The second class constraints, denoted by $s$, are those constraints who have non–vanishing Poisson bracket with at least one other constraint. Because of this definition for second class constraints, one can construct a non–degenerate matrix with all second constraints

$$M_{ij} = \{s_i, s_j\}. \quad (B.16)$$

We then can define Dirac bracket in terms of this matrix and the usual Poisson bracket as

$$\{A, B\}_{DB} = \{A, B\} - \{A, s_i\}M^{-1}_{ij}\{s_j, B\} \quad (B.17)$$

This new defined bracket as Poisson bracket has bilinearity, antisymmetry, product rule and Jacobi identity. To proceed canonical quantization for a constrained Hamiltonian system, one only need to multiply classical Dirac bracket by $i\hbar$ and add the hats on these variables in the bracket.

Let go back to the example we mentioned at the beginning of this section and perform
the canonical quantization on it. The conjugate momenta (B.3) give two primary constraints

\[ \phi_1 = y + px, \quad \phi_2 = x - py \]  

(B.18)

The consistency conditions (B.12) indicates that there is no secondary constraints for this system. The Poisson brackets

\[ \{\phi_1, \phi_2\} = -\{\phi_2, \phi_1\} = -2 \]  

(B.19)

show that these two constraints are second class constraints and thus they give M matrix as

\[ M = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \]  

(B.20)

Noting definition (B.17), Dirac brackets are given by

\[ \{x, y\}_{DB} = -\frac{1}{2}, \quad \{x, px\}_{DB} = \frac{1}{2}, \quad \{y, py\}_{DB} = \frac{1}{2}. \]  

(B.21)

The non–vanished Dirac bracket of two coordinates indicates that when it promote to commutator, \( \hat{x} \) and \( \hat{y} \) are not commute, which describe a noncommutative geometry.

**B.2 Angular momentum derived from supergravity solutions**

Let again start with the definition of the complex harmonic function

\[ H = H_1 + iH_2 = \frac{1}{2\pi} \int \frac{\sqrt{(R - F)(R - F + A)}}{(R - F)^2} dl \]

\[ = \frac{1}{2\pi} \int \frac{\sqrt{r^2 + y^2 - 2rfF_n + f^2 + arA_n - afF_A + iy}}{r^2 + y^2 - 2rfF_n + f^2} dl \]  

(B.22)
and here we set
\[
\mathbf{R} = (r \cos \phi, r \sin \phi, y) \\
\mathbf{F} = (f(\phi_f) \cos \phi_f, f(\phi_f) \sin \phi_f, 0) \\
\mathbf{A} = (a(\phi_a) \cos \phi_a, a(\phi_a) \sin \phi_a, ia(\phi_a)) \\
F_n = \cos(\phi - \phi_f), \quad A_n = \cos(\phi - \phi_a), \quad F_A = \cos(\phi_f - \phi_a). \quad (B.23)
\]

There are two constraints on the complex vector \( \mathbf{A} \)
\[
\mathbf{A} \dot{\mathbf{F}} = 0, \quad \mathbf{A} \mathbf{A} = 0. \quad (B.24)
\]

The second constraint is always satisfied by the setting (B.23), and the first constraint \( \mathbf{A} \dot{\mathbf{F}} = 0 \) gives us the relation
\[
a(\phi_a) \left[ f \sin(\phi_f - \phi_a) - \frac{\partial f}{\partial \phi_f} \cos(\phi_f - \phi_a) \right] = 0. \quad (B.25)
\]

Therefore, \( a(\phi_a) \) can be regarded as a constant but 0. Let \( \beta \equiv \phi_f - \phi_a \), then the above condition becomes
\[
\cos \beta = \frac{f}{\sqrt{f^2 + \left( \frac{\partial f}{\partial \phi_f} \right)^2}}. \quad (B.26)
\]

Considering \( r \gg a, \ r \gg f \), one obtains
\[
H = \frac{1}{2\pi} \int \sqrt{\left(1 + \frac{y^2}{r^2}\right) + (aA_n - 2fF_n + \frac{iay}{r})} \frac{1}{r} + (f^2 - aF_A) \frac{1}{r^2} \\
\left(1 + \frac{y^2}{r^2}\right) - 2fF_n + \frac{f^2}{r^2} \quad dl \\
= \frac{1}{2\pi} \int \sqrt{\left(1 + \frac{y^2}{r^2}\right) + (aA_n - 2fF_n + \frac{iay}{r})\epsilon + (f^2 - aF_A)\epsilon^2} \frac{1}{r} \\
\left(1 + \frac{y^2}{r^2}\right) - 2fF_n\epsilon + f^2\epsilon^2 \quad dl \quad (B.27)
\]

Expanding \( H \) in terms of \( \epsilon \) to third order, the real parts give the multipole expansion of \( H_1 \)
\[
H_1^{(0)} = \frac{Q}{\sqrt{r^2 + y^2}}, \quad Q = \frac{1}{2\pi} \int dl \\
H_1^{(2)} = \frac{(r^2 - 2y^2)W_1 + r^2 [W_{r1} \cos 2\phi + W_{s1} \sin 2\phi]}{16(r^2 + y^2)^{5/2}}, \quad (B.28)
\]
with
\[ W_1 = \frac{1}{2\pi} \int (4f^2 - a^2 + 4af \cos \beta)dl \equiv 4f_s - a^2Q + 4f_\beta, \quad (B.29) \]
while the imaginary parts give the multipole expansion of \( H_2 \)
\[ H_2^{(1)} = \frac{aQy}{2(r^2 + y^2)^{3/2}}, \]
\[ H_2^{(3)} = \frac{ay\left[(3r^2 - 2y^2)W_2 + r^2(W_{c2}\cos 2\phi + W_{s2}\sin 2\phi)\right]}{4(r^2 + y^2)^{7/2}}, \quad (B.30) \]
with
\[ W_2 = \frac{1}{2\pi} \int (12f^2 + a^2 - 4fL \cos \beta)dl \equiv 12f_s + a^2Q - 4f_\beta, \quad (B.31) \]
where \( W_{c1}, W_{s1}, W_{c2}, W_{s2} \) are some integrals, and
\[ Q \equiv \frac{1}{2\pi} \int dl, \]
\[ f_s \equiv \frac{1}{2\pi} \int f^2dl, \]
\[ f_\beta \equiv \frac{1}{2\pi} \int f \cos \beta dl = a \frac{1}{2\pi} \int f \cos(\phi_f - \phi_a)dl. \quad (B.32) \]

Following the procedures (3.48) to (3.51) and picking
\[ p_1(\mu, \phi) = \frac{1}{192Q^3} \left[ (27u^2 - 23)W_{c1} + (5 - 9u^2)W_{c2} \right] \sin 2\phi \]
\[ + \left[ (23 - 27u^2)W_{s1} + (9u^2 - 5)W_{s2} \right] \cos 2\phi \quad (B.33) \]
we find
\[ \frac{g_{t\phi}}{g_{\phi\phi}} \approx \frac{a}{2Q^2} + \frac{a \left[ 4f_s - 20af_\beta + 9a^2Q - 3(12f_s - 28af_\beta + 11a^2Q)u^2 \right]}{32Q^5u^2}. \quad (B.34) \]

As discussed above, \( a(\phi) \) can be any constant (except 0) if condition (B.26) is satisfied. More constraints on \( a(\phi_a) \) are: it is enable the metric to reduce to \( \text{AdS}_2 \times S^2 \) when we consider the circular profile (the sources of the two harmonic function) which the center is located in the origin; it is enable the coefficient of the second order of (B.34) to be constant, i.e. (B.34)
is \( u \)-independent. Taking into account all of these constraints on \( a(\phi_a) \) we can be

\[
a(\phi_a) = \frac{2 \left( 7f_\beta + \sqrt{49f_\beta^2 - 33f_sQ} \right)}{11Q},
\]

(B.35)

which makes \( \frac{g_u}{g_{\phi\phi}} \) is not \( u \)-dependent.

Here, in order to avoid same result as (3.59), we may define something different from (3.53)

\[
D\phi^2 = (d\phi + \frac{g_{\phi\phi}}{g_{\phi\phi}} dt)^2 = (d\tilde{\phi} - \frac{J'}{v^2} dt)^2
\]

(B.36)

where

\[
d\tilde{\phi} = d\phi + \frac{a}{2Q^3} dt
\]

(B.37)

\[
J' \propto \left( W_a + \sqrt{W_a^2 - 3WQ'} \right) \frac{Q'}{Q'^4} \left[ 7WQ' - W_a \left( W_a + \sqrt{W_a^2 - 3WQ'} \right) \right]
\]

(B.38)

Where

\[
Q' = 11^{-3/2}Q, \quad W = 11^{5/2}f_s, \quad W_a = 7f_\beta.
\]

(B.39)

Next, we will apply the expression for angular momentum (or energy due to the supersymmetric condition) into two specific cases. The first case is to consider an extra small circular profile with radius \( r_0 \) centered at \( R_0 \)

\[
F' = (R_0 \cos \phi_0 + r_0 \cos \alpha, R_0 \sin \phi_0 + r_0 \sin \alpha, 0)
\]

(B.40)

with length

\[
f' = \sqrt{R_0^2 + r_0^2 + 2r_0R_0 \cos(\phi_0 - \alpha)},
\]

(B.41)

besides the circular profile with radius \( L \) (the radius of AdS2) centered on the origin. The
integrals $f_s, f_\beta$ in this case can be directly computed

$$
\begin{align*}
    f_s &= L^3 + \int_0^{2\pi} f^2 r_0 d\alpha = L^3 + r_0 (r_0^2 + R_0^2), \\
    f_\beta &= L^2 + r_0 R_0, 
\end{align*}
$$ (B.42)

The second line has been taken into account the constraint $R_0 \gg r_0$. Under the condition $L \gg r_0$ and only keeping the leading order, the Eq. (B.38) becomes

$$
J = \frac{1}{32\pi^2 l_p^8} L^5 (R_0 - L)^2 r_0^2 
$$ (B.43)

Another case we can consider here is the quantized profiles

$$
\begin{align*}
    f(\phi) &= f_0(\phi) + \lambda \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 1}} \left( c_n e^{-in\phi} + c_n^\dagger e^{in\phi} \right), \\
    [c_n, c_m^\dagger] &= \delta_{mn},
\end{align*}
$$ (B.44)

The leading term of energy computed through (B.38) is

$$
E = \frac{\lambda^2 L^6}{32\pi^2 l_p^8} \sum_{n>2} \frac{1}{(n^2 - 1)} \left( c_n c_n^\dagger + \frac{1}{2} \right). 
$$ (B.45)
APPENDIX C
Appendix for Hidden symmetries of superstrata

C.1 Superstrata

In this appendix, we give the review for a new class of solutions describing the microstates of five–dimensional three–charge BPS black holes which can have arbitrarily small angular momenta [17]. These horizonless supergravity solutions have the same mass, charges and angular momenta as general BPS rotating D1–D5–P back holes in five dimensions. These solutions later was generalized to asymptotically–flat backgrounds [91].

C.1.1 Single–mode solutions

These solutions have been considered in Type IIB string theory on $M^{4,1} \times S^1 \times M$, where $M$ is either $T^4$ or K3. The circle $S^1$ here is macroscopic and parametrized by the coordinate $y$: $y \sim y + 2\pi R_y$ where $R_y$ is the radius of the circle. We set that all fields involved have no dependence on $M$ which is considered to be microscopic. These solutions have nontrivial momentum along the circle which is wrapped by both D1 and D5 branes. A theory can be obtained under dimensional reduction on $M$ which is six–dimensional $\mathcal{N} = 1$ supergravity coupled to two tensor multiplets in the low–energy limit. All the fields that are involved the study of D1–D5–P string world–sheet amplitudes are included in this theory [92]. All the 1/8–BPS solutions of this theory are described by the system of equations found in [93], which is a generalization of the system studied in [84, 94]. In [95], this supersymmetric system has been simplified and linearized. For BPS solutions, the well–known form of the six–dimensional metric is given by

$$ds^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv + \beta)\left((du + \omega + \frac{1}{2}\mathcal{F}(dv + \beta)) + \sqrt{\mathcal{P}}ds^2_1(\mathcal{B})\right),$$  \hspace{1cm} (C.1)

where the asymptotically null coordinates $u$ and $v$ can be expressed by $y$ and $t$ as

$$u \equiv \frac{1}{\sqrt{2}}(t - y), \quad v \equiv \frac{1}{\sqrt{2}}(t + y).$$  \hspace{1cm} (C.2)
The generic BPS solutions depends all coordinates but $u$ due to the requirement of supersymmetry.

The discussion in this section only considers the decoupling limit under which one would find the solutions asymptotical to $\text{AdS}_3 \times S^3$. The solutions that tensor fields involved explicitly depend on $v$ and on $S^3$ are called "superstrata" [15, 16, 17, 18, 91, 96]. On the four–dimensional base $B$, the flat metric $ds_4^2$ can be written as the form

$$ds_4^2 = \Sigma \left( \frac{dr^2}{(r^2 + a^2)} + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta d\phi_1^2 + r^2 \cos^2 \theta d\phi_2^2, \quad \text{(C.3)}$$

where $\Sigma \equiv (r^2 + a^2 \cos^2 \theta)$. Since the supergravity solutions [17] whose tensor fields have dependance on a single linear combination of $v, \phi_1, \phi_2$, we call them as single–mode superstrata. There also has isometries along $v, \phi_1, \phi_2$ because this phase dependence cancels in the energy–momentum tensor. Thus the solution we discuss here only have nontrivial dependence on $r$ and $\theta$.

In order to figure out metric (C.1), we need to specify the undetermined quantities, $\mathcal{P}, \mathcal{F}, \beta, \omega$. Due to the superstrata constructed in [17, 96] by adding momentum wave to the background of the circular supertube [97], the vector field $\beta$ then can be the standard magnetic flux of the supertube

$$\beta = \frac{R_y a^2}{\sqrt{2\Sigma}} (\sin^2 \theta d\phi_1 - \cos^2 \theta d\phi_2), \quad \Theta^{(3)} \equiv d\beta. \quad \text{(C.4)}$$

where $\Theta^{(3)}$ is self–dual. Three potential functions $Z_I$ and magnetic self–dual two–forms $\Theta^{(I)}$ where $I = 1, 2, 4$, defining the first part of the solution, obey the so-called "first layer" of the linear system of equations governing BPS solutions

$$\star_4 \mathcal{D}(\partial_v Z_1) = \mathcal{D}\Theta^{(2)}, \quad \star_4 \mathcal{D}(\partial_v Z_2) = \mathcal{D}\Theta^{(1)}, \quad \star_4 \mathcal{D}(\partial_v Z_4) = \mathcal{D}\Theta^{(4)},$$

$$\mathcal{D}\star_4 \mathcal{D} Z_1 = -\Theta^{(2)} \wedge d\beta, \quad \mathcal{D}\star_4 \mathcal{D} Z_2 = -\Theta^{(1)} \wedge d\beta, \quad \mathcal{D}\star_4 \mathcal{D} Z_4 = -\Theta^{(4)} \wedge d\beta,$$

$$\Theta^{(I)} = \star_4 \Theta^{(I)}, \quad \mathcal{D} \equiv d_4 - \beta \wedge \partial_v, \quad \text{(C.5)}$$

where $d_4$ is the exterior derivative on base $B$ while $\star_4$ is the Hodge in this base. The
quadratic form in the potential functions gives the the warp factor in (C.1) as

\[ P = Z_1 Z_2 - Z_4^2. \]  

(C.6)

The quantities \( \mathcal{F}, \omega \) can be found from the "second layer" supersymmetric equations

\[
D\omega + \star_4 D\omega + \mathcal{F} d\beta = Z_1 \Theta^{(1)} + Z_2^{(2)} - 2Z_4 \Theta^{(4)},
\]

\[
\star_4 D \star_4 \left( \partial_v \omega - \frac{1}{2} \mathcal{F} \right) = \partial^2_v (Z_1 Z_2 - Z_4^2) - \left[ \partial_v Z_1 \partial_v Z_2 - (\partial_v Z_4)^2 \right] - \frac{1}{2} \star_4 (\Theta^{(1)} \wedge \Theta^{(2)} - \Theta^{(4)} \wedge \Theta^{(4)}).
\]

(C.7)

By adding a fluctuating mode with strength \( b_{k,m,n} \) and then performing the "coiffuring" technique of [16, 96, 98, 99], the [17] gave the potentials as

\[
Z_1 = \frac{Q_1}{\Sigma} + \frac{R_y^2 b_{k,m,n}^2}{2Q_5} \frac{\Delta_{2k,2m,2n}}{\Sigma} \cos \chi_{2k,2m,2n},
\]

\[
Z_2 = \frac{Q_5}{\sigma}, \quad Z_4 = b_{k,m,n} R_y \frac{\Delta_{k,m,n}}{\Sigma} \cos \chi_{k,m,n},
\]

(C.8)

where the phase dependence \( \chi_{k,m,n} \) and \( \Delta_{k,m,n} \) are given by

\[
\chi_{k,m,n} \equiv \frac{\sqrt{2}}{R_y} (m + n)v + (k - m)\phi_1 - m\phi_2,
\]

\[
\Delta_{k,m,n} \equiv \frac{a^k r^n}{(r^2 + a^2)^{\frac{k+n}{2}}} \sin^{k-m} \theta \cos^m \theta,
\]

(C.9)

where \( k \) is a positive integer and \( m, n \) are non–negative integers with \( m \leq k \), which are required by the smoothness of the solutions. The remaining quantities are given by

\[
\mathcal{F} = b_{k,m,n}^2 \mathcal{F}_{k,m,n}, \quad \omega = \omega_0 + b_{k,m,n} \omega_{k,m,n},
\]

(C.10)

where \( \omega_0 \) is given by the value of \( \omega \) that takes undeformation supertube solution:

\[
\omega_0 = \frac{a^2 R_y^2}{\sqrt{2} \Sigma} (\sin^2 \theta d\phi_1 + \cos^2 \theta d\phi_2),
\]

(C.11)

The expressions for \( \mathcal{F}_{k,m,n}, \omega_{k,m,n} \) are a little bit complicated, thus we do not show here,
which can be found in the appendix of [17]. So far it is clear that when \( k, m, n \) are given, we then can specify the metric \((C.1)\).

### C.1.2 A example: \((1, 0, n)\)

In this part we will give special families of superstrata metrics. The solution to the second layer of BPS equations \((C.7)\) for \(k = 1, m = 0\) and general \(n \geq 0\) can be found directly by applying the expressions \((C.10)\)

\[
\mathcal{F} = -\frac{b^2}{a^2} \left[ 1 - \frac{r^{2n}}{(r^2 + a^2)^n} \right], \quad \omega = \omega_0 + \frac{R_y b^2}{\sqrt{2} \Sigma} \left[ 1 - \frac{r^{2n}}{(r^2 + a^2)^n} \right] \sin^2 \theta d\phi_1. \tag{C.12}
\]

The value of \(\mathcal{P}\) is given by \(Q_1Q_5/\Sigma^2\). For the convenience, one can introduce a quantity

\[
\Lambda = \frac{\sqrt{P} \Sigma}{\sqrt{Q_1Q_5}} = \sqrt{1 - \frac{a^2b^2}{(2a^2 + b)(r^2 + a^2)^{n+1}}} \sin^2 \theta. \tag{C.13}
\]

Now the specific formula for metric \((C.1)\) then can be written as

\[
ds_6^2 = \sqrt{Q_1Q_5} \frac{\Lambda}{F_2(r)} \left[ \frac{F_2(r)}{r^2 + a^2} - \frac{2F_1(r)}{a^2(2a^2 + b^2)^2 R_y^2} \left( dv + \frac{a^2(4 + (2a^2 + b^2)r^2)}{F_1(r)} du \right)^2 + \frac{2a^2r^2(r^2 + a^2)F_2(r)}{F_1(r) R_y^2} du^2 \right]
\]

\[
+ \sqrt{Q_1Q_5} \left[ \Lambda d\theta^2 + \frac{1}{\Lambda} \sin^2 \theta \left( d\phi_2 - \frac{a^2}{(2a^2 + b^2) R_y^2} (du + dv) \right)^2 \right] + \frac{F_2(r)}{\Lambda} \cos^2 \theta \left( d\phi_2 + \frac{1}{(2a^2 + b^2)F_2(r)} \sqrt{2} \left[ a^2(du - dv) - b^2 F_0(r) dv \right] \right)^2, \tag{C.14}
\]

where the functions \(F_i(r)\) are given by

\[
F_0(r) \equiv 1 - \frac{r^{2n}}{(r^2 + a^2)^n}, \quad F_1(r) \equiv a^2 - b^2(2a^2 + b^2)r^2 F_0(r),
\]

\[
F_2(r) \equiv 1 - \frac{a^2b^2}{(2a^2 + b^2)(r^2 + a^2)^{n+1}}. \tag{C.15}
\]
C.2 Eigenvalues of conformal Killing tensor

In this appendix, we would like to outline the procedures of proving that there is no conformal Killing–Yano tensor due to the constraints on the eigenvalues of conformal Killing tensor cannot be satisfied. We start with the conformal killing tensor with the general perturbative form

\[ K_{MN} = -X_{MN} + Y_{MN} + bC_{1}^{MN} + b^{2}C_{2}^{MN} + bf_{1}g_{MN} + b^{2}f_{2}g^{MN}, \]  

(C.16)

where the $X_{MN}$, $Y_{MN}$ are encoded by separating Hamilton–Jacobi equation, and the coefficient $-1$ and $1$ in front of them are from the comparison with F1–NS5 system in near horizon limit. The two $C$–matrices are set as

\[
C_{1}^{MN} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_{1} & g_{5} & g_{8} & g_{9} \\
0 & 0 & g_{5} & g_{2} & g_{6} & g_{9} \\
0 & 0 & g_{8} & g_{6} & g_{3} & g_{7} \\
0 & 0 & g_{9} & g_{9} & g_{7} & g_{4}
\end{bmatrix}, \quad C_{2}^{MN} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_{g_{1}} & g_{g_{5}} & g_{g_{8}} & g_{g_{9}} \\
0 & 0 & g_{g_{5}} & g_{g_{2}} & g_{g_{6}} & g_{g_{9}} \\
0 & 0 & g_{g_{8}} & g_{g_{6}} & g_{g_{3}} & g_{g_{7}} \\
0 & 0 & g_{g_{9}} & g_{g_{9}} & g_{g_{7}} & g_{g_{4}}
\end{bmatrix}.
\]  

(C.17)

We first consider the determinant

\[
det|\lambda I - K_{N}^{M}| = c_{0} + c_{1} \lambda + c_{2} \lambda^{2} + c_{3} \lambda^{3} + c_{4} \lambda^{4} + c_{5} \lambda^{5} + \lambda^{6}.
\]  

(C.18)

where $c_{i}$, $i = 0, 1, 3, 4, 5, 6$ are the function of $r$ and $\theta$. Here we will focus on the ratios of these coefficients

\[
c_{10} \equiv \frac{c_{1}}{c_{0}}, \quad c_{20} \equiv \frac{c_{2}}{c_{0}}, \quad c_{30} \equiv \frac{c_{3}}{c_{0}}, \quad c_{40} \equiv \frac{c_{4}}{c_{0}}, \quad c_{50} \equiv \frac{c_{5}}{c_{0}}, \quad c_{60} \equiv \frac{1}{c_{0}}.
\]  

(C.19)

Substituting the expression of conformal killing tensor $K_{MN}$ and metric $g_{MN}$ into (C.18), one can find $c_{i}$ and the ratios of them $c_{i0}$. Using perturbation method, we expand $c_{i0}$ in terms of $b$ as

\[
c_{i0} = c_{i0}^{(0)} + c_{i0}^{(1)} b + c_{i0}^{(2)} b^{2}.
\]  

(C.20)
On the other hand, we set the 6 solutions to equation

\[ \det |\lambda - K^M_N| = 0 \]  \hfill (C.21)

as \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \). Thus (C.18) can be rewritten as

\[ \det |\lambda - K^M_N| = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)(\lambda - \lambda_5)(\lambda - \lambda_6). \]  \hfill (C.22)

Without perturbation, i.e. for the zero order solution \((b = 0)\), it is known that

\[ \lambda_1 = \lambda_2 = \lambda_3 = Q, \quad \lambda_4 = \lambda_5 = \lambda_6 = -Q. \]  \hfill (C.23)

Noting the constraints on the eigenvalues for the existence of conformal Killing–Yano tensor of rank three

\[ \lambda_1 + \mu_1 = \lambda_2 + \mu_2 = \lambda_3 + \mu_3, \]  \hfill (C.24)

Noting (C.23), there are six possibilities in total

\[ \begin{align*}
\lambda_1 + \lambda_4 &= \lambda_2 + \lambda_5 = \lambda_3 + \lambda_6, \\
\lambda_1 + \lambda_5 &= \lambda_2 + \lambda_6 = \lambda_3 + \lambda_4,
\end{align*} \]  \hfill (C.25)

In this appendix we focus our attention on the first case in (C.25),

\[ \lambda_1 + \lambda_4 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_6, \]  \hfill (C.26)

and other cases can be discussed similarly. Using perturbation theory, the six eigenvalues of conformal Killing tensor then can be written as the following forms:

\[ \begin{align*}
\lambda_i &= Q + q_i^{(1)} b + q_i^{(2)} b^2, \quad i = 1, 2, 3 \\
\lambda_j &= -(Q + q_j^{(1)} b + q_j^{(2)} b^2), \quad j = 4, 5, 6
\end{align*} \]  \hfill (C.27)
where \( q^{(1)} \) and \( q^{(2)} \) are the function of \( r \) and \( \theta \). Therefore, we can rewrite (C.22) as

\[
\det |\lambda - K_N^M| = \prod_{i=1,2,3} \left[ \lambda - (Q + q_i^{(1)} b + q_i^{(2)} b^2) \right] \prod_{i=4,5,6} \left[ \lambda + (Q + q_i^{(1)} b + q_i^{(2)} b^2) \right] \\
\equiv \tilde{c}_0 + \tilde{c}_1 \lambda + \tilde{c}_2 \lambda^2 + \tilde{c}_3 \lambda^3 + \tilde{c}_4 \lambda^4 + \tilde{c}_5 \lambda^5 + \lambda^6
\]  
(C.28)

Similarly to (C.19), for \( \tilde{c}_i \), we have

\[
\tilde{c}_{10} \equiv \frac{\tilde{c}_1}{\tilde{c}_0}, \quad \tilde{c}_{20} \equiv \frac{\tilde{c}_2}{\tilde{c}_0}, \quad \tilde{c}_{30} \equiv \frac{\tilde{c}_3}{\tilde{c}_0}, \quad \tilde{c}_{40} \equiv \frac{\tilde{c}_4}{\tilde{c}_0}, \quad \tilde{c}_{50} \equiv \frac{\tilde{c}_5}{\tilde{c}_0}, \quad \tilde{c}_{60} \equiv \frac{1}{\tilde{c}_0}.
\]  
(C.29)

For every \( \tilde{c}_{i0} \), it has the form

\[
\tilde{c}_{i0} = c_{i0}^{(0)} + c_{i0}^{(1)} b + c_{i0}^{(2)} b^2.
\]  
(C.30)

Comparing (C.30) and (C.19), there 6 constraints on the first order

\[
c_i^{(1)} - \tilde{c}_i^{(1)} = 0, \quad i = 1, 2, 3, 4, 5, 6,
\]  
(C.31)

and 6 constraints on second order

\[
c_i^{(2)} - \tilde{c}_i^{(2)} = 0, \quad i = 1, 2, 3, 4, 5, 6.
\]  
(C.32)

Alternatively, there can be a direct way to give the constraints which are equivalent to the constraints (C.31) and (C.32). Expanding the coefficient \( c_i \) in (C.18) we can write

\[
c_i = c_i^{(0)} + c_i^{(1)} b + c_i^{(2)} b^2, \quad i = 0, 1, 2, 3, 4, 5.
\]  
(C.33)

On the other hand the coefficients in (C.28) have the form

\[
\tilde{c}_i = c_i^{(0)} + c_i^{(1)} b + c_i^{(2)} b^2, \quad i = 0, 1, 2, 3, 4, 5.
\]  
(C.34)

Thus constraints (C.31), (C.32) are equivalent to

\[
c_i^{(1)} - \tilde{c}_i^{(1)} = 0, \\
c_i^{(2)} - \tilde{c}_i^{(2)} = 0.
\]  
(C.35)
We now turn to check if conditions (C.31) and (C.32) can be satisfied. First of all, we use
\[
\begin{align*}
  c^{(1)}_{10} - \tilde{c}^{(1)}_{10} &= 0, \\
  c^{(2)}_{10} - \tilde{c}^{(2)}_{10} &= 0,
\end{align*}
\]
(C.36)
to solve \( f_1 \) and \( f_2 \). Using (C.26) and (C.27), we can solve
\[
\begin{align*}
  q^{(1)}_5 b + q^{(2)}_5 b^2 &= \left[ q^{(1)}_2 + q^{(1)}_4 - q^{(1)}_1 \right] b + \left[ q^{(2)}_2 + q^{(2)}_4 - q^{(2)}_1 \right] b^2, \\
  q^{(1)}_6 b + q^{(2)}_6 b^2 &= \left[ q^{(1)}_3 + q^{(1)}_4 - q^{(1)}_1 \right] b + \left[ q^{(2)}_3 + q^{(2)}_4 - q^{(2)}_1 \right] b^2.
\end{align*}
\]
(C.37)
Substituting (C.37) into (C.28), we then can write \( \tilde{c}^{(1)}_{i0} (i = 1, 2, 3, 4, 5, 6) \), as the function of \( q^{(1)}_i \), \( (i = 1, 2, 3, 4) \), and write \( \tilde{c}^{(2)}_{i0} (i = 1, 2, 3, 4, 5, 6) \) as the function of \( q^{(1)}_i, q^{(2)}_i \) \( (i = 1, 2, 3, 4) \).

Substituting \( f_1 \) into \( c^{(1)}_{30} \) and \( c^{(1)}_{50} \), the conditions
\[
\begin{align*}
  c^{(1)}_{30} - \tilde{c}^{(1)}_{30} &= 0, \\
  c^{(1)}_{50} - \tilde{c}^{(1)}_{50} &= 0,
\end{align*}
\]
(C.38)
are satisfied. The following analysis divide into two cases: \( q^{(1)}_i \) is the function of \( r \) and \( \theta \), and that \( q^{(1)}_i \) is not the function of \( r \) and \( \theta \). For the first case, \( q^{(1)}_i \) is the function of \( r \) and \( \theta \). To satisfy the other three conditions (the equations with \( i \) is even) in (C.31), we solve these equations and find
\[
\begin{align*}
  g_5 &= 0, \\
  g_0 &= -g_1 - g_2 + R_y \left( \sqrt{2} g_9 - g_4 R_y \right) \over \sqrt{2} R_y, \\
  g_8 &= g_1 + g_2 + R_y \left( -\sqrt{2} g_6 + g_3 R_y \right) \over \sqrt{2} R_y, \\
  q^{(1)}_4 &= {1 \over 3} (q^{(1)}_1 - 2q^{(1)}_2 - 2q^{(1)}_3).
\end{align*}
\]
(C.39)
Next we move to equations of second order. Substituting these values (C.39) and \( f_2 \) into
constraints (C.32), The odd equations

\begin{align*}
\tilde{c}_{30}^{(2)} - \tilde{c}_{30}^{(2)} &= 0, \\
\tilde{c}_{50}^{(2)} - \tilde{c}_{50}^{(2)} &= 0,
\end{align*}

(C.40)
can be satisfied by solving \(g_1, g_2, gg_5, gg_0\). However, the even equations, considering simple situation \(n = 1, \theta = 0\), reduce to

\begin{align*}
-\frac{r^2}{Q^2 (a^2 + r^2)^2} &= 0, \quad i = 2 \\
\frac{2r^2}{Q^4 (a^2 + r^2)^2} &= 0, \quad i = 4 \\
-\frac{r^2}{Q^6 (a^2 + r^2)^2} &= 0, \quad i = 6.
\end{align*}

(C.41)

These equations are not correct unless \(r = 0\) which is condition that cannot be satisfied, which implies that the conditions (C.24) on eigenvalues are not possible.

Next we consider the case that \(q_i^{(1)}\) is not the function of \(r\) and \(\theta\). To satisfy the even equations, we find the solution

\begin{align*}
g_5 &= 0 \\
g_0 &= -g_1 - g_2 + R_y \left( \sqrt{2}g_9 - g_4 R_y + \tilde{c}_{20}^{(1)} Q^2 R_y \right) \\
g_8 &= g_1 + g_2 + R_y \left( -\sqrt{2}g_6 + g_3 R_y - \tilde{c}_{20}^{(1)} Q^2 R_y \right)
\end{align*}

(C.42)

where

\[ \tilde{c}_{20}^{(1)} = (-q_1^{(1)} + 2(q_2^{(1)} + q_3^{(1)}) + 3q_4^{(1)})/Q^3. \]

(C.43)

When \(\tilde{c}_{20}^{(1)} = 0\), (C.42) will reduce to (C.39) due to \(\tilde{c}_{20}^{(1)} = 0\) is equivalent to last equation in (C.39).

When comes to second order, we find odd equations can be satisfied. However, the
even equations reduce to

\[-\frac{r^2}{Q^2(a^2 + r^2)^2} = 0, \quad i = 2\]

\[-\frac{2r^2}{Q^4(a^2 + r^2)^2} = 0, \quad i = 4\]

\[-\frac{r^2}{Q^6(a^2 + r^2)^2} = 0, \quad i = 6\]

(C.44)

This is same as (C.41).

As a check, we can consider constraints (C.35). Using solutions (C.39), all first order equations \(c_i^{(1)} - \tilde{c}_i^{(1)} = 0\) are satisfied. The odd second order equations \(c_i^{(2)} - \tilde{c}_i^{(2)} = 0\) are easy to reach by solving more elements of C–matrix. However, when one considers the simple situations \((n = 1, \theta = 0)\), the even second order equations reduce to

\[-\frac{Q^6r^2}{(a^2 + r^2)^2} = 0, \quad i = 0\]

\[-\frac{2Q^4r^2}{(a^2 + r^2)^2} = 0, \quad i = 2\]

\[-\frac{Q^2r^2}{(a^2 + r^2)^2} = 0, \quad i = 4.\]

(C.45)

We can conclude that, from (C.41), (C.44) and (C.45), There does not exist such matrices \(C_1^{MN}\) and \(C_2^{MN}\) which are Killing tensor constructed from Killing vectors and such functions \(f_1\) and \(f_2\) that the constraints (C.24) are satisfied. It can be checked that other possible cases in (C.25) are also terminated by equalities banned. Thus there are no conformal Killing–Yano tensor in the current situation.
APPENDIX D

Appendix for (Conformal) Killing(–Yano) tensors of
\( \text{AdS}_p \times S^p \times (S^p) \)

In this appendix, we will outline the procedures to solve equations for all components of CKT in \( \text{AdS}_2 \times S^2 \times S^2 \)

\[
d s^2 = L^2 \left( d\rho^2 - \cosh^2 \rho d t^2 \right) + R^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + \tilde{R}^2 \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right) . \quad (D.1)
\]

Since coordinates \( t, \phi, \tilde{\phi} \) are cyclic coordinates, we can write the every component of CKYT as

\[
K_{MN}(\rho, t, \theta, \phi, \tilde{\theta}, \tilde{\phi}) = K_{MN}(\rho, \theta, \tilde{\theta}) e^{int+im\phi+i\tilde{m}\tilde{\phi}} , \quad (D.2)
\]

where \( n, m, \tilde{m} \) are three integers. Here we neglect the prime on \( K_{MN} \) which appears in section 5.5.1. Thanks to the symmetries there are only have 21 components but 36:

\[
K_{MN}(\rho, t, \theta, \phi, \tilde{\theta}, \tilde{\phi}) = \begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\
K_{12} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\
K_{13} & K_{23} & K_{33} & K_{34} & K_{35} & K_{36} \\
K_{14} & K_{24} & K_{34} & K_{44} & K_{45} & K_{46} \\
K_{15} & K_{25} & K_{35} & K_{45} & K_{55} & K_{56} \\
K_{16} & K_{26} & K_{36} & K_{46} & K_{56} & K_{66}
\end{bmatrix} \times e^{int+im\phi+i\tilde{m}\tilde{\phi}} . \quad (D.3)
\]

There are \( 6 \times 6 \times 6 = 216 \) partial differential equations in total. However, recall that the CKT equation

\[
\nabla_L K_{MN} = W_{(Lg_{MN)}} , \quad (D.4)
\]

is completely symmetric. There are only

\[
\binom{6}{3} + \binom{6}{2} + \binom{6}{1} = 20 + 15 + 6 = 41 \quad (D.5)
\]

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independent PDEs. It is easy to solve the six components of the associated vector \( W_L \) from six equations \((M,N,L) = (k,k,k), \ k = 1, \cdots , 6.\) After this, there are 35 PDEs left.

The combination of \((M,N,L) = (1,1,2)\) and \((M,N,L) = (1,2,2)\) gives

\[
\cosh \rho \partial_\rho \left[ \cosh \rho \partial_\rho \tilde{K}_{12} \right] + n^2 \tilde{K}_{12} = 0 \]
\[
\cosh \rho \partial_\rho \left[ \cosh \rho \partial_\rho (\tilde{K}_{11} + \tilde{K}_{22}) \right] + n^2 (\tilde{K}_{11} + \tilde{K}_{22}) = 0, \tag{D.6}
\]

where

\[
\tilde{K}_{12} = \frac{K_{12}}{\cosh^3 \rho}, \quad \tilde{K}_{11} = \frac{K_{11}}{\cosh^2 \rho}, \quad \tilde{K}_{22} = \frac{K_{12}}{\cosh^4 \rho} \tag{D.7}
\]

The solution of (D.6) is given by

\[
\tilde{K}_{12} = c_1 \sin \left[ 2n \tan^{-1} \left( \tanh \left( \frac{\rho}{2} \right) \right) \right] + c_2 \cos \left[ 2n \tan^{-1} \left( \tanh \left( \frac{\rho}{2} \right) \right) \right] \]
\[
\tilde{K}_{11} + \tilde{K}_{22} = c'_1 \sin \left[ 2n \tan^{-1} \left( \tanh \left( \frac{\rho}{2} \right) \right) \right] + c'_2 \cos \left[ 2n \tan^{-1} \left( \tanh \left( \frac{\rho}{2} \right) \right) \right] \tag{D.8}
\]

where \(c_1, \ c_2, \ c'_1, \ c'_2\) are integral constants which are function of \(\rho, \hat{\theta}\).

The combination of \((M,N,L) = (3,3,4)\) and \((M,N,L) = (3,4,4)\) gives

\[
\sin \theta \partial_\theta \left[ \sin \theta \partial_\theta \tilde{K}_{34} \right] - m^2 \tilde{K}_{34} = 0 \]
\[
\sin \theta \partial_\theta \left[ \sin \theta \partial_\theta (\tilde{K}_{33} + \tilde{K}_{44}) \right] - m^2 (\tilde{K}_{33} + \tilde{K}_{44}) = 0 \tag{D.9}
\]

where

\[
\tilde{K}_{34} = \frac{K_{34}}{\sin^3 \hat{\theta}}, \quad \tilde{K}_{33} = \frac{K_{33}}{\sin^2 \hat{\theta}}, \quad \tilde{K}_{44} = \frac{K_{44}}{\sin^4 \hat{\theta}} \tag{D.10}
\]

The solution of (D.9) is given by

\[
\tilde{K}_{34} = c_3 \cosh \left[ m \log \left( \cot \left( \frac{\theta}{2} \right) \right) \right] - ic_4 \sinh \left[ m \log \left( \cot \left( \frac{\theta}{2} \right) \right) \right] \]
\[
(\tilde{K}_{33} + \tilde{K}_{44}) = c'_3 \cosh \left[ m \log \left( \cot \left( \frac{\theta}{2} \right) \right) \right] - ic'_4 \sinh \left[ m \log \left( \cot \left( \frac{\theta}{2} \right) \right) \right] \tag{D.11}
\]

where \(c_3, \ c_4, \ c'_3, \ c'_4\) are integral constants which are function of \(\rho, \hat{\theta}\).
Similarly, \((M, N, L) = (5, 5, 6)\) and \((M, N, L) = (5, 6, 6)\) gives

\[
\tilde{K}_{56} = c_5 \cosh \left[ m \log \left( \cot \left( \frac{\theta}{2} \right) \right) \right] - ic_6 \sinh \left[ m \log \left( \cot \left( \frac{\theta}{2} \right) \right) \right]
\]

\[
(\tilde{K}_{55} + \tilde{K}_{66}) = c'_5 \cosh \left[ m \log \left( \cot \left( \frac{\theta}{2} \right) \right) \right] - ic'_6 \sinh \left[ m \log \left( \cot \left( \frac{\theta}{2} \right) \right) \right]
\]

where \(c_5, c_6, c'_5, c'_6\) are integral constants which are function of \(\rho, \theta\). and where

\[
\tilde{K}_{56} = \frac{K_{56}}{\sin^3 \theta}, \quad \tilde{K}_{55} = \frac{K_{55}}{\sin^2 \theta}, \quad \tilde{K}_{66} = \frac{K_{66}}{\sin^4 \theta}.
\]

So far, we have worked out exactly how \(K_{12}\) depends on \(\rho\) and \(K_{34}\) depends on \(\theta\) and \(K_{56}\) depends on \(\tilde{\theta}\).

Equations \((M, N, L) = (1, 1, 3)\) and \((M, N, L) = (1, 3, 3)\) give

\[
\partial_3 \left[ \frac{K_{11}}{L^2} - \frac{K_{33}}{R^2} \right] + \frac{2}{L^2} \partial_1 K_{13} = 0
\]

\[
\partial_1 \left[ \frac{K_{11}}{L^2} - \frac{K_{33}}{R^2} \right] - \frac{2}{R^2} \partial_3 K_{13} = 0
\]

Equations \((M, N, L) = (3, 3, 5)\) and \((M, N, L) = (3, 5, 5)\) give

\[
\partial_5 \left[ \frac{K_{33}}{R^2} - \frac{K_{55}}{R^2} \right] + \frac{2}{R^2} \partial_3 K_{35} = 0
\]

\[
\partial_3 \left[ \frac{K_{33}}{R^2} - \frac{K_{33}}{R^2} \right] - \frac{2}{R^2} \partial_5 K_{35} = 0
\]

Equations \((M, N, L) = (1, 1, 5)\) and \((M, N, L) = (1, 5, 5)\) give

\[
\partial_1 \left[ \frac{K_{55}}{R^2} - \frac{K_{11}}{L^2} \right] + \frac{2}{R^2} \partial_5 K_{51} = 0
\]

\[
\partial_5 \left[ \frac{K_{55}}{R^2} - \frac{K_{11}}{L^2} \right] - \frac{2}{L^2} \partial_1 K_{51} = 0
\]
By defining new function we can rewrite (D.14), (D.15) and (D.16) as

\[ K_{13} \equiv \partial_1 f_{13} \quad \frac{K_{11}}{L^2} - \frac{K_{33}}{R^2} = \frac{2}{R^2} \partial_3 f_{13}, \quad (D.17) \]

\[ K_{15} \equiv \partial_1 f_{15} \quad \frac{K_{11}}{L^2} - \frac{K_{55}}{R^2} = \frac{2}{R^2} \partial_5 f_{15}, \quad (D.18) \]

\[ K_{35} \equiv \partial_3 f_{35} \quad \frac{K_{33}}{R^2} - \frac{K_{55}}{R^2} = \frac{2}{R^2} \partial_5 f_{35}, \quad (D.19) \]

and

\[ \left( \frac{\partial^2 \rho}{L^2} + \frac{\partial^2 \theta}{R^2} \right) f_{13} = 0, \quad \left( \frac{\partial^2 \rho}{R^2} + \frac{\partial^2 \theta}{R^2} \right) f_{35} = 0, \quad \left( \frac{\partial^2 \rho}{R^2} + \frac{\partial^2 \bar{\theta}}{L^2} \right) f_{51} = 0. \quad (D.20) \]

We can impose the ansatz of the equations (D.20) as

\[ f_{13} = H_{13}(L\rho + iR\theta, \bar{R}\bar{\theta}) + G_{13}(L\rho - iR\theta, \bar{R}\bar{\theta}) \]

\[ f_{51} = H_{51}(L\rho + i\bar{R}\bar{\theta}, R\theta) + G_{51}(L\rho - i\bar{R}\bar{\theta}, R\theta) \]

\[ f_{35} = H_{35}(R\theta + i\bar{R}\bar{\theta}, L\rho) + G_{35}(R\theta - i\bar{R}\bar{\theta}, L\rho) \quad (D.21) \]

Adding (D.17) and (D.19) and then subtracting (D.18), we have

\[ \frac{1}{R} \partial_5 f_{35} + \frac{1}{R} \partial_3 f_{13} + \frac{1}{R} \partial_5 f_{15} = 0 \quad (D.22) \]

On the other hand the equation for \((M, N, L) = (1, 3, 5)\) gives

\[ \partial_1 \partial_5 f_{13} + \partial_5 \partial_1 f_{15} + \partial_1 \partial_3 f_{15} = 0 \quad (D.23) \]

Substituting ansatz (D.21) into (D.22) and (D.23), we can work out \(f_{13}, f_{15}, f_{35}\). Finally, noting the definitions of \(f_{13}, f_{15}, f_{35}\), we can find \(K_{13}, K_{15}, K_{35}\) as the polynomials of \(\rho, \theta, \bar{\theta}\) with lots of integral constants.

The equation \((M, N, L) = (1, 2, 3)\) and the combination of equations \((M, N, L) = (2, 2, 3)\) and \((M, N, L) = (2, 3, 3)\) give us

\[ \text{i}nK_{13} + \partial_6 K_{12} + \cosh^2 \rho \partial_\rho K_{23} = 0, \quad K_{23} = K_{23}/\cosh^2 \rho \]

\[ \text{i}nK_{13} - \partial_6 K_{12} + \coth \rho \left( \frac{L^2 \cosh^2 \rho}{R^2} \partial_\rho^2 + n^2 \right) K_{23} = 0 \quad (D.24) \]
Remember that $K_{13}$ is known and how $K_{12}$ depends on $\rho$ is also known from (D.8) and thus we can work out how $K_{23}$ depends on $\rho$ from the first equation. Adding these two equations, $K_{12}$ is cancelled out such that we can work out how $K_{23}$ depends on $\theta$. Substituting $K_{23}$ back to the first equation, we then know $K_{12}$ depends on $\theta$.

The equation $(M, N, L) = (3, 4, 1)$ and the combination of equations $(M, N, L) = (1, 1, 4)$ and $(M, N, L) = (1, 4, 4)$ give us

$$\begin{align*}
im K_{13} + \partial_{\rho} K_{34} + \sin^{2} \theta \partial_{\theta} K_{14} &= 0, \quad K_{14} = K_{14}/\sin^{2} \theta \\
im K_{13} - \partial_{\rho} K_{34} + \tan \theta \left( \frac{R^{2} \sin^{2} \theta}{L^{2}} \partial_{\rho}^{2} - m^{2} \right) K_{14} &= 0 \quad (D.25)
\end{align*}$$

Similarly, we can work out how $K_{14}$ depends on $\theta$ from the first equation, while the combination of these two equation give how $K_{14}$ depends on $\rho$. Substituting $K_{14}$ back to the first equation, we then know how $K_{34}$ depends on $\rho$.

The equation $(M, N, L) = (1, 2, 5)$ and the combination of equations $(M, N, L) = (2, 2, 5)$ and $(M, N, L) = (2, 5, 5)$ give us

$$\begin{align*}
in K_{15} + \partial_{\tilde{\theta}} K_{12} + \cosh^{2} \rho \partial_{\rho} K_{25} &= 0, \quad K_{25} = K_{25}/\cosh^{2} \rho \\
in K_{15} - \partial_{\tilde{\theta}} K_{12} + \coth \rho \left( \frac{L^{2} \cosh^{2} \rho}{R^{2}} \partial_{\tilde{\theta}}^{2} + n^{2} \right) K_{25} &= 0 \quad (D.26)
\end{align*}$$

What we can work out from these two equations are: how $K_{25}$ depends on $\rho, \tilde{\theta}$; how $K_{12}$ depends on $\tilde{\theta}$.

The equation $(M, N, L) = (5, 6, 1)$ and the combination of equations $(M, N, L) = (1, 1, 6)$ and $(M, N, L) = (1, 6, 6)$ give us

$$\begin{align*}
im K_{15} + \partial_{\rho} K_{56} + \sin^{2} \tilde{\theta} \partial_{\tilde{\theta}} K_{16} &= 0, \quad K_{16} = K_{16}/\sin^{2} \tilde{\theta} \\
im K_{15} - \partial_{\rho} K_{56} + \tan \tilde{\theta} \left( \frac{R^{2} \sin^{2} \tilde{\theta}}{L^{2}} \partial_{\rho}^{2} - \tilde{m}^{2} \right) K_{16} &= 0 \quad (D.27)
\end{align*}$$

What we can work out from these two equations are: how $K_{16}$ depends on $\rho, \tilde{\theta}$; how $K_{56}$ depends on $\rho$.

The equation $(M, N, L) = (3, 4, 5)$ and the combination of equations $(M, N, L) =
(4, 4, 5) and \((M, N, L) = (4, 5, 5)\) give us

\[
\begin{align*}
im K_{35} + \partial_\rho K_{34} + \sin^2 \hat{\theta} \partial_\rho \hat{K}_{45} &= 0, \quad \hat{K}_{45} = K_{45} / \sin^2 \theta \\
im K_{35} - \partial_\rho K_{34} + \tan \theta \left( \frac{R^2 \sin^2 \theta}{R^2} \partial_\rho^2 - m^2 \right) \hat{K}_{45} &= 0
\end{align*}
\]

(D.28)

What we can work out from these two equations are: how \(K_{45}\) depends on \(\theta, \tilde{\theta}\); how \(K_{34}\) depends on \(\tilde{\theta}\).

The equation \((M, N, L) = (5, 6, 3)\) and the combination of equations \((M, N, L) = (3, 3, 6)\) and \((M, N, L) = (3, 6, 6)\) give us

\[
\begin{align*}
im \tilde{m} K_{35} + \partial_\rho K_{56} + \sin^2 \theta \partial_\rho \tilde{K}_{36} &= 0, \quad \tilde{K}_{36} = K_{36} / \sin^2 \tilde{\theta} \\
im \tilde{m} K_{35} - \partial_\rho K_{56} + \tan \tilde{\theta} \left( \frac{R^2 \sin^2 \tilde{\theta}}{R^2} \partial_\rho^2 - \tilde{m}^2 \right) \tilde{K}_{36} &= 0
\end{align*}
\]

(D.29)

What we can work out from these two equations are: how \(K_{36}\) depends on \(\theta, \tilde{\theta}\); how \(K_{56}\) depends on \(\theta\).

The equations \((M, N, L) = (1, 2, 4)\) and \((M, N, L) = (3, 4, 2)\) give us

\[
\begin{align*}
im K_{12} + \im K_{14} + \sin^2 \theta \cosh^2 \rho \partial_\rho \tilde{K}_{24} &= 0, \quad \tilde{K}_{24} = K_{24} / (\sin^2 \theta \cosh^2 \rho) \\
\im K_{34} + \im K_{23} + \sin^2 \theta \cosh^2 \rho \partial_\rho \tilde{K}_{24} &= 0.
\end{align*}
\]

(D.30)

What we can work out from these two equations is how \(K_{24}\) depends on \(\rho, \tilde{\theta}\).

The equations \((M, N, L) = (1, 2, 6)\) and \((M, N, L) = (5, 6, 2)\) give us

\[
\begin{align*}
im \tilde{m} K_{12} + \im K_{16} + \sin^2 \tilde{\theta} \cosh^2 \rho \partial_\rho \tilde{K}_{26} &= 0, \quad \tilde{K}_{26} = K_{26} / (\sin^2 \tilde{\theta} \cosh^2 \rho) \\
\im K_{56} + \im \tilde{m} K_{25} + \sin^2 \tilde{\theta} \cosh^2 \rho \partial_\rho \tilde{K}_{26} &= 0.
\end{align*}
\]

(D.31)

What we can work out from these two equations is how \(K_{26}\) depends on \(\rho, \tilde{\theta}\).

The equations \((M, N, L) = (3, 4, 6)\) and \((M, N, L) = (5, 6, 4)\) give us

\[
\begin{align*}
im \tilde{m} K_{34} + \im K_{36} + \sin^2 \tilde{\theta} \sin^2 \theta \partial_\rho \tilde{K}_{46} &= 0, \quad \tilde{K}_{46} = K_{46} / (\sin^2 \tilde{\theta} \sin^2 \theta) \\
im K_{56} + \im \tilde{m} K_{45} + \sin^2 \tilde{\theta} \sin^2 \theta \partial_\rho \tilde{K}_{46} &= 0.
\end{align*}
\]

(D.32)
What we can work out from these two equations is how $K_{46}$ depends on $\theta, \tilde{\theta}$.

So far we know $K_{12}, K_{34}$ and $K_{56}$ completely. The mixed components that we do not know are: $K_{23}(\tilde{\theta}), K_{14}(\tilde{\theta}), K_{25}(\theta), K_{16}(\theta), K_{45}(\rho), K_{36}(\rho), K_{24}(\tilde{\theta}), K_{26}(\theta), K_{46}(\rho)$. These can be worked out through the following ordinary equations:

$$2 \left( m^2 - \frac{R^2 \sin^2 \theta}{L^2 \cosh^2 \rho} n^2 \right) K_{24} + \im \left( \sin 2\theta K_{34} - \frac{R^2 \sin^2 \theta}{L^2 \cosh^2 \rho} \sinh 2\rho K_{14} \right)$$

$$+ \im \left( \frac{R^2 \sin^2 \theta}{L^2 \cosh^2 \rho} \sinh 2\rho K_{12} - \sin 2\theta K_{23} \right) = 0$$

$$2 \left( \tilde{m}^2 - \frac{\tilde{R}^2 \sin^2 \tilde{\theta}}{L^2 \cosh^2 \rho} n^2 \right) K_{26} + \im \left( \sin 2\tilde{\theta} K_{56} - \frac{\tilde{R}^2 \sin^2 \tilde{\theta}}{L^2 \cosh^2 \rho} \sinh 2\rho K_{16} \right)$$

$$+ \im \left( \frac{\tilde{R}^2 \sin^2 \tilde{\theta}}{L^2 \cosh^2 \rho} \sinh 2\rho K_{12} - \sin 2\theta K_{25} \right) = 0$$

$$2 \left( \tilde{m}^2 - \frac{\tilde{R}^2 \sin^2 \tilde{\theta}}{\tilde{R}^2 \sin^2 \tilde{\theta} m^2} \right) K_{46} + \im \left( \sin 2\tilde{\theta} K_{56} - \frac{\tilde{R}^2 \sin^2 \tilde{\theta}}{\tilde{R}^2 \sin^2 \tilde{\theta}} \sin 2\theta K_{36} \right)$$

$$+ \im \left( \frac{\tilde{R}^2 \sin^2 \tilde{\theta}}{\tilde{R}^2 \sin^2 \tilde{\theta}} \sin 2\theta K_{34} - \sin 2\theta K_{45} \right) = 0, \quad (D.33)$$

These three equations come from the combinations of equations $(M, N, L) = (2, 2, 4)$ and $(M, N, L) = (2, 4, 4)$, equations $(M, N, L) = (2, 2, 6)$ and $(M, N, L) = (2, 6, 6)$ and equations $(M, N, L) = (6, 6, 4)$ and $(M, N, L) = (4, 6, 6)$. All the mixed components are worked out so far. Substituting these known components into conformal Killing tensor equations one then can solve $K_{11}, K_{22}, K_{33}, K_{44}, K_{55}, K_{66}$. 

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BIBLIOGRAPHY


I. Bena and P. Kraus, “Microstates of the D1-D5-KK system,” Phys. Rev. D 72, 025007


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B. Hoare, A. Pittelli and A. Torrielli, “Integrable S-matrices, massive and massless modes


B 610, 49 (2001) [hep-th/0105136].

