Coarse geometric coherence

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Coarse Geometric Coherence

by

Jonathan L. Grossman

A Dissertation
Submitted to the University at Albany, State University of New York
in Partial Fulfillment of
the Requirements for the Degree of
Doctor of Philosophy

College of Arts & Sciences
Department of Mathematics and Statistics

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Abstract

This dissertation establishes three coarse geometric analogues of algebraic coherence: geometric coherence, coarse coherence, and relative coarse coherence. Each of these coarse geometric coherence notions is a coarse geometric invariant. Several permanence properties of these coarse invariants are demonstrated, elementary examples are computed, and the relationships that these properties have with one another and with other previously established coarse geometric invariants are investigated. Significant results include that the straight finite decomposition complexity of A. Dranishnikov and M. Zarichnyi implies coarse coherence, and that M. Gromov’s finite asymptotic dimension implies coherence, coarse coherence, and relative coarse coherence. Further, as a consequence of a theorem of D. Kasprowski, A. Nicas, and D. Rosenthal, the collection of countable groups with coarse coherence is closed under extensions and free products, and includes all elementary amenable, all linear, and subgroups of virtually connected Lie groups.
Acknowledgments

We all exist in context, and it is cognizance of that fact that leads me to acknowledge the contributions many individuals have made to my personal and academic successes.

I came to the University at Albany, in part, because of the promise on the department webpage that the people here were like a family. I am happy to have found and become a part of that family, and, in particular, would like to thank a few of my fellow graduate students: Jen H., Kseniya K., Doug L., and Janis R.. I would especially like to thank Mike M., who was more than willing to entertain far-flung hypothetical scenarios and semantic arguments with me, as well as engage in jolly cooperation, when the flow of time permitted. I’d also like to thank Mike N., for spending a few of his evenings online with me despite his far superior hand-eye coordination; Bryan G. and Mai T., for inviting me to take up residence in their office whenever I felt like reaching out; Rob S., for competing with me to see how long it is possible to stretch what for anyone else would be an hour; and James C., for allowing me to bounce ideas off of him and think through teaching, research, and personal dilemmas together. Finally, I’d like to thank Roy C., for being a good friend, excellent office-mate, and generally chill guy, whose deadpan sarcasm is exactly my speed.
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Chapter 1

Introduction

The study of large-scale or coarse geometry began with the work of M. Gromov [22] to establish and investigate what he termed the “asymptotic invariants” of groups. He defined the asymptotic dimension of a metric space, and, after using the word-metric on groups to represent them as metric spaces, asserted that asymptotic dimension is an invariant of groups.

Since Gromov’s pioneering work, others have investigated the notion of “large-scale” or “coarse” geometry. In particular, A. Dranishnikov and G. Bell have done much work to examine and extend the notion (finite) asymptotic dimension (FAD) ([3],[4],[5],[15],[16]). E. Guentner, R. Tessera, and G. Yu have worked extensively on the notion of (finite) decomposition complexity (FDC) ([24],[25]). Dranishnikov and M. Zarichnyi have introduced and study a generalization of FDC, known as straight finite decomposition complexity (sFDC) ([18]). Yu further invented the notions of asymptotic property A (APA) and coarse embeddability into Hilbert space ([39]), and M. Dadarlat and E. Guentner crafted “exactness” as a comparable (and perhaps more generative) property to APA ([14]).

Meanwhile, a common object of study in algebraic K-theory is that of assembly maps,
and there are a number of conjectures (Borel, Farrell-Jones, Novikov, etc.) that are concerned with determining when such maps constitute equivalences. G. Carlsson and B. Goldfarb have devoted much time to crafting and investigating conditions on groups and metric spaces that, once ascertained, yield that the group or space satisfies the Novikov or Borel conjectures. In particular in [9], they introduced the notion of “weak coherence,” which functions as an analogue to Waldhausen’s notion of coherence [38]. Carlsson and Goldfarb have since demonstrated that groups satisfying FAD or sFDC conditions are weakly coherent, and that (in many instances) weakly coherent groups for which the Novikov conjecture holds are groups for which the Borel conjecture necessarily holds.

This dissertation is a merging of these two veins of study. It is devoted to investigating how these coarse geometric notions interact with various coherence properties. Three coherence properties are described: coherence, coarse coherence, and relative coarse coherence. Each will be defined in Section 2.3. The results discussed will generally be statements of the form “If a metric space $X$ possesses property $\mathcal{P}$, then $X$ is coherent/coarsely coherent/relatively coarsely coherent.” While the assorted properties $\mathcal{P}$ will not defined until Section 2.2, several significant results are previewed below.

**Theorem 1** (108) ————

*Coarse coherence satisfies fibering permanence.*

———

**Theorem 2** (141) ————

*Coarse coherence satisfies coarse permanence, finite amalgamation permanence, finite union permanence, union permanence, and limit permanence.*

———
Theorem 3 (143)

The class of (countable) groups with coarse coherence is closed under extensions, direct unions, free products (with amalgam), and relative hyperbolicity. Furthermore, all elementary amenable, all linear, and all subgroups of virtually connected Lie groups are coarsely coherent.

Theorem 4 (145)

If $X$ has finite asymptotic dimension, then $X$ is coherent, coarsely coherent, and relatively coarsely coherent.

Theorem 5 (154,149)

If $X$ has straight finite decomposition complexity, then $X$ is (coarsely) coherent.

The following table summarizes the results presented in this dissertation.

<table>
<thead>
<tr>
<th>Permanence</th>
<th>Geometric Coherence</th>
<th>Coarse Geometric Coherence</th>
<th>Relative Coarse Coherence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse Invariance</td>
<td>✓ [98]</td>
<td>✓ [102]</td>
<td>✓ [115]</td>
</tr>
<tr>
<td>Subspace Perm.</td>
<td>✓ [97]</td>
<td>✓ [104]</td>
<td>✓ [117]</td>
</tr>
<tr>
<td>Union Permanence</td>
<td>✓ [99]</td>
<td>✓ [105]</td>
<td>✓ [119]</td>
</tr>
<tr>
<td>Fibering Permanence</td>
<td>✓ [107]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Limit Permanence</td>
<td>✓ [141]</td>
<td></td>
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</tr>
<tr>
<td>Direct Product Perm.</td>
<td>✓ [109]</td>
<td></td>
<td>✓ [123]</td>
</tr>
<tr>
<td>Free Product Perm.</td>
<td>✓ [141]</td>
<td></td>
<td></td>
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<tr>
<td>Group Extension Perm.</td>
<td>✓ [139]</td>
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</table>
This dissertation is intended to be reasonably self-contained. Thus, many elementary definitions and concepts are included. In **Chapter 2, Section 2.1** is entirely devoted to basic terminology regarding metric spaces, modules, and geometric group theory. **Section 2.2** introduces coarse geometry of metric spaces, including key coarse invariants like asymptotic dimension, decomposition complexity, and asymptotic property A. **Section 2.3** contains the first original content in the thesis, specifically the three coherence notions that this dissertation engages. There is a subsection pertaining to a category informed by these notions.

**Chapter 3** consists of arguments about maps between coherent (coarsely coherent, relatively coarsely coherent) metric spaces (**Section 3.1**), as well as determining some of the coarse geometric properties possessed by coherent (**Section 3.2**), coarsely coherent (**Section 3.3**), and relatively coarsely coherent (**Section 3.4**) spaces.

**Chapter 4** compares the coherence notions to one another, as well as with other significant coarse invariants. **Section 4.1** focuses on the former. In **Subsection 4.1.1**, it is
determined that
\[
\text{relative coarse coherence} \implies \text{coherence} \implies \text{coarse coherence},
\]
the arguments for which provide a method to calculate appropriate necessary constants. For example, the argument that relative coarse coherence implies coherence yields that the leanness constant required for coherence is obtained in a particular way from the $\pi$-scattering constant required for relative coarse coherence. **Subsection 4.1.2** includes arguments that the real numbers $\mathbb{R}$ are relatively coarsely coherent and coherent (and thus coarsely coherent). The two parallel proofs illustrate the many computational similarities found when working with these properties, while also allowing us to tease out the discrepancies that make one preferable to the other in various circumstances. **Subsection 4.1.3** brings in the recent works of other mathematicians and applies them to the coherence notions, demonstrating additional permanence properties that these notions possess.

**Section 4.2** is devoted to relating these coherence notions to other coarse geometric invariants that have been developed toward similar ends. **Subsection 4.2.1** demonstrates that finite asymptotic dimensional spaces are coherent (coarsely coherent, relatively coarsely coherent). **Subsection 4.2.2** illustrates similar results for spaces with finite decomposition complexity. **Subsection 4.2.3** differs from the preceding two; it demonstrates by example that coarse coherence does not imply asymptotic property A.

Finally, **Chapter 5** concludes the dissertation. **Section 5.1** is comprised of a discussion of what the thesis has accomplished, as well as which issues remain unresolved. **Section**
5.2 considers the questions raised by the preceding section, as well as ongoing work in the field, and suggests future avenues of inquiry that may be of interest in addressing these issues.
Chapter 2

Background and Preliminaries

2.1 Basic Terminology and Concepts

In the discussion that follows, we will work primarily with metric spaces, modules, and coarse geometric notions. It is therefore necessary to first recall some basic definitions from algebra and topology.

2.1.1 Metric Spaces

Much of the information that follows can be found in James R. Munkres seminal text [33].

A metric space \((X, d_X)\) is a set \(X\) equipped with a function \(d_X : X \times X \rightarrow [0, \infty)\), known as a metric, such that

1. \(d_X(x, x') = 0 \iff x = x'\),

2. \(d_X(x, x') = d_X(x', x)\), and

3. \(d_X(x, x') \leq d_X(x, x'') + d_X(x'', x')\)
for all $x, x', x'' \in X$. We refer to $A \subseteq X$ equipped with the metric $d_A$ as a **metric subspace** $(A, d_A)$ of $(X, d_X)$. The metric $d_A := d_X|_{A \times A}$ is the **subspace metric** or the **inherited metric**. The **direct product** of two metric spaces $(X, d_X)$ and $(Z, d_Z)$, $(X \times Z, d_{X \times Z})$, consists of the set

$$X \times Z := \{(x, z): x \in X, z \in Z\}$$

and equipped with the **max metric**, defined by

$$d_{\text{max}}((x, z), (x', z')) := \max\{d_X(x, x'), d_Z(z, z')\}$$

for all $x, x' \in X$, $z, z' \in Z$.

A **disjoint union** of two metric spaces $(X, d_X)$, $(Z, d_Z)$, is denoted $(X \sqcup Z, d_{X \sqcup Z})$, where

$$X \sqcup Z := \{(w, X): w \in X\} \cup \{(w, Z): w \in Z\}$$

and $d_{X \sqcup Z}$ is any metric on $X \sqcup Z$. In other words, the disjoint union is a union in which the elements are indexed to indicate from which of the original spaces the element of the union came. In an abuse of notation, we often write simply $w \in X \sqcup Z$, without the index, knowing that we have kept distinct any elements common to both $X$ and $Z$. We will later
consider $X \sqcup Z$ equipped with the metric

$$d_{X \sqcup Z}(w, w') = \begin{cases} 
  d_X(w, w') & \text{if } w, w' \text{ both from } X \\
  d_Z(w, w') & \text{if } w, w' \text{ both from } Z \\
  \infty & \text{else.}
\end{cases}$$

We refer to $X \sqcup Z$ with the above metric as the coarse disjoint union of $X$ and $Z$, as per the conventions in [23].

Henceforth, when the context makes the metric under consideration clear, any metric may simply be denoted $d$, without a subscript to denote the space with which it is paired.

**Hilbert Space**

The discussion that follows is informed by Peter D. Lax’s foundational text [30].

Let $X$ be a metric space, and let $x_1, x_2, \ldots \in X$ be a sequence in $X$ (that is, define a function from $\mathbb{Z}_+$ to $X$ so that the image of 1 is $x_1$, the image of 2 is $x_2$, etc.). A sequence $\{x_i\}_{i \in \mathbb{Z}_+}$ converges to a point $x$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}_+$ such that for all $n \geq N$, $d(x_n, x) < \epsilon$. Such sequences are convergent sequences. A sequence is Cauchy if for any $\epsilon > 0$ there exists an $N \in \mathbb{Z}_+$ so that for all $n, m \geq N$, $d(x_n, x_m) < \epsilon$. A Cauchy sequence need not converge to a point, but a sequence that converges to a point is clearly Cauchy. Metric spaces in which every Cauchy sequence is convergent with respect to the given metric on that space are complete or Cauchy metric spaces (with respect to that metric).

One of the properties of coarse geometric spaces we will ultimately consider relies on
the class of metric space known as Hilbert space. To discuss Hilbert space, we must first understand the notion of an inner product.

An inner product on a real (complex) vector space $V$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ ($\mathbb{C}$) satisfying for all $u, v, w \in V$ and all $\alpha \in \mathbb{R}$ ($\mathbb{C}$),

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,

2. $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$,

3. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ ($\langle v, w \rangle = \langle w, v \rangle$),

A vector space together with an inner product is an inner product space.

Inner product spaces are metric spaces, with a metric given by

$$d(v, w) = \langle v - w, v - w \rangle.$$ 

If an inner product space is complete with respect to the metric given by the inner product, it is a Hilbert space.

---

**Example 6**

The space $\ell^2(X)$ of a countable set $X$ is a Hilbert space given by

$$\ell^2(X) = \{ f : X \to \mathbb{R} : \sum_{x \in X} |f(x)|^2 < \infty \}.$$ 

The metric on $\ell^2(X)$ is given by

$$d(f, g) = \|f - g\|_2$$
where $\|h\|_2$ is defined as
\[
\|h\|_2 = \left(\sum_{x \in X} |h(x)|^2\right)^{\frac{1}{2}}.
\]
When $X = \{1, 2, ..., n\}$, $\ell_2(X)$ is $\mathbb{R}^n$ with the Euclidean metric.

The real Hilbert space, which we will simply refer to as “the” Hilbert space, is $\ell^2(\mathbb{N})$, the collection of square-summable sequences of real numbers. When we later define coarse embeddability into Hilbert space, it is to this Hilbert space that we refer.

Metric Families

In this section, many of the definitions are those given by Guentner in [23]. At the end of this chapter and in the chapters that follow, several permanence properties for coarse geometric spaces will be stated in the language used in [23]. Therefore, his terminology will be utilized from the outset to avoid confusion.

A metric family $\mathcal{X}$ is a set of metric spaces $\{X_1, X_2, X_3, \ldots\}$. One can place conditions on metric spaces and on metric families. For example, a metric space is bounded if there exists $M > 0$ such that $d(x, x') \leq M$ for all $x, x' \in X$. A metric family $\mathcal{X}$ is bounded if for each $X_i \in \mathcal{X}$, there is an $M_i > 0$ such that $d_i(x_i, x'_i) \leq M$ for all $x_i, x'_i \in X_i$. A metric family is uniformly bounded if there exists an $M > 0$ such that for all $X_i \in \mathcal{X}$, $d(x_i, x'_i) \leq M$ for all $x_i, x'_i \in X_i$.

Let $\{X_\alpha\}_{\alpha \in A}$, $\{Y_\beta\}_{\beta \in B}$ be metric families. A function of families $f$ is a collection of functions $\{f_\gamma\}_{\gamma \in \mathcal{G}}$ with an associated structure map $\sigma : G \to A \times B$, $\sigma(\gamma) = (\alpha(\gamma), \beta(\gamma))$. 

11
satisfying

\[ f_\gamma : X_{\alpha(\gamma)} \to Y_{\beta(\gamma)}, \]

where the projection of the structure map onto \( A \) is onto (that is, for all \( \alpha \in A \) there exists at least one \( \gamma \in C \) so that \( X_{\alpha} \) is the domain of \( f_\gamma \)). If \( \mathcal{X} := \{ X_{\alpha} \} \) and \( \mathcal{Y} := \{ Y_{\beta} \} \), we write \( f : \mathcal{X} \to \mathcal{Y} \). When composing functions of families \( f : \mathcal{Y} \to \mathcal{Z} \) and \( g : \mathcal{X} \to \mathcal{Y} \) to form \( f \circ g \), we assume the indexing set (say \( C \)) is the index set for both \( f \) and \( g \) and for every index \( \gamma \in C \), the domain of \( f_\gamma \) and the range of \( g_\gamma \) agree. The composite collection is then \( \{ f_\gamma \circ g_\gamma \} \).

Let \( \mathcal{X} = \{ X_{\alpha} \} \) (a convention used frequently in this paper) be a metric family and let \( \sigma : \gamma \mapsto \alpha(\gamma) \) be a structure map. The disjoint union

\[ X = \bigsqcup_{\gamma} X_{\alpha(\gamma)} = \{(x, \gamma) : x \in X_{\alpha(\gamma)}\} \]

can be equipped with the familiar metric

\[ d_X(x_\gamma, y_{\gamma'}) = \begin{cases} 
    d_{X_{\alpha(\gamma)}}(x, y) & \gamma = \gamma' \\
    \infty & \text{else,}
\end{cases} \]

where \( x_\gamma = (x, \gamma) \) when \( x \in X_{\alpha(\gamma)} \). \((X, d_\sqcup)\) is known as the total space of \( \mathcal{X} \) with respect to the structure map \( \sigma \).

When the structure map \( \sigma \) is the identity function\( (\sigma(\gamma) = \gamma) \), \( \sigma \) is the standard structure map, and \( X \) defined above is then the standard total space.

Let \( \{ f_\gamma \} : \mathcal{X} \to \mathcal{Y} \) be a family of functions, let \( X \) the total space of \( \mathcal{X} \) with respect to
some given structure map, and $Y$ the total space of $\mathcal{Y}$ with respect to some structure map. $f(x, \gamma) := (f_\gamma(x), \gamma)$ defines a function between the total spaces $f : X \to Y$, see [23]. We will make use of this construction later.

**Additional Key Terms**

The **diameter** of a subset $Y$ of a metric space $(X, d)$ is

$$\text{diam}(Y) := \sup_{y, y' \in Y} \{d(y, y')\}.$$ 

A **partition of unity** of a space $X$ is a set $R$ of continuous functions from $X$ to $[0, 1]$ such that for every $x \in X$, there is a $C$-neighborhood

$$x[C] := \{y \in X : d(x, y) < C\}$$

for some $C > 0$ where all but a finite number of the functions of $R$ are zero, and the sum of the function values at $x$ is one; that is,

$$\sum_{\rho \in R} \rho(x) = 1.$$ 

### 2.1.2 Modules

In this section, the necessary background on groups, rings, and modules is supplied. Much of the content is found in Thomas W. Hungerford’s popular text [26].

Let $R$ be a ring. A **left $R$-module** $M$ is an abelian group $(M, +)$ and an operation
\[ R \times M \to M, \text{ often called scalar multiplication, such that for all } r, s \in R \text{ and } m, n \in M \]

1. \[ r \cdot (m + n) = r \cdot m + r \cdot n, \]

2. \[ (r + s) \cdot m = r \cdot m + s \cdot m, \]

3. \[ (r \cdot s) \cdot m = r \cdot (s \cdot m). \]

A **right** \( R \)-module \( M \) is defined similarly with an operation \( \cdot : M \times R \to M \). An \( R \)-**bimodule** \( M \) is a module \( M \) that is both a left and right \( R \) module such that the scalar multiplications commute. If \( R \) is commutative, then there is no distinction between left and right \( R \)-modules and they are termed simply **\( R \)-modules**. If \( R \) is a field, \( R \)-modules are known as **vector spaces**. An **\( R \)-submodule** \( N \) of \( M \) is a subgroup of \( M \) that is closed under the scalar multiplication.

---

**Example 7**

\( \mathbb{R} \) is a ring with the usual addition and multiplication. \( \mathbb{R} \) is, in fact, a field. \( \mathbb{R} \)-modules are known as **real vector spaces**. Similarly, \( \mathbb{C} \) (the complex numbers), form a ring that is a field, and \( \mathbb{C} \)-modules are known as **complex vector spaces**. \( \mathbb{R} \) is a subring (actually, a subfield) of \( \mathbb{C} \).

---

A **group homomorphism** is a function \( f : G \to H, (G, \ast) \text{ and } (H, \cdot) \) groups, satisfying

\[ f(g \ast g') = f(g) \cdot f(g'). \]

A group homomorphism is **injective** if its **kernel** (the elements of the domain which are sent to the identity in the codomain) consists only of the identity element of the domain.
Such a group homomorphism may be termed a **monomorphism**. A group homomorphism is **surjective** if its **image** (the elements of the codomain to which the elements of the domain are sent) consists of all elements of the codomain. Such a homomorphism may be termed an **epimorphism**. A homomorphism that is both injective and surjective is an **isomorphism**. If there is an isomorphism between two groups, those groups are termed **isomorphic**. The relation “is isomorphic to” is an equivalence relation, and we often view isomorphic groups as interchangeable with one another. An **equivalence relation** is a rule, in the case of isomorphisms denoted \( \cong \), that associates (in this case two groups) \( G \) to \( H \) (written \( G \cong H \)) if and only if \( G \) is isomorphic to \( H \). Equivalence relations are

- **Reflexive**: \( G \cong G \)
- **Symmetric**: \( G \cong H \iff H \cong G \)
- **Transitive**: \( G \cong H \) and \( H \cong F \) implies \( G \cong F \)

A **sequence** is a series of groups and group homomorphisms such that the domain of each function is the codomain of the preceding function:

\[
\ldots \rightarrow G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} G_{i+2} \rightarrow \ldots
\]

A sequence is **exact** if \( \text{im} f_k = \ker f_{k+1} \) for all \( k \). A **short exact sequence** is a five group sequence of the form

\[
\{0\} \rightarrow G' \xrightarrow{f} G \xrightarrow{\phi} G'' \rightarrow \{0\}
\]
that is exact \((\text{im } f = \ker g)\). In this situation, \(f\) is a monomorphism and \(g\) is an epimorphism. We will generally refer to the group consisting of one element \(\{0\}\) as simply 0, and use context to distinguish it from the additive identity of a ring.

Projective resolutions are another type of exact sequence, and will be of interest later. Let \(R\) be a ring. An \(R\)-module \(P\) is **projective** if for any surjective homomorphism \(\phi : N \rightarrow M\) between two \(R\)-modules, if there is a homomorphism \(\psi : P \rightarrow M\), then there also exists a homomorphism \(\psi' : P \rightarrow N\) such that the following diagram commutes.

\[
\begin{array}{ccc}
P & \xrightarrow{\exists \psi'} & P \\
\downarrow{\psi} & & \downarrow{\psi} \\
N & \xrightarrow{\phi} & M
\end{array}
\]

An \(R\)-module \(M\) has a **projective resolution** if there exists a surjective homomorphism \(\epsilon : P_0 \rightarrow M\) and an exact sequence of projective modules

\[
... \rightarrow P_{n+1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow ... \rightarrow P_2 \rightarrow P_1 \rightarrow P_0
\]

such that

\[
... \rightarrow P_{n+1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow ... \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{\epsilon} M \rightarrow 0
\]

is an exact sequence. This sequence is **finite** if there exists an \(N \geq 0\) such that for all \(i > N\), \(P_i = 0\). Note that if \(N = 0\), the resolution is of the form

\[
0 \rightarrow P_0 \xrightarrow{\epsilon} M \rightarrow 0.
\]
Since the sequence is exact, \( \epsilon \) is injective and surjective, so \( P_0 \cong M \) and \( M \) is itself a projective \( R \)-module.

### 2.1.3 Geometric Group Theory

The definitions in this section may all be found in John Meier’s fundamental text [32]

A **graph** is constituted by two associated sets: the **vertices**, which consists of an assortment of points; and the **edges**, which consists of an assortment of pairs where each element of the pair is an element of the set of vertices.

A group \( G \) is **finitely generated** if every element of \( G \) is a product of elements from a finite set \( S \subseteq G \). The set \( S \) is said to be **symmetric** if whenever \( s \in S \), \( s^{-1} \) is in \( S \).

The **Cayley graph** of a group \( G \) finitely generated by a symmetric generating set \( S \) is the graph whose vertices are the elements of \( G \), and the vertices \( g, h \) are connected by an edge if \( g^{-1} h \in S \). The Cayley graph of a group \( G \) may be denoted \( \Gamma(G) \).

---

**Example 8**

The Cayley graph of \( \mathbb{Z} \) with respect to \( S = \{ \pm 1 \} \) is

```
... -4 -3 -2 -1 0 1 2 3 4 ...
```

---

A **geodesic** in a metric space \((X, d)\) is the image of a function \( \gamma : [0, d(x, x')] \to X \) such that \( \gamma(0) = x, \gamma(d(x, x')) = x' \), and

\[
d(x_1, x_2) = d(\gamma(x_1), \gamma(x_2))
\]
for all $x_1, x_2 \in \text{im} \gamma$. Such a distance-preserving function (a function satisfying the equation above) is an isometry. A metric space $X$ is geodesic if for all $x, x' \in X$, there exists a geodesic $\gamma$ joining them. A quasi-geodesic in a metric space $(X, d)$ is the image of a function $\gamma' : [0, d(x, x')] \to X$ satisfying $\gamma'(0) = x, \gamma'(d(x, x')) = 0$, and

$$\frac{1}{L}d(x, x') - C \leq d(\gamma'(x), \gamma'(x')) \leq Ld(x, x') + C$$

for some $L, C \geq 0$. A function satisfying the equation above is an $(L, C)$-quasi-isometry (this definition will be revisited in greater generality later). A metric space $X$ is quasi-geodesic if there exists $L, C \geq 0$ such that for all $x, x' \in X$, there exists a quasi-geodesic $\gamma'$ that is an $(L, C)$-quasi-isometry joining them.

A graph $T$ is a (metric) tree if between any two vertices there is exactly one geodesic joining them. The metric on a metric tree is the path metric, and is defined by

$$d(v, v') = \inf\{|\gamma| : \gamma : [0, 1] \to X \text{ continuous}, \ \gamma(0) = v, \gamma(1) = v'\}.$$

The $\gamma$ above are paths, and $|\gamma|$ is the number of edges traversed by $\gamma$ as a path from $v$ to $v'$.

---

Example 9

The Cayley graph of any finitely generated free group (a finitely generated group with no relations, denoted $\mathbb{F}_n$) is a tree for some choice of generators, and any Cayley graph of a finitely generated free group is quasi-isomorphic to a tree. $\mathbb{Z}$ is the free group generated by one element.
By viewing groups as graphs and then metric spaces, we are able to use our knowledge of metric spaces to manipulate and make statements about groups. To this end, and in particular to make use of a theorem of Dranishnikov [16] in a subsequent chapter, we supply the following definition.

Trees, as we have defined them, are simplicial: they consist of a set $\Delta \subseteq P(G)$ (where $P(G)$ is the power set of $G$) that is closed under the taking of subsets. That is, if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. For trees, the elements of $\Delta$ would be the collection of edges and of vertices.

---

**Example 10**

A very simple example would be the tree describing $\mathbb{Z}/2\mathbb{Z}$.

```
0 --- 1
```

The set $E$ of edges just consists of $e = \{0, 1\}$, which would then be in $\Delta$, and both 0, 1 in $e$ implies 0, 1 in $\Delta$, yielding, for this graph $\Delta = \{\{0\}, \{1\}, \{0, 1\}\}$.

---

A simplicial tree is **locally finite** if every vertex is an element of finitely many edges. Alternatively, their exists a neighborhood of every vertex that intersects finitely many edges.

Combining our knowledge of groups with our knowledge of metric spaces, we can consider the notion of **group actions** on metric spaces, as follows.

Let $G$ be a group and $X$ a metric space. A **(left) $G$-action on $X$** is a function $\psi : \Delta \subseteq P(G)$ (where $P(G)$ is the power set of $G$) that is closed under the taking of subsets. That is, if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. For trees, the elements of $\Delta$ would be the collection of edges and of vertices.
$G \times X \to X$ satisfying for all $g, g' \in G$ and all $x \in X$,

$$
\psi(e, x) = x \quad \text{and} \quad \psi(g, \psi(g', x)) = \psi(gg', x),
$$

where $e$ is the identity element in $G$. Often, rather than writing the group action as a function $\psi$ evaluated at $(g, x)$, we will simply write $gx$, where it is clear that $g$ is the group element and $x$ is the metric space element.

A group $G$ acts **isometrically** on a metric space $X$ if

$$
d_X(x, x') = d_X(gx, gx')
$$

for all $x, x' \in X$ and for all $g \in G$. That is, the $G$-action preserves distances: points in the metric space that are a given distance from one another are translated by the $G$-action to points equally as far away from each other.

The **$G$-orbit of $x$ in $X$** is given by

$$
G(x) = \{gx|g \in G\}.
$$

We may sometimes simply write $Gx$, without the parentheses. The **stabilizer of $x$**, **isotropy group of $x$**, or **subgroup fixing $x$ in $G$** is given by

$$
G_x = \{g|gx = x\}.
$$

Orbits give rise an **equivalence relation**. That is, if we define a rule $\sim$ that associates $x$ to
Reflexive: \( x \sim x \)

Symmetric: \( x \sim x' \Leftrightarrow x' \sim x \)

Transitive: \( x \sim x' \) and \( x' \sim x'' \) implies \( x \sim x'' \)

Equivalence relations partition groups and spaces. That is, any element of \( x \) belongs to exactly one orbit - there are no nonempty intersections of orbits. These non-intersecting components of the partitions are known as equivalence classes, and, at times, when we are thinking of \( G(x) \) not only as the orbit of \( x \), but as an equivalence class, it may be denoted instead as \( \overline{x} \) or as \([x]\), which are common notations for equivalence classes. We write the collection of these equivalence classes resulting from the group action of \( G \) on \( X \) as \( G\backslash X \).

We define a map \( q \), the (canonical) quotient map, that sends \( x \in X \) to its orbit or equivalence class \( Gx \in G\backslash X \) under the \( G \)-action. If \( F \) is a finite group that acts on \( X \) isometrically, \( F\backslash X \) is a metric space with metric

\[
d_{F\backslash X}(Fx, Fx') := \min_{g \in F} d_X(x, gx').
\]

A function \( \psi : X \to Y \) between two metric spaces that are both acted upon by a group \( G \) is \( G \)-equivariant if for all \( x, x' \in X \) and for all \( g \in G \),

\[
g\psi(x) = \psi(gx).
\]
2.2 Coarse Geometric Properties

The key principle underlying large-scale or coarse geometry is to disregard all local or infinitesimal structure (topological or geometric) and consider only geometric properties that are global. The content that comprises this section is found in Piotr W. Nowak and Guoliang Yu's excellent introduction to the subject [35].

2.2.1 Coarse Geometry

A metric space $X$ with a given metric $d$ is a discrete metric space if there exists a constant $C$ such that for all $x, x' \in X$, $d(x, x') \geq C$. A $C$-net of a metric space $X$ is a discrete subset $A \subseteq X$ such that for every $x \in X$ there exists a point $a \in A$ with $d(x, a) \leq C$. When the particular value for $C$ is irrelevant, we say that $A$ is a net. Further, functions $f, g : X \rightarrow Y$, $X, Y$ both metric spaces, are $C$-close if for all $x \in X$, $d_X(f(x), g(x)) \leq C$. Functions of families $f, g : \mathcal{X} \rightarrow \mathcal{Y}$ are $C$-close if for all indexes $\gamma$, $f_\gamma, g_\gamma$ have the same domain and satisfy the definition of $C$-close.

Definition 11 A quasi-isometry is a map $\phi : X \rightarrow Y$ between metric spaces such that there exist constants $L, C > 0$ satisfying the following:

$$\frac{1}{L}d_X(x, x') - C \leq d_Y(\phi(x), \phi(x')) \leq Ld_X(x, x') + C$$

and the image $\phi(X)$ is a net in $Y$. Without the latter condition, we say that $\phi$ is a quasi-isometric embedding. Two metric spaces are quasi-isometric if there is a quasi-isometry between them.
Definition 12 A map \( f : X \to Y \) is a coarse equivalence if there are non-decreasing functions \( \rho_- \), \( \rho_+ : [0, \infty) \to [0, \infty) \) such that \( \rho_-(t) \to \infty \) as \( t \to \infty \) and for all \( x, x' \in X \),

\[
\rho_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x'))
\]

and if \( f(X) \) is a net in \( Y \). Without the latter condition, we say \( f \) is a coarse embedding. Two metric spaces are coarsely equivalent if there is a coarse equivalence between them.

Definition 13 Two metric families \( X \) and \( Y \) are coarsely equivalent as families if there is a function of families \( f : X \to Y \) that is a coarse embedding of families and is coarsely onto as a function of families. \( f \) is then termed a coarse equivalence of families.

Analogously with functions of metric spaces, \( f \) a coarse equivalence of families implies that \( f \) has at least one coarse inverse \( g \).

Example 14 Quasi-isometries are coarse equivalences where the \( \rho_- \), \( \rho_+ \) are linear. Quasi-isometric spaces are therefore coarsely equivalent.

As a result of the definitions of quasi-isometry and coarse equivalence, coarse geometry is generally determined up to nets in metric spaces. Therefore, we frequently assume that the spaces we are working with are discrete metric spaces, knowing that the results will hold for all quasi-isometric or coarsely equivalent spaces.

A function satisfying the "latter condition" referred to in the definitions of quasi-isometry and coarse equivalence is termed coarsely onto. Formally, if \( f : X \to Y \) is a
function of metric spaces and \( f(X) \) is a net in \( Y \), we say that \( f \) is **coarsely onto**. A function of families \( f : X \to Y \) is **coarsely onto** if for all \( Y_\beta \in \mathcal{Y} \) there is a \( \gamma \) so that \( f_\gamma : X_\alpha \to Y_\beta \) for some \( X_\alpha \in \alpha \) and \( f_\gamma(X_\alpha) \) is a net in \( Y_\beta \).

We will refer to the functions \( \rho_-, \rho_+ \) as **control functions**.

**Definition 15** A map \( \phi : X \to Y \) between two metric spaces is **uniformly expansive** or **coarse** if there exists a non-decreasing function \( \rho_+ : [0, \infty) \to [0, \infty) \) such that for every \( x, x' \in X \),

\[
d(\phi(x), \phi(x')) \leq \rho_+(d(x, x')).
\]

In other words, the right-hand inequality from the definition of coarse equivalence holds.

A function of families \( f : X \to Y \) is **uniformly expansive** or **coarse** if each \( f_\gamma \in f \) is uniformly expansive with respect to the same control function \( \rho_+ \).

**Example 16** (Kasprowski-Nicas-Rosenthal, [27])

Let \( X \) be a metric space and let \( F \) be a finite group that acts on \( X \) isometrically. Then the canonical quotient map \( q : X \to F\backslash X \) is a uniformly expansive map with \( \rho_+ = \text{id}_{[0,\infty)} \).

Further, a \( F \)-equivariant coarse map \( \psi : X \to Y \) with control function \( \rho_+ \) induces a coarse map \( \overline{\psi} : F\backslash X \to F\backslash Y \) that also has control function \( \rho_+ \) and makes the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{\psi} & Y \\
\downarrow{q_X} & & \downarrow{q_Y} \\
F\backslash X & \xrightarrow{\overline{\psi}} & F\backslash Y
\end{array}
\]

A fact we will make use of later is that a function \( f : X \to Y \) is a coarse equivalence if and only if \( f \) is uniformly expansive and there exists some \( C > 0 \) and a function \( g : Y \to X \)
that is uniformly expansive, known as a coarse inverse of $f$, satisfying that $f \circ g$ is $C$-close to the identity on $Y$ and $g \circ f$ is $C$-close to the identity on $X$. If a function of families $f : X \to Y$ is uniformly expansive and there exists some $C > 0$ and a function of families $g : Y \to X$ that is uniformly expansive satisfying that $f \circ g$ is $C$-close to the identity on $Y$ and $g \circ f$ is $C$-close to the identity on $X$, we say $g$ is a coarse inverse of $f$ as a function of families.

**Definition 17** A map $\phi : X \to Y$ is effectively proper if there exists a non-decreasing function $\rho_- : [0, \infty) \to [0, \infty)$ satisfying $\rho_-(t) \xrightarrow{t \to \infty} \infty$ such that for every $x, x' \in X$,

$$\rho_-(d(x, x')) \leq d(\phi(x), \phi(x')).$$

That is, the left-hand inequality from the definition of coarse equivalence holds.

A function of families $f : X \to Y$ is effectively proper if each $f_\gamma \in f$ is effectively proper with respect to the same control function $\rho_-$. 

It is clear from the preceding definitions that $\phi : X \to Y$ is both uniformly expansive and effectively proper if and only if $\phi$ is a coarse embedding. $\phi$ is a coarse equivalence if and only if $\phi$ is uniformly expansive, effectively proper, and coarsely onto.

Two facts regarding any uniformly expansive map $\phi$ or effectively proper map $\psi$ with respective control functions both $\rho$ that appear often in the arguments that follow are that for any subset $T$ of the domain of $\phi, \psi$ and for any subset $U$ of the codomain of $\phi, \psi$,

$$\phi(T[b]) \subseteq \phi(T)[\rho(b)] \quad \phi^{-1}(U)[b] \subseteq \phi^{-1}(U[\rho(b)])$$
and

\[ \psi(T)[b] \subseteq \psi(T[\rho(b)]) \quad \psi^{-1}(T[b]) \subseteq \psi^{-1}(T)[\rho(b)]. \]

### 2.2.2 Key Coarse Invariants

#### Finite Asymptotic Dimension

There are several equivalent definitions of asymptotic dimension. The two we will find most useful are given below. First,

**Definition 18** A collection of subsets of a metric space \( X \) are \( R \)-bounded if each subset is contained in a ball of radius \( R \).

**Definition 19** A collection of subsets of a metric space \( X \) are \( d \)-disjoint if the infimum of the distance between any pair of points from different subsets is at least \( d \).

**Definition 20** A metric space \( X \) has **finite asymptotic dimension** if for every \( R > 0 \) there exists a family \( \{U_i\}_{i=0}^n \) and a \( d > 0 \) such that the \( U_i \) are \( R \)-bounded, \( d \)-disjoint, and cover \( X \). The least such \( n \) for which this definition holds is the **asymptotic dimension** of \( X \). If there are no \( n \) satisfying this definition, \( X \) has **infinite asymptotic dimension**.

Alternatively, we describe asymptotic dimension as follows:

**Definition 21** A cover \( \mathcal{U} = \{U_\alpha\}_{\alpha \in A} \) of a metric space \( X \) has **\( d \)-multiplicity** \( n \) if \( n \) is the least such integer so that for every \( x \in X \) the ball of radius \( d \) centered at \( x \) intersects at most \( n \) of the \( U_\alpha \).

**Definition 22** A metric space \( X \) has **finite asymptotic dimension** if there exists an \( n \) such that for every \( 0 < d < \infty \) there exists a uniformly bounded cover \( \mathcal{U} \) of \( X \) with \( d \)-
multiplicity $n+1$. The least such $n$ is the **asymptotic dimension** of $X$. If there are no such $n$, $X$ has **infinite asymptotic dimension**.

---

**Example 23**

All metric spaces $X$ with bounded diameter have asymptotic dimension 0. However, the converse does not hold. The space $X = \{n^4 : n \in \mathbb{N}\}$, which is not of bounded diameter, also has asymptotic dimension 0.

---

**Example 24** (Nowak-Yu [35])

Trees, and therefore finitely generated free groups, have finite asymptotic dimension. Trees have asymptotic dimension 1. Specifically $\mathbb{Z}$ has asymptotic dimension 1.

---

**Example 25** (Nowak-Yu [35])

$\mathbb{R}^n$ has asymptotic dimension $n$ for all finite $n > 0$, $n \in \mathbb{Z}$.

---

**Example 26** (Nowak-Yu [35])

$\mathbb{Z}^{(\infty)} = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ has infinite asymptotic dimension since (1) the inclusion of each $\mathbb{Z}^n$ is a quasi-isometry, (2) finite asymptotic dimension is a large-scale geometry invariant, and (3) subspaces inherit asymptotic dimension. Therefore, the asymptotic dimension of $\mathbb{Z}^{(\infty)}$ is at least that of $\mathbb{Z}^n$ for all $n \in \mathbb{N}$, and is thus infinite.
Finite Decomposition Complexity and sFDC

The following definitions, culminating in a definition of (finite) decomposition complexity, are adapted from [35].

**Definition 27** Let $R > 0$. A family of metric spaces $\mathcal{X}$ is \textit{$R$-decomposable} over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits a decomposition

$$X = X_0 \cup X_1,$$

such that

$$X_i = \prod_{i=0,1} Y_{ij},$$

where $Y_{ij} \in \mathcal{Y}$ and each $\{Y_{ij}\}_{j \in \mathbb{N}}$ is an $R$-disjoint family for $i = 0, 1$.

When we want to discuss a specific metric space $X$, we define $\mathcal{X} = \{X\}$, the metric family with one element.

**Definition 28** Consider a ‘game’ in which a challenger and a defender take turns choosing integers $R_i$ and $R_i$-decomposing a metric family $\mathcal{Y}_i$, respectively.

- **Round 1** Challenger gives Defender an integer $R_1$. The first round ends when Defender $R_1$-decomposes the family $\mathcal{Y}_0$ over a family $\mathcal{Y}_1$.

- **Round $k$** Challenger gives Defender an integer $R_k$. The $k^{th}$ round ends when Defender $R_k$-decomposes $\mathcal{Y}_{k-1}$ over $\mathcal{Y}_k$.

A collection of metric families $\mathcal{Y} = \{\mathcal{Y}_i\}$ has \textit{finite decomposition complexity} if the defender can always decompose $\mathcal{Y}_i$ over a \textit{bounded family} (all members are bounded in
size by a common bound) in finitely many rounds.

**Definition 29** A metric family $\mathcal{X}$ has **finite decomposition complexity** (FDC) if Defender has a winning strategy in the decomposition game with $\mathcal{Y}_0 = \mathcal{X}$. A metric space $X$ has **finite decomposition complexity** if the family $\mathcal{X} = \{X\}$ has finite decomposition complexity.

---

**Example 30 (Nowak-Yu [35])** $\mathbb{Z}^{(\infty)}$ has finite decomposition complexity. Let Challenger give a number $R_0$. Consider the copy of $\mathbb{Z}$ indexing $\mathbb{Z}^{(\infty)}$ (technically $\mathbb{N}$, but $\# \mathbb{N} = \# \mathbb{Z}$ so we can re-index). asdim $\mathbb{Z} \leq 1$, so we decompose it into two $R_0$-disjoint families $\mathcal{Z}_0$ and $\mathcal{Z}_1$ via the appropriate definition of finite asymptotic dimension. We then take the decomposition

$$\mathcal{X}_i = \bigoplus_{n \in \mathbb{Z}_i} \mathbb{Z}, \quad i = 0, 1$$

and note that the families $\mathcal{X}_i$ are both $R_0$ disjoint. The first round ends. Challenger gives a second number $R_1$. Defender takes each set

$$\mathcal{X}^j_i = \bigoplus_{z_j^i \in \mathcal{Z}_i} \mathbb{Z}.$$  

---

**Example 31 (Nowak-Yu [35])** Any space $X$ with finite asymptotic dimension has finite decomposition complexity.
As previously mentioned, straight finite decomposition complexity was developed by Dranishnikov and Zarichnyi as follows (see [18]).

**Definition 32** A metric family $\mathcal{X}$ has **straight finite decomposition complexity** (sFDC) if for any sequence $R_1 \leq R_2 \leq ...$ of positive numbers, there exists a finite sequence of metric families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ such that

1. $\mathcal{X}$ is $R_1$-decomposable over $\mathcal{V}_1$,

2. $\mathcal{V}_{i-1}$ is $R_i$-decomposable over $\mathcal{V}_i$ for all $i > 1$, and

3. the family $\mathcal{V}_n$ is bounded.

A single metric space $X$ has sFDC if $\{X\}$ has sFDC.

**Property A**

Guoliang Yu defined Property A in his argument that if a discrete metric space with bounded geometry admits a uniform embedding into a Hilbert space, then the coarse Baum-Connes conjecture holds for the original space (see [39]). That is, he defined it as a property that would imply coarse embeddability - the topic of the next section.

**Definition 33** Let $X$ be a discrete metric space. $X$ has **Property A** if for every $\epsilon, R > 0$ there exists a collection of finite subsets $\{A_x\}_{x \in X}$, $A_x \subseteq X \times \mathbb{N}$, and a constant $S > 0$ such that

\[
\frac{\#(A_x \triangle A_{x'})}{\#(A_x \cap A_{x'})} \leq \epsilon \quad \text{when} \quad d(x, x') \leq R \quad \text{and} \quad (2) \quad A_x \subseteq B(x, S) \times \mathbb{N}.
\]
Example 34

Finite groups have Property A. Take $A_x = G \times \{1\}$.

Example 35 (Nowak-Yu [35])

$\mathbb{Z}$ has Property A. Take $\epsilon_n = \frac{1}{n}$, $A_x = B(x, R(n + 1)) \times \{1\}$. Then if $d(x, x') < R$ we have $\#(A_x \triangle A_{x'}) \leq 2R$ and $\#(A_x \cap A_{x'}) \geq 2Rn$. The result follows.

Example 36 (Nowak-Yu [35])

Finite asymptotic dimensional spaces have Property A. Therefore, trees have Property A, and thus finitely generated free groups have Property A. Consequently, all fundamental groups of negatively curved manifolds have Property A.

Example 37 (Nowak-Yu [35], Bell-Moran [6])

Spaces with (straight) finite decomposition complexity have Property A.

Example 38 (Nowak-Yu [35])

Finitely generated abelian groups and countable abelian groups have Property A.

Subspaces inherit Property A. Thus, Property A will have “subspace permanence,” as defined at the end of this section. Importantly, as is relevant for many non-examples,
the contrapositive statement (if \( A \subseteq X \) does not have Property A then \( X \) does not have Property A) holds.

**Definition 39** As per [23], a metric space is **exact** if for every \( R > 0 \) and every \( \epsilon > 0 \) there exists a partition of unity \( \{ \phi_U \}_{U \in \mathcal{U}} \) subordinate to a uniformly bounded cover \( \mathcal{U} \) such that for \( x, y \in X \),

\[
d(x, y) \leq R \Rightarrow \sum_{U \in \mathcal{U}} |\phi_U(x) - \phi_U(y)| < \epsilon.
\]

A metric family is **exact** if one, equivalently each, of its total spaces is exact.

Dadarlat and Guentner crafted exactness as a roughly equivalent notion to Property A, in the hope that it would be easier to manipulate to produce results. They showed that exactness and Property A are equivalent for metric spaces of bounded geometry.

**Coarse Embeddability**

**Definition 40** A metric space is **coarsely embeddable** if it admits a coarse embedding into a Hilbert space. A metric family is **coarsely embeddable** if it admits a coarse embedding into the family comprised of a single Hilbert space. That is, for all \( X_\alpha \in \mathcal{X}, \) \( X_\alpha \) admits a coarse embedding into \( H. \)

A metric family \( \mathcal{X} \) is coarsely embeddable if and only if all \( X_\alpha \in X \) admit a \( \rho_+ \)-uniformly expansive and \( \rho_- \)-effectively proper coarse embedding, where \( \rho_+, \rho_- \) are independent of \( \alpha. \)

**Example 41**

Spaces with Property A are coarsely embeddable. Nowak and Yu provide a proof in [35] that any discrete space \( X \) with Property A can be coarsely embedded into \( \bigoplus_{n=1}^\infty \ell_2(X \times \mathbb{N}). \)
2.2.3 Coarse Permanence

Consider any of the coarse geometric properties described thus far, or some other property with which you may be familiar. Following the notation of Guentner [23], denote your chosen property by $\mathcal{P}$, and express that a chosen metric space $X$ has this property by $X \in \mathcal{P}$. If, instead, we wish to make a statement about a metric family $\mathcal{X}$, we stipulate that the standard total space of $\mathcal{X}$ satisfies the property if and only if each total space of $\mathcal{X}$ satisfies the property. Therefore, $\mathcal{X} \in \mathcal{P}$ if and only if its total spaces satisfy the property. We also say that the metric spaces comprising $\mathcal{X}$ have the property uniformly.

The following are notions Guentner [23] provides as coarse permanence properties. Some are slightly adapted to fit our purposes.

1. **Coarse Invariance.** If $\mathcal{X}$ is coarsely equivalent to $\mathcal{Y}$, then $\mathcal{X} \in \mathcal{P} \iff \mathcal{Y} \in \mathcal{P}$.

2. **Subspace Permanence.** If $\mathcal{X} \leq \mathcal{Y}$, $\mathcal{Y} \in \mathcal{P}$, then $\mathcal{X} \in \mathcal{P}$.

3. **Union Permanence.** Define $Z = \mathcal{X} \cup \mathcal{Y}$, $\mathcal{X}, \mathcal{Y} \in \mathcal{P} \Rightarrow Z \in \mathcal{P}$.

4. **General Fibering Permanence.** For all $F : \mathcal{X} \to \mathcal{Y}$ a coarse map of metric families, where $\mathcal{Y} \in \mathcal{P}$ and for all bounded $Z \leq \mathcal{Y}$, $F^{-1}(Z) \in \mathcal{P}$, then $\mathcal{X} \in \mathcal{P}$.

5. **Fibering Permanence.** For $f : \mathcal{X} \to \mathcal{Y}$ uniformly expansive, $\mathcal{Y} \in \mathcal{P}$, and for every bounded family $Z \leq \mathcal{Y}$ where the elements of $Z$ are of the form $y[r]$ for an arbitrary fixed $r \geq 0$, $f^{-1}(Z) \in \mathcal{P}$. Then $\mathcal{X} \in \mathcal{P}$.
6. **Limit Permanence.** If for every $R > 0$ there exists a decomposition $\mathcal{Z} = \bigcup_i \mathcal{X}_i$ such that for all $\alpha \in A$, $\{X_{i,\alpha}\}$ is $R$-disjoint and $\{X_{i,\alpha}\} \in \mathcal{P}$, then $\mathcal{Z} \in \mathcal{P}$.

What we call “general fibering permanence,” Guentner calls simply “fibering permanence,” and our notion of fibering permanence is a special case of his.

The following permanence property, finite quotient permanence, is stated in the Kasprzak-Nicas-Rosenthal paper [27].

**Definition 42** A collection of metric spaces $\mathcal{P}$ satisfies **finite quotient permanence** if for any $\mathcal{X} \in \mathcal{P}$ and any finite group $F$ that acts isometrically on every $X_\alpha \in \mathcal{X}$, the family $F \setminus \mathcal{X} := \{F \setminus X \mid X \in \mathcal{X}\}$ is also in $\mathcal{P}$.

To remain compatible with Guentner’s paper [23], we state the following definitions:

**Definition 43** If more than one $Z_\gamma \in \mathcal{Z}$ can be a subset of the same $Y_{\beta(\gamma)} \in \mathcal{Y}$, $\mathcal{Z}$ is referred to as a **family of subspaces** of $\mathcal{Y}$. If instead $\mathcal{Z} = \{Z_\beta\}$ and for all $\beta$, $Z_\beta \subseteq Y_\beta$ (allowing that $Z_\beta = \emptyset$), $\mathcal{Z}$ is instead a **subspace** of $\mathcal{Y}$. Note that in the case of a **subspace**, the same index set is applied to both $\mathcal{Z}$ and $\mathcal{Y}$, whereas when discussing a **family of subspaces**, a structure map is used.

**Definition 44** The **direct product of families** $\mathcal{X} = \{X_\alpha\}$ and $\mathcal{Z} = \{Z_\beta\}$ is the family $\mathcal{X} \times \mathcal{Z}$ whose elements are the spaces $X_\alpha \times Z_\beta$ where $X_\alpha \in \mathcal{X}$ and $Z_\beta \in \mathcal{Z}$.

**Remark 45** Often in the chapters that follow, a result will first be proven with respect to a particular space, and then that argument will be utilized as a step toward proving a result holds for the family version of the condition. However, there will be instances where the result for the family condition is shown without a separate or independent argument for
the condition on spaces. Under such circumstances, it is understood that the result for spaces can be demonstrated by taking the family to be a single space.

2.3 Coherence

Let $R$ be a ring with a unit. A left $R$-module has a projective resolution of finite type if it has a projective resolution in which all of the modules in the resolution are finitely generated. If a left $R$-module $M$ has a projective resolution by finitely generated $R$-modules (i.e., is of finite type), then $M$ is coherent. If such a projective resolution of $M$ can be chosen to be finite, $M$ is regular coherent and has finite (homological) dimension. In discussing group rings $R[\Gamma]$, regular coherent modules are a useful feature to consider.

In his work on algebraic $K$-theory, F. Waldhausen [38] detected that many fundamental types of discrete groups $\Gamma$ satisfy that all finitely presented modules over the group ring $R[\Gamma]$ are regular coherent. Groups of specific interest for which he proved this statement include free and free abelian groups, torsion-free one relator groups, and fundamental groups of submanifolds of the three-dimensional sphere. Waldhausen termed this property of the group $\Gamma$ regular coherence. He used regular coherence as a tool with which he computed the algebraic $K$-theory of these groups.

However, homologically finite dimensional modules over arbitrary group rings are rare. Not much is known in this setting beyond what Waldhausen discovered. In light of this observation, a weaker notion of coherence (weak coherence) was defined by Gunnar Carlsson and Boris Goldfarb in [9], along with a new method for constructing finite dimensional modules using the coarse geometric properties of the group $\Gamma$. Carlsson and
Goldfarb required this weaker coherence property when they computed the K-theory of geometrically finite groups of finite asymptotic dimension in [11].

We now formulate Carlsson and Goldfarb’s weak coherence condition. To discuss it, we require some background terminology, in particular, the definition of a bicontrolled map, which can be found in the following section. These definitions may be found in [9].

**Definition 46** (1.6, [9]) An \( R[\Gamma] \)-module is **finitely presented** if it is the cokernel of a homomorphism, known as a **presentation**, between free finitely generated \( R[\Gamma] \)-modules. If the homomorphism is boundedly bicontrolled, the presentation is **admissible**.

**Definition 47** (1.7, [9]) The group ring \( R[\Gamma] \) is **weakly coherent** if every \( R[\Gamma] \)-module with an admissible presentation has a projective resolution of finite type. The ring \( R[\Gamma] \) is **weakly regular coherent** if every \( R[\Gamma] \)-module with an admissible presentation has finite homological dimension.

We will return to a discussion of weak coherence after introducing the necessary terminology.

### 2.3.1 Filtered Modules

Let \( X \) be a metric space, \( R \) be a ring.

**Definition 48** An **\( X \)-filtered \( R \)-module** \( F \) is a covariant functor \( f : P(X) \rightarrow \text{Mod}_R(F) \) from the power set of \( X \) ordered by inclusion to the category of \( R \)-submodules of \( F \) and injective homomorphisms such that \( f(X) = F \) and \( f(T) \) is finitely generated for all bounded \( T \subseteq X \). In an abuse of notation, we denote \( f \) by \( F \): for all \( T \subseteq X \), denote \( f(T) \) by \( F(T) \). An \( X \)-filtered \( R \)-module \( F \) is **reduced** if \( F(\emptyset) = 0 \).
Remark 49 In this paper, all filtered modules are reduced.

Definition 50 If $X = \Gamma(G)$, the Cayley graph of some group $G$, we may write that $F$ is a $G$-filtered $R$-module, rather than a $X$- or $\Gamma(G)$-filtered $R$-module.

Remark 51 Let $G$ be a finitely generated group. As per the discussion in [9], a finitely generated $R[G]$-module $F = \langle \Sigma \rangle$, $\Sigma$ finite, is also an $R$-module, and is generated by $G \times \Sigma$. There is, therefore, an associated action of $G$ on $F$ as an $R$-module that is given by multiplication by $G$ on the first factor of the new generating set.

Definition 52 Let $F$ be as in the preceding remark. $F$ is an equivariant $G$-filtered $R$-module if for all $\gamma \in G, S \subseteq \Gamma(G)$,

$$F(\gamma S) = \gamma F(S).$$

Definition 53 The discrete trivial filtration is the $X$-filtration of an $R$-module $F$ given by $F(\emptyset) = 0$ and $F(S) = F$ for all nonempty $S \subseteq X$. The indiscrete trivial filtration is the $X$-filtration of an $R$-module $F$ given by $F(S) = 0$ for all proper subsets of $X$ and $F(X) = F$.

Clearly, any $R$-module can be thought of as $X$-filtered with either of the trivial filtrations.

Definition 54 The standard filtration of a submodule $F'$ of an $X$-filtered $R$-module $F$ is given by $F'(S) := F' \cap F(S)$ for all $S \subseteq X$. 

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The above definition agrees with the following definition for a submodule of $F$ supported on a subset $Z \subseteq X$.

**Definition 55** The external standard filtration with respect to a subset $Z \subseteq X$ for some $X$-filtered $R$-module $F$ is given by

$$F_Z(T) := F_X(T) \cap F_X(Z),$$

where the subscripts denote whether $F$ is viewed as $Z$-filtered or $X$-filtered, respectively. In other words, if we take $F' := F_X(Z)$, $F'(T) = F_X(T) \cap F'$, as in the standard filtration.

A competing notion for an inherited filtration for subsets is given in the following definition.

**Definition 56** The internal standard filtration with respect to a subset $Z \subseteq X$ for some $X$-filtered $R$-module $F$ is given by

$$F_Z(T) := F_X(T \cap Z),$$

where the subscripts denote whether $F$ is viewed as $Z$-filtered or $X$-filtered, respectively.

**Theorem 57**

Let $Z \subseteq X$, $F$ be as in the above definitions. Denote $F$ as $Z$-filtered with the internal standard filtration by $F_I$ and denote $F$ as $Z$-filtered with the external standard filtration by $F_E$. Then for all $T \subseteq X$, $F_I(T) \subseteq F_E(T)$.

**Proof.**

$$F_I(T) = F(T \cap Z) \subseteq F(T) \cap F(Z) = F_E(T)$$

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where the inclusion above results from the requirement that filtrations respect inclusion and \( T \cap Z \subseteq T, Z \). \( \square \)

**Definition 58 ([20])** Let \( Z \subseteq X \), and let \( F \) be an \( X \)-filtered \( R \)-module. For any \( D \geq 0 \) and \( S \subseteq X \), the submodules \( F_{Z;D}(S) = \sum_{x \in Z \cap S} F(x[D]) \) yield an \( X \)-filtration of the module \( F_{Z;D} := \sum_{x \in Z} F(x[D]) \). By design, \( F_{Z;D} \) is always a \( D \)-lean \( X \)-filtered \( R \)-module, and will be referred to as the \( D \)-**leanly constructed** \( X \)-filtered submodule.

**Definition 59** Let \( \phi : Y \to X \) be a map between metric spaces, and let \( F \) be a \( Y \)-filtered \( R \)-module. The \( \phi \)-**induced** \( X \)-**filtration** of \( F \) is given by

\[
F_X(S) := F_Y(\phi^{-1}(S))
\]

for all \( S \subseteq X \).

We may view \( F \) as the value \( F(X) \) filtered by submodules associated to subsets \( S \subseteq X \). Often, the submodule we will be concerned with is the kernel of a map from between \( X \)-filtered \( R \)-modules. As is consistent with our notation for \( C \)-neighborhoods, will use the notation \( S[b] \) for the metric \( b \)-enlargement of \( S \) in \( X \). Thus, as before, \( x[b] \) is the \( b \)-neighborhood of \( x \).

We will use the above notation to define several relevant notions regarding \( X \)-filtered \( R \)-modules.

**Definition 60** \( F \) is \( D \)-lean if there is a number \( D \geq 0 \) such that

\[
F(S) \subseteq \sum_{x \in S} F(x[D])
\]
for every subset $S$ of $X$.

**Definition 61** $F$ is $\delta$-scattered if there is a number $\delta \geq 0$ such that

$$F(X) \subseteq \sum_{x \in X} F(x[\delta]).$$

**Definition 62** $F$ is $d$-insular if there is a number $d \geq 0$ such that

$$F(S) \cap F(U) \subseteq F(S[d] \cap U[d])$$

for every pair of subsets $S, U$ of $X$.

**Definition 63** $F$ is $\delta'$-split if there is a number $\delta' \geq 0$ such that

$$F(S \cup U) \subseteq F(S[\delta']) + F(U[\delta'])$$

for every pair of subsets $S, U$ of $X$.

As with finite asymptotic dimension, when we are not concerned with what specific $D$ (respectively $\delta, d, \delta'$) a given module is lean (respectively scattered, insular, split), we will simply state that a module is **lean** (respectively **scattered**, **insular**, **split**).

If a module is lean, clearly it is also scattered, though the converse need not hold. If a module is $D$-lean, it is also $D$-split.

**Definition 64** An $R$-homomorphism $f : F \to F'$ of $X$-filtered $R$-modules is $b$-controlled if there exists $b \geq 0$ such that $f(F(S))$ is a submodule of $F'(S[b])$ for all subsets $S \subseteq X$. 
The collection of lean and insular \( X \)-filtered \( R \)-modules constitute the objects of a category \( \text{LI}(X, R) \), where the morphisms are the \textbf{controlled} (each \( b \)-controlled for some function-specific \( b \)) \( R \)-homomorphisms. This category will be discussed further in Section 2.3.3. Among the controlled \( R \)-homomorphisms of \( \text{LI}(X, R) \) are morphisms whose “control” (containment) extends in both directions:

**Definition 65** A homomorphism \( f : F \to F' \) of \( X \)-filtered \( R \)-modules is \textbf{\( b \)-bicontrolled} if there exists \( b \geq 0 \) such that \( f \) is \( b \)-controlled (\( f(F(S)) \subseteq F'(S[b]) \)) and there are inclusions

\[
f(F) \cap F'(S) \subseteq fF(S[b])
\]

for all subsets \( S \subseteq X \).

The preceding list of conditions on \( X \)-filtered \( R \)-modules and the morphisms between them come together to produce the following theorem of Carlsson and Goldfarb [12].

**Theorem 66**

Let

\[
0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0
\]

be an exact sequence of \( X \)-filtered \( R \)-modules where \( f \) and \( g \) are bicontrolled.

1. If the object \( E \) is lean then \( E'' \) is lean.

2. If \( E \) is insular then \( E' \) is insular.

3. If \( E \) is insular and \( E' \) is lean then \( E'' \) is insular.

4. If both \( E' \) and \( E'' \) are lean then \( E \) is lean.

5. If both \( E' \) and \( E'' \) are insular then \( E \) is insular.
Proof. This is a summary of results from section 3.1 in [12]. □

If one reads carefully, there seems to be a “missing” 6th statement. This statement, along with the following remark, was the inspiration for the definitions that comprise the next subsection.

**Remark 67** In practice, the “first step” in arguments asserting the weak coherence of some group or space \( X \) is that the kernel of any surjective bicontrolled homomorphisms between lean \( X \)-filtered \( R \)-modules is lean (see Lemma 2.4, [9]). This is almost precisely the definition of coherence that follows. The image of a boundedly bicontrolled homomorphism of lean filtered modules is always lean, as seen above.

### 2.3.2 Types of Coherence

**Definition 68** A metric space \( X \) is *(geometrically) coherent* if in any exact sequence

\[
0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0
\]

of \( X \)-filtered \( R \)-modules where \( f \) and \( g \) are bicontrolled maps, \( E \) lean, and \( E'' \) insular imply that \( E' \) is lean.

**Definition 69** A metric family \( \{X_\alpha\} \) is *(geometrically) coherent as a family* if for any exact sequences

\[
0 \rightarrow E'_\alpha \xrightarrow{f_\alpha} E_\alpha \xrightarrow{g_\alpha} E''_\alpha \rightarrow 0
\]

of \( X_\alpha \)-filtered \( R \)-modules where \( E_\alpha \ D \)-lean, \( E''_\alpha \ d \)-insular, \( f_\alpha, g_\alpha \ b \)-bicontrolled maps for
some fixed constants $D, d, b \geq 0$, then $E'_\alpha$ is necessarily $D'$-lean for some uniform constant $D' \geq 0$.

When we say that a family is in $\mathcal{C}$, we mean that the family is coherent. This is a stronger assumption than that each space in the family be coherent because of the requirement that the constants involved in the definition be common to all appropriate maps, modules.

This definition of coherence evolved and developed into the following slightly broader concept in an attempt to formulate a more easily manipulated, roughly equivalent concept.

**Definition 70** A metric space $X$ is **coarsely coherent** if in any exact sequence

$$0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0$$

of $X$-filtered $R$-modules where $f$ and $g$ are bicontrolled maps, $E$ is lean, and $E''$ is insular imply that $E'$ is scattered.

**Definition 71** A metric family $\{X_\alpha\}$ is **coarsely coherent as a family** if for any exact sequence

$$0 \rightarrow E'_\alpha \xrightarrow{f_\alpha} E_\alpha \xrightarrow{g_\alpha} E''_\alpha \rightarrow 0$$

of $X_\alpha$-filtered $R$-modules where $E_\alpha D$-lean, $E''_\alpha d$-insular, $f_\alpha, g_\alpha b$-bicontrolled maps for some fixed constants $D, d, b \geq 0$, then $E'_\alpha$ is necessarily $\delta$-scattered for some uniform constant $\delta \geq 0$.

When we say that a family is in $\mathcal{C}$, we mean that the family is coarsely coherent.
As with coherence, this is a stronger assumption than that each space in the family be coarsely coherent because of the requirement that the constants involved in the definition be common to all appropriate maps, modules.

When $R$ is a Noetherian ring (as in the equivariant setting of [9]), coarse coherence of a group $\Gamma$ yields that “the kernel of any $R[\Gamma]$-equivariant surjection between finitely generated, lean, insular modules is finitely generated” [20].

Ultimately, coarse coherence was not as straightforward to generate results with as was hoped, and a new property that is stronger than coarse coherence but more generative was defined. This new property makes use of uniformly expansive maps, as defined previously.

**Definition 72**  A metric space $X$ is **relatively coarsely coherent** if for any metric space $Y$ equipped with a uniformly expansive map $\pi: Y \to X$ and for any exact sequence

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

of $Y$-filtered $R$-modules where $f$ and $g$ are both bicontrolled maps, $E$ is lean, $E''$ is insular imply $E'$ is $\pi$-**coarsely scattered**. That is, there exist $\partial \geq 0$ and $\partial' \geq 0$ such that for all $S \subseteq Y$,

$$E'(S) \subseteq \sum_{x \in X} E'(\pi^{-1}(x[\partial]) \cap S[\partial']).$$

We will use $\mathcal{C}^{\text{rel}}$ to denote the family of all relatively coarsely coherent metric spaces.
Note that if \( E \) is insular, then \( E' \) is \( d \)-insular as a \( Y \)-filtered \( R \)-module for some \( d \),

\[
E'(\pi^{-1}(x[\partial])) \cap E'(S[\partial']) \subset E' \left( (\pi^{-1}(x[\partial]))[d] \cap S[\partial' + d] \right),
\]

since \( \pi \) is a uniformly expansive map.

As in our discussion of coarse permanence properties (to which collection of properties we would like to promote the coherences to membership), there is a formulation of relative coarse coherence for metric families. This metric family definition makes use of the terminology and conceptual framework from Guentner [23].

**Definition 73** A metric family \( \{X_\alpha\} \) is **relatively coarsely coherent as a family** if for any family of metric spaces \( \{Y_\delta\} \) with uniformly expansive maps \( \pi_\gamma: Y_\delta(\gamma) \to X_\alpha(\gamma) \) with the same control function and for any exact sequence

\[
0 \to E'^\prime_\delta(\gamma) \xrightarrow{f_\delta(\gamma)} E_\delta(\gamma) \xrightarrow{g_\delta(\gamma)} E''_\delta(\gamma) \to 0
\]

of \( Y_\delta(\gamma) \)-filtered lean, insular \( R \)-modules where \( f_\delta(\gamma) \) and \( g_\delta(\gamma) \) are bicontrolled maps with \( \text{fil}(f_\delta(\gamma)), \text{fil}(g_\delta(\gamma)) \) both bounded by some fixed number, \( E'_\delta(\gamma) \) is **uniformly \( \pi_\gamma \)-coarsely scattered**. That is, there exist \( \partial \geq 0 \) and \( \partial' \geq 0 \) such that for all \( S \subseteq Y_\delta(\gamma) \),

\[
E'_\delta(\gamma)(S) \subseteq \sum_{x \in X_\alpha(\gamma)} E'_\delta(\gamma)(\pi^{-1}_\gamma(x[\partial]) \cap S[\partial']).
\]

We say that a family is in \( \mathcal{C}^{\text{rel}} \), if that the family is coarsely coherent in this context, rather than the alternative choice that each space in the family is in \( \mathcal{C}^{\text{rel}} \)
Clearly, if $\{X_\alpha\} \in \mathcal{C}$, each $X_\alpha \in \mathcal{C}$. That is, the definition above is more restrictive. This exclusivity will allow us to make use of the results of Guentner’s paper [23] in the next chapter.

However, it is important to note that what we mean by $\mathcal{X} \in \mathcal{C}$ is not what Guentner means by $\mathcal{X} \in \mathcal{P}$ for some coarse property $\mathcal{P}$. In [23], $\mathcal{X} \in \mathcal{P}$ means that all of the total spaces of $\mathcal{X}$ have property $\mathcal{P}$, while our definition does not make any mention of total spaces. It turns out, as will be shown in the next section, these conditions are equivalent. When instead Guentner’s notion of a metric family $\mathcal{X}$ having a coarse property is meant, we will denote it $\mathcal{X} \in \mathcal{C}$.

It may be helpful to note later that if $\{X_\alpha\} \in \mathcal{C}$ as a family, then any subcollection $\{X_{\alpha,\beta}\}$ is coarsely coherent as a family.

The following is the main permanence result of the Goldfarb and Grossman paper [20].

**Theorem 74**

Assume $f : X \to M$ is a uniformly expansive map with $M \in \mathcal{C}$. If for all $r > 0$ the family $\{f^{-1}(m[r]) \mid m \in M\}$ is in $\mathcal{C}$, then $X$ is in $\mathcal{C}$.

A similar statement for families $\mathcal{X}, \mathcal{M}$ would constitute the fibering permanence described by Guentner [23]. Guentner also states in [23] that if a coarse property $\mathcal{P}$ satisfies coarse invariance, subspace permanence, and fibering permanence if and only if each of its factors satisfy the same three conditions.
2.3.3 Lean Insular Filtered Modules as a Category

The category of lean, insular, $X$-filtered $R$-modules with the morphisms bicontrolled homomorphisms is denoted $LI(X, R)$. When we are considering the coherence properties defined above, we are in essence investigating the following phenomena: a bicontrolled homomorphism between two objects in $LI(X, R)$ may have a kernel that is not an object in $LI(X, R)$. That is, kernels can “escape” this category. Here, a kernel of a bicontrolled $X$-filtered $R$-module homomorphism $g$ is the $R$-module $\ker(g)$ equipped with the standard filtration inherited from the domain of the homomorphism (Definition 54). We are interested in determining what types of conditions on the filtration space $X$ result in this category being “closed” in the sense that all kernels of all bicontrolled maps between objects in the category are also lean and insular. Note that the inclusion $f$ of the kernel $E' := \ker(g)$ of a bicontrolled homomorphism $g : E \to E''$ into the domain $E$ is always 0-bicontrolled: $f$ is a monomorphism and for any $T \subseteq X$,

$$f(E'(T)) = f(E(T) \cap E') = f(E(T)) \cap f(E') = E(T) \cap E' \subseteq E(T)$$

and

$$f(E') \cap E(T) = E' \cap E(T) = E'(T) = f(E'(T)).$$

By Theorem 66, the kernels of bicontrolled $R$-module homomorphisms between elements of $LI(X)$ are always insular. A metric space $X$ is coherent exactly when the kernels of all such homomorphisms are necessarily lean. Consequently, coherence is (by con-
struction) a sufficient condition on an arbitrary metric space $X$ to render $LI(X)$ closed under kernels. Since relative coarse coherence implies coherence (Theorem 126), relative coarse coherence is also a sufficient condition on $X$ for $LI(X)$ to be closed under kernels. The combination of Theorem 66 and $X$ coherent implies that for $X$-filtered $R$-modules $E, E', E''$, such that we have a short exact sequence

$$0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0$$

where $f, g$ are bicontrolled, whenever any two of $E, E', E''$ are lean and insular, the third module is also lean and insular.

The uniformly expansive maps $\pi : Y \rightarrow X$ relate $LI(Y, R)$ to $LI(X, R)$, and are associated to functors between the two categories. These functors merit some discussion.

A (covariant) functor $\Pi$ between two categories $LI(Y, R)$ and $LI(X, R)$ associates to each $F \in LI(Y, R)$ a module $\Pi(F) \in LI(X, R)$, and associates to each morphism $\phi : F \rightarrow F'$ in $LI(Y, R)$ a morphism $\Pi(\phi) : \Pi(F) \rightarrow \Pi(F')$ such that $\Pi(id_F) = id_{\Pi(F)}$ and $\Pi(\phi \circ \psi) = \Pi(\phi) \circ \Pi(\psi)$. In our situation, given a uniformly expansive map $\pi : Y \rightarrow X$ with control function $\rho$, the functor $\Pi$ associates to each $Y$-filtered $R$-module $F$ the $X$-filtered $R$-module $F$ with the $\pi$-induced filtration defined above. That is, and as can be seen from upcoming Section 3.1, $\Pi$ sends any $D$-lean, $d$-insular $R$-module $F$ with respect to a $Y$-filtration to the same $R$-module $F$ now $\rho(D)$-lean and $\rho(d)$-insular with respect to and equipped with an $X$-filtration. Similarly, $b$-controlled homomorphisms between any two $R$-modules $F$ and $F'$ each equipped with $Y$-filtrations are sent to themselves and are $\rho(b)$-controlled, while $F$ and $F'$ are now equipped with $X$-filtrations as just described.
Define for finitely generated $G$ with respect to some fixed finite generating set (realized as a metric space via its Cayley graph with respect to that fixed generating set), the subcategory $\text{LI}_{fg}^{\text{eq}}(G, R)$, the category of lean, insular, finitely generated, equivariant $G$-filtered $R$-modules. The collection $\text{obj}(\text{LI}_{fg}^{\text{eq}}(G, R))$ contains all free, finitely generated $R[G]$-modules [9]. Carlsson and Goldfarb in [9] and Goldfarb in [19] have worked to establish for what $R, G$ the category $\text{LI}_{fg}^{\text{eq}}(G, R)$ is closed under kernels ($R$ Noetherian, $G$ with finite asymptotic dimension and with straight finite decomposition complexity, respectively). Their condition weak coherence of the group ring $R[G]$ is, by construction, a sufficient condition for $\text{LI}_{fg}^{\text{eq}}(G, R)$ to be closed under kernels.
3.1 The Induced Filtrations

In this section, it is demonstrated that properties of $Y$-filtered $R$-modules are preserved by $X$-filtered $R$-modules with respect to a uniformly expansive map $\pi : Y \to X$ equipped with control function $\rho$. Specifically, insularity, leanness, scatteredness, and coarse-scatteredness are preserved. Further, module-homomorphisms that are bicontrolled with respect to the $Y$-filtration are bicontrolled with respect to the $X$-filtration induced by $\pi$. Also included in this section are several corollaries that are adapted from results in [19]. The results in this section will be utilized as lemmas in the following sections that comprise this chapter.

It will often be the case that a space or group of interest can be mapped to or embedding in a space or group for which the coherence property under investigation is known to hold. With such maps in mind, we define a filtration induced by such maps and observe some of the properties of filtered modules equipped with these induced filtrations.
First, we need to know that the properties of modules are preserved under uniformly expansive maps. That is, under the induced filtration, insularity, leanness, scatteredness, and coarse scatteredness are inherited. The following lemmas detail these processes explicitly.

Recall the definition of a \( \pi \)-induced filtration for a map \( \pi : Y \to X \): If \( F \) is a \( Y \)-filtered \( R \)-module, we can view it as \( X \)-filtered by taking

\[
F_X(T) := F_Y(\pi^{-1}(T)),
\]

for all \( T \subseteq X \).

**Lemma 75 (Permanence of Insularity)**

Let \( \pi : Y \to X \) be a uniformly expansive map with control function \( \rho \), and let \( E \) be \( d \)-insular as a \( Y \)-filtered \( R \)-module. Then \( E \) is \( \rho(d) \)-insular as an \( X \)-filtered \( R \)-module under the \( \pi \)-induced filtration.

**Proof.** Let \( T, U \subseteq X \) be given. We will demonstrate that \( E_X(T) \cap E_X(U) \subseteq E_X(T[\rho(d)] \cap U[\rho(d)]) \).
\[ E_X(T) \cap E_X(U) = E_Y(\pi^{-1}(T)) \cap E_Y(\pi^{-1}(U)) \]
\[ \subseteq E_Y(\pi^{-1}(T)[d] \cap \pi^{-1}(U)[d]) \]
\[ \subseteq E_Y(\pi^{-1}(T[\rho(d)]) \cap \pi^{-1}(U[\rho(d)])) \]
\[ = E_Y(\pi^{-1}(T[\rho(d)] \cap U[\rho(d)])) \]
\[ = E_X(T[\rho(d)] \cap U[\rho(d)]). \]

\[ \square \]

**Lemma 76 (Permanence of Leanness)**

Let \( \pi : Y \to X \) be a uniformly expansive map with control function \( \rho \), and let \( E \) be \( D \)-lean as a \( Y \)-filtered \( R \)-module. Then \( E \) is \( \rho(d) \)-lean as an \( X \)-filtered \( R \)-module under the \( \pi \)-induced filtration.

**Proof.** Let \( S \subseteq Y \), \( \pi : Y \to X \) uniformly expansive. Let \( E \) be a \( D \)-lean \( Y \)-filtered \( R \)-module. There is a filtration induced on \( X \) given by

\[ E_X(T) := E_Y(\pi^{-1}(T)) \]

for all \( T \subseteq X \), where the subscripts \( Y \) and \( X \) denote by which space \( E \) is a filtered \( R \)-
where the last equality is a result of recognizing (1) that each \(\pi(y)\) is an \(x\) for some \(x\) in \(T\), (2) that \(\{\pi(y) : y \in \pi^{-1}(T)\} = T\), (3) modules are closed under addition. □

**Lemma 77 (Permanence of Scatteredness)**

Let \(\pi : Y \to X\) be a uniformly expansive map with control function \(\rho\), and let \(E\) be \(\delta\)-scattered as a \(Y\)-filtered \(R\)-module. Then \(E\) is \(\rho(\delta)\)-scattered as an \(X\)-filtered \(R\)-module under the \(\pi\)-induced filtration.

**Proof.** Let \(\pi : Y \to X\) uniformly expansive. Let \(E\) be a \(\delta\)-scattered \(Y\)-filtered \(R\)-module.

There is a filtration induced by \(\pi\) such that we can view \(E\) as \(X\)-filtered as follows:

\[
E_X(T) := E_Y(\pi^{-1}(T))
\]

for all \(T \subseteq X\), where the subscripts \(Y\) and \(X\) denote by which space \(E\) is a filtered \(R\)-
\[ E_X(X) = E_Y(\pi^{-1}(X)) \]
\[ = E_Y(Y) \]
\[ \subseteq \sum_{y \in Y} E_Y(y[\delta]) \]
\[ \subseteq \sum_{y \in Y} E_X(\pi(y[\delta])) \]
\[ \subseteq \sum_{y \in Y} E_X(\pi(y)[\rho(\delta)]) \]
\[ \subseteq \sum_{x \in X} E_X(x[\rho(\delta)]). \]

\[ \square \]

**Lemma 78 (Permanence of Coarse Scatteredness)**

Let \( \phi : Z \to X \) be a uniformly expansive map with control function \( \rho \), and let \( E \) be \( \pi_Z \)-coarsely scattered with coarse scattering constants \( \partial, \partial' \geq \) as a \( Y \)-filtered \( R \)-module with respect to an arbitrary metric space \( Y \) and uniformly expansive map \( \pi_Z : Y \to Z \). Then \( E \) is \( \pi_X \)-coarsely scattered with coarse scattering constants \( \rho(\partial) \) and \( \partial' \) as a \( Y \)-filtered \( R \)-module, where \( \pi_X \) is the uniformly expansive map \( \pi_X = \phi \circ \pi_Z : Y \to X \).

**Proof.** Let \( \phi, \pi_Z, \pi_X, Y \) be given as above, and let \( E \) be a \( \pi_Z \)-coarsely scattered \( Y \)-filtered \( R \)-module with coarse scattering constants \( \partial, \partial' \). That is, for any \( S \subseteq Y \), it is the case that

\[ E_Y(S) \subseteq \sum_{z \in Z} E_Y(\pi_Z^{-1}(z[\partial]) \cap S[\partial']). \]
There are two induced filtrations to consider: the $\pi_Z$-induced filtration

$$E_Z(T) := E_Y(\pi_Z^{-1}(T))$$

and the $\pi_X$-induced filtration

$$E_X(U) := E_Y(\pi_X^{-1}(U)) = E_Y(\pi_Z^{-1} \phi^{-1}(U)).$$

It is of note then that

$$E_X(U) = E_Z(\phi^{-1}(U)).$$

We now demonstrate that $E$ is $\pi_Z$-coarsely scattered.

Observe that for any $z \in Z$, $\pi_Z^{-1}(z) \subseteq \pi_X^{-1}(\phi(z))$, and, therefore,

$$\pi_Z^{-1}(z[\partial]) \subseteq \pi_X^{-1}(\phi(z[\partial])).$$

Thus,

$$E_Y(S) \subseteq \sum_{z \in Z} E_Y(\pi_Z^{-1}(z[\partial]) \cap S[\partial'])$$

$$\subseteq \sum_{z \in Z} E_Y(\pi_X^{-1}(\phi(z[\partial])) \cap S[\partial'])$$

$$\subseteq \sum_{z \in Z} E_Y(\pi_X^{-1}(\phi(z)[\rho(\partial)]) \cap S[\partial'])$$

$$\subseteq \sum_{x \in X} E_Y(\pi_X^{-1}(x[\rho(\partial)]) \cap S[\partial']).$$
It is also helpful to know that bicontrolled homomorphisms between filtered $R$-modules remain bicontrolled when those same modules are instead filtered via the induced filtration.

**Lemma 79** (Permanence of (Bi)Controlled Homomorphisms)

Let $\pi : Y \to X$ be a uniformly expansive map. If a morphism $\phi : F \to F'$ of $Y$-filtered $R$-modules is $b$-controlled, it is $\rho(b)$-controlled with respect to the $\pi$-induced $X$-filtration, where $\rho : [0, \infty) \to [0, \infty)$ is the control function for $\pi$. Further, if $\phi$ is $b$-bicontrolled with respect to the $Y$-filtration, $\phi$ is $\rho(b)$-bicontrolled with respect to the $\pi$-induced $X$-filtration.

**Proof.** This is adjustment is clear if you recall that a $b$-controlled map $\phi : F \to F'$ satisfies

$$\phi(F_Y(S)) \subseteq F'_Y(S[b])$$

where the subscript $Y$ denotes that we are viewing $F, F'$ as $Y$-filtered. The induced filtration $F_X(T) := F_Y(\pi^{-1}(T))$ renders the statement

$$\phi(F_X(T)) = \phi(F_Y(\pi^{-1}(T))) \subseteq F'_Y(\pi^{-1}(T)[b]) \subseteq F'_Y(\pi^{-1}(T[\rho(b)])) = F'_X(T[\rho(b)]).$$

If $\phi$ is $b$-bicontrolled, then

$$\phi(F_X) \cap F'_X(T) = \phi(F_Y) \cap F'_Y(\pi^{-1}(T))$$

$$\subseteq \phi(F_Y(\pi^{-1}(T)[b]))$$

$$\subseteq \phi(F_Y(\pi^{-1}(T[\rho(b)])))$$

$$= \phi(F_X(T[\rho(b)])),$$
The following lemmas will be required in later arguments, particularly the coarse coherence of a direct product of coarsely coherent spaces. By an $\ell^p$-norm induced metric on the product $X \times Z$, it is meant that

$$d_{X \times Z}((x, z), (x', z')) = \sqrt[p]{d_X(x, x')^p + d_Z(z, z')^p}$$

for $1 \leq p < \infty$, and the $\ell^\infty$ or max metric

$$d_{X \times Z}((x, z), (x', z')) = \max\{d_X(x, x'), d_Z(z, z')\}$$

for $p = \infty$. In the arguments that follow, let $E$ be a $Z$-filtered $R$-module, and let $\mathcal{E}$ be $E$ equipped with the $x[B] \times Z$-filtration induced by the projection $\phi : x[B] \times Z \to Z$. That is,

$$\mathcal{E}(T) := E(\pi(T))$$

for all $T \subseteq x[B] \times Z$. Write

$$T_Z = \{z \in Z | (y, z) \in x[B] \times Z \text{ for some } y \in x[B]\} = \phi(T)$$

$$T_X = \{y \in x[B] | (y, z) \in x[B] \times Z \text{ for some } z \in Z\}$$

Observe that $\mathcal{E}(x[B] \times Z) = E(Z) = E$, so we are filtering the entirety of the module $E$, rather than a submodule. Further, note that for any $x \in X$ and for any $y \in x[B]$,
Lemma 80

If $E$ is $D$-lean as a $Z$-filtered $R$-module, then $E$ is lean as a $x[B] \times Z$-filtered $R$-module with respect to the filtration induced by projection onto $Z$.

Proof. Let $S \subseteq x[B] \times Z$.

\[
E(S) = E(S_Z) \\
\subseteq \sum_{z \in S_Z} E(z[D]) \\
= \sum_{z \in S_Z} \mathcal{E}(x[B] \times z[D]) \\
\subseteq \sum_{(y,z) \in S} \mathcal{E}((y,z)[2B + D])
\]

Therefore, $E$ is $(2B + D)$-lean as a $x[B] \times Z$-filtered $R$-module. \qed

Lemma 81

If $E$ is $d$-insular as a $Z$-filtered $R$-module, then $E$ is insular as a $x[B] \times Z$-filtered $R$-module with respect to the filtration induced by projection onto $Z$.

Proof. Let $S, T \subseteq x[B] \times Z$.

\[
\mathcal{E}(S) \cap \mathcal{E}(T) = E(S_Z) \cap E(T_Z) \\
\subseteq E(S_Z[d]) \cap T_Z[d]) \\
= \mathcal{E}(x[B] \times (S_Z[d] \cap T_Z[d])) \\
= \mathcal{E}((x[B] \times S_Z[d]) \cap (x[B] \times T_Z[d])) \\
\subseteq \mathcal{E}(S[2B + d] \cap T[2B + d])
\]
Therefore, $E$ is $(2B + d)$-insular as a $x[B] \times Z$-filtered $R$-module. \hfill \square

**Lemma 82**

If $f : E \rightarrow F$ is a $b$-bicontrolled $Z$-filtered $R$-module homomorphism, then $f$ is bicontrolled as a $x[B] \times Z$-filtered $R$-module homomorphism with respect to the filtration induced by projection onto $Z$.

**Proof.** Let $S \subseteq x[B] \times Z$ be given. $f$ is controlled, since

$$f(\mathcal{E}(S)) = f(E(S_Z))$$

$$\subseteq F(S_Z[b])$$

$$= \mathcal{F}(x[B] \times S_Z[b])$$

$$\subseteq \mathcal{F}(S[2B + b]).$$

Further, $f$ is bicontrolled:

$$f(\mathcal{E}) \cap \mathcal{F}(S) = f(E) \cap F(S_Z)$$

$$\subseteq f(E(S_Z[b]))$$

$$= f(\mathcal{E}(x[B] \times S_Z[b]))$$

$$\subseteq f(\mathcal{E}(S[2B + b])).$$

Consequently, $f$ is $(2B + b)$-bicontrolled. \hfill \square
Lemma 83

If $E$ is $\delta$-scattered as a $Z$-filtered $R$-module, then $E$ is scattered as a $x[B] \times Z$-filtered $R$-module with respect to the filtration induced by projection onto $Z$.

Proof.

\[
\mathcal{E}(x[B] \times Z) = E(Z) 
\subseteq \sum_{z \in Z} E(z[\delta]) 
= \sum \mathcal{E}(x[B] \times z[\delta]) 
= \sum \mathcal{E}((x, z)[2B + \delta]) 
\subseteq \sum_{(y, z) \in x[B] \times Z} \mathcal{E}((y, z)[2B + \delta])
\]

Therefore, $E$ is $2B + \delta$-scattered as a $x[B] \times Z$-filtered $R$-module. \qed

Lemma 84

If $E$ is $\pi$-coarsely scattered as a $Y$-filtered $R$-module, with $\pi : Y \to Z$ an arbitrary uniformly expansive map, then $E$ is $\pi_{x[B] \times Z}$-coarsely scattered as a $x[B] \times Z$-filtered $R$-module, where $\pi_{x[B] \times Z} = \phi \circ \pi : Y \to x[B] \times Z$ and with respect to the filtration induced by projection onto $Z$.

Proof. Let $S \subseteq Y$ be given, and let $\vartheta, \vartheta'$ be the $\pi$-coarse scattering constants associated to
Thus, $E$ is $\pi_{x[B] \times Z}$-coarsely scattered with coarse scattering constants $2B + \partial, \partial'$. □

Consider the following proposition and resulting consequence from Goldfarb [19].

**Theorem 85** (Proposition 2.3 of [19])

If $\phi : F \to G$ is a $b$-bicontrolled epimorphism between $X$-filtered $R$-modules, $F$ is $D$-lean, and $G$ is $d$-insular, then the kernel $K := \ker(\phi)$ is $(D + 2b + d)$-split.

The resulting corollary for our purposes is as follows.

**Corollary 86**

Let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of $X$-filtered $R$-modules with $f, g$ $b$-bicontrolled, $E$ $D$-lean, $E''$ $d$-insular.

Then $E'$ is $(D + 2b + d)$-split.

The proposition immediately following Proposition 2.3 of [19] yields an additional consequence which will be made use of here.
Theorem 87 (Proposition 2.4 of [19])

Let $\phi : F \to G$ be a $b$-controlled homomorphism between $X$-filtered $R$-modules, let $F$ be $D$-lean, and let $G$ be $d$-insular. If $U$ is a $(2D + 2b + 2d)$-disjoint collection $\{U_\alpha\}$ of subsets of $X$, then the kernel $K$ of $\phi$ with the standard filtration ($K(S) = K \cap F(S)$) obeys $K(U) \subseteq \sum_\alpha K(U_\alpha[D])$.

To adapt this theorem to our coherence calculations, we state:

Corollary 88

Let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of $X$-filtered $R$-modules with $f, g$-bicontrolled, $E$ $D$-lean, $E''$ $d$-insular.

If $U$ is a $(2D + 2b + 2d)$-disjoint collection $\{U_\alpha\}$ of subsets of $X$, then $E'(U) \subseteq \sum_\alpha E'(U_\alpha[D])$.

Further, in light of the coherence family conditions, we obtain:

Corollary 89

Let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of $X$-filtered $R$-modules with $f, g$-bicontrolled, $E$ $D$-lean, $E''$ $d$-insular.

If $X$ can be decomposed as a $(2D + 2b + 2d)$-disjoint collection $\{X_\alpha\}$ of subsets of $X$, then $E' = E'(X) \subseteq \sum_\alpha E'(X_\alpha[D])$. 
Corollary 90

Let \( \pi : Y \to X \) uniformly expansive, and let

\[
0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0
\]

be a short exact sequence of \( Y \)-filtered \( R \)-modules with \( f, g \) \( b \)-bicontrolled, \( E \) \( D \)-lean, \( E'' \) \( d \)-insular.

Apply the \( \pi \)-induced \( X \)-filtration to the short exact sequence. If \( E \) is lean with respect to \( Y \), then \( E' \) is \((\rho(D) + 2b + d)\)-split with respect to \( X \).

As above, we analogously find that:

Corollary 91

Let \( \pi : Y \to X \) uniformly expansive, and let

\[
0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0
\]

be a short exact sequence of \( Y \)-filtered \( R \)-modules with \( f, g \) \( b \)-bicontrolled, \( E \) \( D \)-lean, \( E'' \) \( d \)-insular. Apply the \( \pi \)-induced \( X \)-filtration to the short exact sequence. If \( X \) can be decomposed as a \((2\rho(D)+2b+2d)\)-disjoint collection \( \{X_\alpha\} \) of subsets of \( X \), then \( E' = E'(X) \subseteq \sum_\alpha E'(X_\alpha[\rho(D)]) \).

The facts proven in this section will be utilized frequently throughout the rest of this thesis.
3.2 Coherence Results

We proceed to consider results specific to each coherence notion, and present these results in the order these notions were introduced. There will be resemblance between arguments in this section and in the following two. While seemingly repetitious, the resulting leanness/scattering/\(\pi\)-scattering constants computed in these proofs are necessary for the results in the following chapters.

Included in this section are the results that total spaces of coherent families are coherent, that coherence is a coarse invariant, that subsets of coherent metric spaces are coherent (with respect to the internal standard filtration), that subspaces of coherent metric families are coherent metric families, and that the union of two coherent families is coherent.

We ultimately intend to rely on some of the consequences from Guentner’s paper [23]. It is therefore necessary to argue that our framework and his framework are compatible (and, in fact, equivalent). The following result yields exactly this compatibility. Recall that we denote the collection of coherent metric spaces by \(\mathcal{C}\).

**Theorem 92 (Equivalence of \(\mathcal{C}\) and Guentner Family Criteria)**

\(\mathcal{X} = \{X_\alpha\}\) is coherent as a metric family if and only if it is coherent in the sense of Guentner [23]: all total spaces \(X\) of \(\mathcal{X}\) are coherent.

**Proof.** Let \(\mathcal{X} = \{X_\alpha\}\) be a family of metric spaces, and let \(X\) be the total space of \(\mathcal{X}\) with respect to some arbitrary structure map \(\sigma : \gamma \to \alpha(\gamma)\). Recall that the total space of \(\mathcal{X}\) is
the disjoint union

\[ X = \bigsqcup_{\gamma} X_{\alpha(\gamma)} = \{ (x, \gamma) : x \in X_{\alpha(\gamma)} \} \]

equipped with the metric

\[ d_X(x_\gamma, y_{\gamma'}) = \begin{cases} 
    d_{X_{\alpha(\gamma)}}(x, y), & \gamma = \gamma' \\
    \infty, & \text{else,}
\end{cases} \]

where \( x_\gamma = (x, \gamma) \) when \( x \in X_{\alpha(\gamma)} \).

\( \implies \) Let \( \mathcal{X} \) be coherent as a metric family. Let

\[ 0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0 \]

be a short exact sequence of \( X \)-filtered \( R \)-modules with \( E \) \( D \)-lean, \( E'' \) \( d \)-insular, \( f, g \) \( b \)-bi-controlled. We want to show that

\[ E'(S) \subseteq \sum_{x \in S} E'(x[D']) \]

for some \( D' \geq 0 \) and for any \( S \subseteq X \).

By Corollary 89,

\[ E'(X) \subseteq \sum_{\alpha} E'(X_{\alpha}[D]). \]

The construction of the total space yields that \( X_{\alpha}[D] = X_{\alpha} \). Each \( X_{\alpha} \) is coherent, so we
can apply the standard filtration and observe that

\[ E'_\alpha(X_\alpha) = E'(X_\alpha). \]

Let \( D'' \) be the leanness constant associated to the \( E'_\alpha(X_\alpha) \), since \( \mathcal{X} \) is coherent as a family. Further, we observe that \( S = \sqcup_\alpha S_\alpha \), where \( S_\alpha = S \cap X_\alpha \). We have by Corollary 88 that

\[ E'(S) \subseteq \sum_\alpha E'(S_\alpha[D]). \]

We analogously observe that

\[ E'_\alpha(S_\alpha) = E'(S_\alpha). \]

Therefore, we have that

\[ E'(S) \subseteq \sum_\alpha E'_\alpha(S_\alpha[D]) \]
\[ \subseteq \sum_\alpha \sum_{x \in S_\alpha[D]} E'_\alpha(x[D'']) \]
\[ \subseteq \sum_\alpha \sum_{x \in S_\alpha} E'_\alpha(x[D + D'']) \]
\[ \subseteq \sum_{x \in S} E'(x[D + D'']) \]
\[ = \sum_{x \in S} E'(x[D + D'']). \]

\[ \Leftarrow \text{ Let } X \text{ be coherent, and consider for an arbitrary index } \alpha, \]
\[ 0 \rightarrow E'_\alpha \xrightarrow{f_\alpha} E_\alpha \xrightarrow{g_\alpha} E''_\alpha \rightarrow 0, \]
a short exact sequence of \(X_\alpha\)-filtered \(R\)-modules with \(E_\alpha\) \(D\)-lean, \(E''_\alpha\) \(d\)-insular, \(f_\alpha, g_\alpha\) \(b\)-bi-controlled. We want to show that

\[
E'_{\alpha}(S_\alpha) \subseteq \sum_{x \in S_\alpha} E'_{\alpha}(x[D'])
\]

for some \(D' \geq 0\) and for any \(S_\alpha \subseteq X_\alpha\). Since \(X\) is coherent and \(X_\alpha\) is a subspace of \(X\), by Lemma 95, \(X_\alpha\) is coherent and we have the result. The leanness constant is that inherited from \(X\), and so is common to all \(X_\alpha\). \(\square\)

**Corollary 93**

The (possibly infinite) coarse disjoint union of coherent metric spaces \(\{X_\alpha\}\) is coherent if \(\{X_\alpha\}\) is coherent as a metric family.

**Theorem 94** (Coarse Invariance for Spaces)

Let \(X, Y\) be two coarsely equivalent spaces. If \(Y\) is coherent, then \(X\) is coherent.

**Proof.** Let \(\psi : X \to Y\) be a coarse equivalence with \(\rho_\psi\) the upper control function. Let \(\phi : Y \to X\) be the coarse inverse of \(\psi\), equipped with upper control function \(\rho_\phi\). Assume that \(Y\) is coherent, and let

\[
0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0
\]

be a short exact sequence of \(X\)-filtered \(R\)-modules where \(f\) and \(g\) are bicontrolled maps, \(E\) is lean, and \(E''\) is insular. We will demonstrate that \(E'\) is lean with respect to \(X\).

Clearly, \(\psi(X) \subseteq Y\). Since \(Y\) is coherent, by the following Lemma 95, \(\psi(X)\) is coherent. Consider the power set functor \(P : \text{Set} \to \text{Set}\) that sends a set to its power set. We have
the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{P} & P(X) \\
| & | & | \\
\psi & \downarrow & \psi(P) \\
\psi(X) & \xrightarrow{P} & P(\psi(X))
\end{array}
\]

where \( P(\psi) : A \in P(X) \mapsto \psi(A) \in P(\psi(X)) \). We can then view \( E', E, E'' \) as \( \psi(X) \)-filtered \( R \)-modules with induced filtration

\[
E'^{(j)}_{\psi(X)}(B) := E'^{(j)}_{\psi^{-1}}(B)
\]

for any set \( B \) in \( P(\psi(X)) \), where the subscript \( X \) denotes the original \( X \)-filtered \( E' \) and the subscript \( \psi(X) \) indicates that we are instead viewing \( E' \) as \( \psi(X) \)-filtered. Let \( D \geq 0 \) be a constant such that \( E_{\psi(X)} \) is \( D \)-scattered, as per the definition of coherence. By construction, this yields the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{P} & P(X) \\
| & | & | \\
\psi & \downarrow & \psi(P) \\
\psi(X) & \xrightarrow{P} & P(\psi(X))
\end{array}
\]

\[
\xrightarrow{E'_{\psi(X)}} \quad \xrightarrow{E'_{\psi(X)}} \quad \xrightarrow{\text{Mod}_R E'}
\]

Therefore, we have that for any \( S \subseteq X \),

\[
E'_X(S) = E'_{\psi(X)}(\psi(S)) \subseteq \sum_{y \in \psi(S)} E'_{\psi(X)}(y[D]).
\]

Consider each of these neighborhoods \( y[D], y \in \psi(S) \). Let \( \pi, \hat{x} \) be in the preimage
Then there exist \( \overline{y}, \overline{z} \) such that \( \psi(\overline{x}) = \overline{y}, \psi(\overline{z}) = \overline{z} \). Composing with \( \phi \) yields that \( \phi \circ \psi(\overline{x}) = \phi(\overline{y}) \) and \( \phi \circ \psi(\overline{z}) = \phi(\overline{z}) \). \( \phi \) is the coarse inverse of \( \psi \) and thus their composition is \( C \)-close to the identity \( \text{id}_X \) for some \( C \geq 0 \). \( C \)-closeness implies that \( d_X(\phi \circ \psi(\overline{x}), \overline{x}) \leq C \) and \( d_X(\phi \circ \psi(\overline{z}), \overline{z}) \leq C \), and thus that \( d_X(\phi(\overline{y}), \overline{x}) \leq C \) and \( d_X(\phi(\overline{z}), \overline{z}) \leq C \). \( d_X \) is a metric and therefore satisfies the triangle inequality, yielding

\[
d_X(\overline{x}, \overline{z}) \leq d_X(\overline{x}, \phi(\overline{y})) + d_X(\phi(\overline{y}), \phi(\overline{z})) + d_X(\phi(\overline{z}), \overline{z})
\]

\[\leq C + \rho_\phi(d_Y(\overline{y}, \overline{z})) + C\]

\[\leq 2C + \rho_\phi(2D).\]

Thus, \( \psi^{-1}(y[D]) \subseteq x[2C + \rho_\phi(2D)] \) for any \( x \in \psi^{-1}(y[D]) \cap S \), and we have that

\[
E'_{\psi(X)}(y[D]) = E'_X(\psi^{-1}(y[D])) \subseteq E'_X(x[2C + \rho_\phi(2D)]), \quad x \in \psi^{-1}(y[D]) \cap S,
\]

implying

\[
E'_X(S) \subseteq \sum_{x \in S} E'_X(x[2C + \rho_\phi(2D)]).
\]

Therefore, \( E' \) is \( 2C + \rho_\phi(2D) \)-lean as an \( X \)-filtered \( R \)-module and \( X \) is coherent. \( \square \)

**Lemma 95 (Subspace Permanence)**

If \( X \) is coherent and \( Z \subseteq X \), then \( Z \) is coherent with the internal standard filtration.

**Proof.** Let \( X \) be a coherent metric space and let \( Z \subseteq X \). Let

\[
0 \to E' \to E \to E'' \to 0
\]
be a short exact sequence of $Z$-filtered $R$-modules, and let $S \subseteq Z$ be given. Define the filtration of $X$ induced by the inclusion $i$ of $Z$ in $X$ to be

$$E_X^i(S) := E_Z^i(i^{-1}(S)),$$

where $S$ is any subset of $X$, and the subscripts $X$ and $Z$ indicate that we are viewing a module as $X$- or $Z$-filtered, respectively. Effectively, this means that

$$E_X(T) = E_Z(T \cap Z)$$

for any $T \subseteq X$. We then have that we can consider the short exact sequence as an exact sequence of $X$-filtered $R$-modules. Since $X$ is coherent, we have that $E_X^i$ is $D$-lean for some $D \geq 0$. Let $S$ be an arbitrary subset of $Z$. Then $S$ is a subset of $X$ as well and $i^{-1}(S) = S$, yielding

$$E_Z^i(S) = E_X^i(S) \subseteq \sum_{x \in S} E_X^i(x[D]) = \sum_{x \in S} E_Z^i(i^{-1}(x[D])) = \sum_{x \in S} E_Z^i(x[D] \cap Z).$$

However, the $x \in S$ are precisely the $z \in S$ since $S \subseteq Z$. Therefore,

$$E_Z^i(S) \subseteq \sum_{z \in S} E_Z^i(z[D] \cap Z) \subseteq \sum_{z \in S} E_Z^i(z[D]),$$

as desired. \hfill \square

**Corollary 96**

If $X$ is coherent and $Z \subseteq X$, then $Z$ is coherent with respect to the external standard filtration.
Corollary 97

If \( Z \) is a subspace of a family of subspaces of a coherent metric family \( \mathcal{X} \), then \( Z \) is coherent.

Theorem 98 (Coarse Invariance for Families)

Let \( \mathcal{X} = \{ X_\alpha \} \) and \( \mathcal{Z} = \{ Z_\beta \} \) be coarsely equivalent metric families. If \( \mathcal{Z} \) is coherent as a metric family, then \( \mathcal{X} \) is coherent as a metric family.

Proof. Let \( \psi : \mathcal{Z} \to \mathcal{X} \), \( \psi = \{ \psi_\gamma \} \) be a coarse equivalence between \( \mathcal{Z} \) and \( \mathcal{X} \), and let \( \phi = \{ \phi_\gamma \} \) be the coarse inverse of \( \psi \). Let \( \gamma \) be given, and let

\[
0 \to E'_{\alpha(\gamma)} \xrightarrow{f_{\alpha(\gamma)}} E_{\alpha(\gamma)} \xrightarrow{g_{\alpha(\gamma)}} E''_{\alpha(\gamma)} \to 0
\]

be a short exact sequence of \( X_{\alpha(\gamma)} \)-filtered \( R \)-modules, with \( E_{\alpha(\gamma)} \) \( D \)-lean, \( E''_{\alpha(\gamma)} \) \( d \)-insular, \( f_{\alpha(\gamma)} \) and \( g_{\alpha(\gamma)} \) \( b \)-bicontrolled for uniform constants \( D, d, b \) associated to \( \mathcal{X} \). We must demonstrate that \( E'_{\alpha(\gamma)} \) is \( D' \)-lean for some \( D' \) independent of \( \gamma \).

We can view the three \( X_{\alpha(\gamma)} \)-filtered \( R \)-modules as \( Z_{\beta(\gamma)} \)-filtered \( R \)-modules via the induced \( \psi_\gamma \)-filtration, as in Theorem 94.

Theorem 99 (Union Permanence)

Let \( \mathcal{X} = \{ X_\alpha \} \) and \( \mathcal{Z} = \{ Z_\beta \} \) be two coherent families. Then the union of \( \mathcal{X} \) and \( \mathcal{Z} \) is coherent.

Proof. For a given short exact sequence

\[
0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0
\]

of \( Y \)-filtered \( R \)-modules, where \( Y \) is in (at least) one of \( \mathcal{X}, \mathcal{Z} \), \( E \) \( D \)-lean and \( E'' \) \( d \)-insular,
\[ f \text{ and } g \text{ b-bicontrolled, } E' \text{ is } D'-\text{lean for } D' := \max\{D'_X, D'_Z\}, \text{ since } E' \text{ is (at least one of) } D'_X-, D'_Z\text{-lean.} \]

3.3 Coarse Coherence Results

Included in this section are the results that coarse coherence is a coarse invariant, that subsets of coarsely coherent metric spaces are coarsely coherent (with respect to the internal standard filtration), that families of subspaces of coarsely coherent families are coarsely coherent, that the union of coarsely coherent families is coarsely coherent, that total spaces of coarsely coherent families are coarsely coherent, that coarse coherence satisfies (general) fibering permanence, that finite direct products of coarsely coherent metric spaces/families are coarsely coherent, and that coarse coherence satisfies finite quotient permanence.

We ultimately intend to rely on some of the consequences from Guentner’s paper [23]. It is therefore necessary to argue that our framework and his framework are compatible (and, in fact, equivalent). The following result yields exactly this compatibility. Recall that we denote by \( \mathcal{C} \) the collection of metric spaces that are coarsely coherent.

**Theorem 100** (Equivalence of \( \mathcal{C} \) and Guentner Family Criteria)

\( X = \{X_\alpha\} \) is coarsely coherent as a metric family if and only if it is coarsely coherent in the sense of Guentner [23]: all total spaces \( X \) of \( \mathcal{X} \) are coarsely coherent.

**Proof.** Let \( \mathcal{X} = \{X_\alpha\} \) be a family of metric spaces, and let \( X \) be the total space of \( \mathcal{X} \) with respect to some arbitrary structure map \( \sigma : \gamma \rightarrow \alpha(\gamma) \). Recall that the total space of \( \mathcal{X} \) is...
the disjoint union

\[ X = \sqcup_{\gamma} X_{\alpha(\gamma)} = \{(x, \gamma) : x \in X_{\alpha(\gamma)}\} \]

equipped with the metric

\[
d_X(x_\gamma, y_{\gamma'}) = \begin{cases} 
d_{X_{\alpha(\gamma)}}(x, y) & \gamma = \gamma' \\
\infty & \text{else,} \end{cases}
\]

where \( x_\gamma = (x, \gamma) \) when \( x \in X_{\alpha(\gamma)} \).

\[ \Rightarrow \] Let \( \mathcal{X} \) be coarsely coherent as a metric family. Let

\[ 0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0 \]

be a short exact sequence of \( X \)-filtered \( R \)-modules with \( E \) \( D \)-lean, \( E'' \) \( d \)-insular, \( f, g \) \( b \)-bi-controlled. We want to show that

\[ E'(X) \subseteq \sum_{x \in X} E'(x[\delta]) \]

for some \( \delta \geq 0 \).

By Corollary 89,

\[ E'(X) \subseteq \sum_{\alpha} E'(X_{\alpha}[D]). \]

The construction of the total space yields that \( X_{\alpha}[D] = X_{\alpha} \). Each \( X_{\alpha} \) is coarsely coherent,
so we can apply the standard filtration and observe that

\[ E'_\alpha(X_\alpha) = E'(X_\alpha). \]

Therefore, we have that

\[
E'(X) \subseteq \sum_\alpha E'_\alpha(X_\alpha) \\
\subseteq \sum_\alpha \sum_{x \in X_\alpha} E'_\alpha(x[\delta]) \\
\subseteq \sum_\alpha \sum_{x \in X_\alpha} E'(x[\delta]) \\
= \sum_{x \in X} E'(x[\delta]).
\]

\[ \Leftarrow \text{ Let } X \text{ be coarsely coherent, and consider for an arbitrary index } \alpha, \]

\[ 0 \to E'_\alpha \xrightarrow{f_\alpha} E_\alpha \xrightarrow{g_\alpha} E''_\alpha \to 0, \]

a short exact sequence of \( X_\alpha \)-filtered \( R \)-modules with \( E_\alpha \) \( D \)-lean, \( E''_\alpha \) \( d \)-insular, \( f_\alpha, g_\alpha \) \( b \)-bi-controlled. We want to show that

\[ E'_\alpha(X_\alpha) \subseteq \sum_{x \in X_\alpha} E'_\alpha(x[\delta]) \]

for some \( \delta \geq 0 \). Since \( X \) is coarsely coherent and \( X_\alpha \) is a subspace of \( X \), by Lemma 103, \( X_\alpha \) is coarsely coherent and we have the result. The scattering constant is that inherited from \( X \), and so is common to all \( X_\alpha \). \( \Box \)
As in Section 3.2 above, $\mathcal{C}$ is a coarse geometric invariant.

**Theorem 101 (Coarse Invariance for Spaces)**

Coarse coherence is invariant under coarse equivalences.

**Proof.** Let $\psi : X \to Y$ be a coarse equivalence with $\rho_\psi$ the upper control function. Let $\phi : Y \to X$ be the coarse inverse of $\psi$, equipped with upper control function $\rho_\phi$. Assume that $Y$ is coarsely coherent, and let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of $X$-filtered $R$-modules where $f$ and $g$ are bicontrolled maps, $E$ is lean, and $E''$ is insular. We will demonstrate that $E'$ is scattered.

Clearly, $\psi(X) \subseteq Y$. Since $Y$ is coarsely coherent, by the following Lemma 103, $\psi(X)$ is coarsely coherent. Consider the power set functor $P : \text{Set} \to \text{Set}$ that sends a set to its power set. We have the following commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{P} & P(X) \\
\downarrow{\psi} & & \downarrow{P(\psi)} \\
\psi(X) & \xrightarrow{P} & P(\psi(X))
\end{array}
$$

where $P(\psi) : A \in P(X) \mapsto \psi(A) \in P(\psi(X))$. We can then view $E', E, E''$ as $\psi(X)$-filtered $R$-modules with induced filtration

$$E'^{(j)}_{\psi(X)}(B) := E'^{(j)}_{\psi^{-1}(B)}$$

75
for any set $B$ in $P(\psi(X))$, where the subscript $X$ denotes the original $X$-filtered $E'$ and the subscript $\psi(X)$ indicates that we are instead viewing $E'$ as $\psi(X)$-filtered. Let $\delta \geq 0$ be a constant such that $E_{\psi(X)}$ is $\delta$-scattered, as per the definition of coarse coherence. By construction, this yields the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{P} & P(X) \\
\downarrow \psi & & \downarrow \psi \\
\psi(X) & \xrightarrow{P} & P(\psi(X)) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \longrightarrow E'_X \\
& & \uparrow \psi \\
& & \Mod^R E'
\end{array}
\]

Therefore, we have that

\[
E'_X(X) = E'_{\psi(X)}(\psi(X)) \subseteq \sum_{y \in \psi(X)} E'_{\psi(X)}(y[\delta]).
\]

Consider each of these neighborhoods $y[\delta]$. Let $\bar{x}, \hat{x}$ be in the preimage $\psi^{-1}(y[\delta])$. Then there exist $\bar{y}, \hat{y}$ such that $\psi(\bar{x}) = \bar{y}, \psi(\hat{x}) = \hat{y}$. Composing with $\phi$ yields that $\phi \circ \psi(\bar{x}) = \phi(\bar{y})$ and $\phi \circ \psi(\hat{x}) = \phi(\hat{y})$. $\phi$ is the coarse inverse of $\psi$ and thus their composition is $C$-close to the identity $\id_X$ for some $C \geq 0$. $C$-closeness implies that $d_X(\phi \circ \psi(\bar{x}), \bar{x}) \leq C$ and $d_X(\phi \circ \psi(\hat{x}), \hat{x}) \leq C$, and thus that $d_X(\phi(\bar{y}), \bar{x}) \leq C$ and $d_X(\phi(\hat{y}), \hat{x}) \leq C$. $d_X$ is a metric and therefore satisfies the triangle inequality, yielding

\[
d_X(\bar{x}, \hat{x}) \leq d_X(\bar{x}, \phi(\bar{y}))) + d_X(\phi(\bar{y}), \phi(\hat{y})) + d_X(\phi(\hat{y}), \hat{x}) \\
\leq C + \rho(\phi(d_Y(\bar{y}, \hat{y}))) + C \\
\leq 2C + \rho(2\delta).
\]
Thus, $\psi^{-1}(y[\delta]) \subseteq x[2C + \rho_\phi(2\delta)]$ for any $x \in \psi^{-1}(y[\delta])$, and we have that

$$E'_{\psi(X)}(y[\delta]) = E'_X(\psi^{-1}(y[\delta])) \subseteq E'_X(x[2C + \rho_\phi(2\delta)]), \quad x \in \psi^{-1}(y[\delta]),$$

implying

$$E'_X(X) \subseteq \sum_{x \in X} E'_X(x[2C + \rho_\phi(2\delta)]).$$

Therefore, $E'$ is $2C + \rho_\phi(2\delta)$-scattered as an $X$-filtered $R$-module and $X$ is coarsely coherent.

**Theorem 102** (Coarse Invariance for Families)

$\mathcal{C}$ is a coarse invariant of metric families.

**Proof.** By the proof of **Theorem 98**, replacing “lean” with “scattered” and referring to **Theorem 101**, we obtain the result (analogously).

**Lemma 103** (Subspace Permanence for Spaces)

If $X$ is coarsely coherent and $Z \subseteq X$, then $Z$ is coarsely coherent with the internal standard filtration.

**Proof.** Let $X$ be a coarsely coherent metric space and let $Z \subseteq X$. Let

$$0 \to E' \to E \to E'' \to 0$$

be a short exact sequence of $Z$-filtered $R$-modules, and define the filtration of $X$ induced by the inclusion $i$ of $Z$ in $X$ to be

$$E^{(i)}_X(T) := E^{(i)}_Z(i^{-1}(T)),$$
where $T$ is any subset of $X$, and the subscripts $X$ and $Z$ indicate that we are viewing a module as $X$- or $Z$-filtered, respectively. Effectively, this means that

$$E_Z(T) = E_X(T \cap Z)$$

and clearly $E_Z(T) = E_Z(T \cap Z)$, so

$$E'_X(T) = E'_Z(i^{-1}(T)) = E'_Z(T \cap Z) = E'_X(T \cap Z)$$

for any $T \subseteq X$. Observe that $E'_X(X) = E'_Z(X \cap Z) = E'_Z(Z) = E'$, so this is in fact a filtration. We then have that we can consider the short exact sequence as an exact sequence of $X$-filtered $R$-modules. Thus,

$$E'_Z(Z) = E'_X(Z) \subseteq \sum_{x \in X} E'_X(x[\delta]) = \sum_{x \in X} E'_Z(i^{-1}(x[\delta])) \subseteq \sum_{z \in Z} E'_Z(z[\delta])$$

for some $\delta \geq 0$ associated to $X$, since $X$ is coarsely coherent.

\[\square\]

**Corollary 104**

If $Z$ is a family of subspaces of a coarsely coherent metric family $X$, then $Z$ is coarsely coherent.

---

We now begin our investigation of the permanence properties from Section 2.2.3.

**Theorem 105** (Union Permanence)

Let $X = \{X_\alpha\}$ and $Z = \{Z_\beta\}$ be two coarsely coherent families. Then the union of $X$ and $Z$ is coarsely coherent.
Proof. For a given short exact sequence

\[ 0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0 \]

of \( Y \)-filtered \( R \)-modules, where \( Y \) is in (at least) one of \( X, Z, E \) \( D \)-lean and \( E'' \) \( d \)-insular, \( f \) and \( g \) \( b \)-bicontrolled, \( E' \) is \( \delta \)-scattered for \( \delta := \max \{ \delta_X, \delta_Z \} \), since \( E' \) is (at least one of) \( \delta_X-, \delta_Z \)-scattered. \( \square \)

Corollary 106

The (possibly infinite) coarse disjoint union of coarsely coherent metric spaces \( \{ X_\alpha \} \) is coarsely coherent if \( \{ X_\alpha \} \) is coarsely coherent as a metric family.

Theorem 107 (General Fibering Permanence)

\( \mathcal{C} \) has general fibering permanence.

Proof. We formulate the argument with respect to total spaces, noting the equivalence of the notions for total spaces and for metric families. We also comment that, the way these permanence properties are formulated (\( Z \leq Y \) with \( Z = \{ Z_\gamma \} \) and \( Y = \{ Y_\beta \} \)) more than one \( Z_\gamma \) can be a subset of some given \( Y_\beta \). In the literature, \( Z \) in this situation is referred to as a family of subspaces of \( Y \), where as the situation \( Z = \{ Z_\beta \} \) and for all \( \beta, Z_\beta \subseteq Y_\beta, Z \) is instead a subspace of \( Y \).

Recall that \( \mathcal{C} \) has general fibering permanence if for all \( F : \mathcal{X} \to \mathcal{Y} \) a coarse map of metric families, where \( \mathcal{Y} \in \mathcal{C} \) and for all bounded \( Z \leq \mathcal{Y} \), \( F^{-1}(Z) \in \mathcal{C} \), then \( \mathcal{X} \in \mathcal{C} \). Let \( X \) be the total space of \( \mathcal{X} \), \( Y \) the total space of \( \mathcal{Y} \). Then we view \( F \) as a uniformly expansive map, \( F : X \to Y \), with control function \( \rho \), and we have general fibering permanence if
for all $F : X \to Y$ a coarse map of total spaces, $Y \in \mathcal{C}$, and for all bounded families $\{Z_\gamma\}$, $F^{-1}\{\{Z_\gamma\}\} \in \mathcal{C}$, then $X \in \mathcal{C}$ as a metric space.

Let

$$0 \to E'_X \xrightarrow{f} E_X \xrightarrow{g} E''_X \to 0$$

be a short exact sequence of $X$-filtered $R$-modules where $f$ and $g$ are bicontrolled maps, $E_X$ is lean, and $E''_X$ is insular. We will demonstrate that $E'_X$ is scattered.

Consider the $Y$-filtration induced by $F$:

$$E'_Y(T) := E'_X(F^{-1}(T)),$$

where $T$ is any subset of $Y$. Observe that

$$E'_Y(Y) = E'_X(F^{-1}(Y)) = E'_X(X).$$

Since $Y \in \mathcal{C}$, we know then that

$$E'_X(X) = E'_Y(Y) \subseteq \sum_{y \in Y} E'_Y(\delta[y]) = \sum_{y \in Y} E'_X(F^{-1}(\delta[y]))$$

for some $\delta \geq 0$.

We consider now the submodule $E'_X(F^{-1}(\delta[y]))$. Define $K_y = F^{-1}(\delta[y] + \rho(b)))$, and let

$$\mathcal{E} := E_{K,D} = \sum_{x \in K} E_x(x[D]),$$
a $D$-leanly constructed $K_y$-filtered $R$-module. Recall that the $K_y$-filtration associated to $E$ is given by

$$E(S) = \sum_{x \in K_y \cap S} E_X(x[D])$$

for all $S \subseteq X$. By design, $E$ is lean and contains $f(E'_X(y[\delta])) = f(E'_X(\pi^{-1}(y[\delta])))$. Further define $E'' = g(E)$, and equip it with the standard filtration as a submodule of $E'_X$, which is also a $K_y$-filtration. Consequently, $E''$ is again insular. Finally, define $E' := f^{-1}(\ker(g|_E)) = f^{-1}(\text{im}(f) \cap E) = E'_X \cap f^{-1}(E) = f^{-1}(E)$, and equip it with the standard $K_y$-filtration as a submodule of $E'_X$.

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

is therefore a short exact sequence of $K_y$-filtered $R$-modules with $E$ $D$-lean, $E''$ $d$-insular. $f$ is $b$-bicontrolled, and hence $D + b$-bicontrolled, since for any $S \subseteq K_y$,

$$f(E'(S)) = \mathcal{E} \cap f(E'_X(S)) \subseteq \mathcal{E} \cap E_X(S[b]) = \sum_{x \in K_y \cap S} E_X(x[D]) = \mathcal{E}(S[b])$$

and

$$f(E') \cap \mathcal{E}(S) = \mathcal{E}(S) = \sum_{x \in K_y \cap S} E_X(x[D]) \subseteq \sum_{x \in K_y \cap S} E_X(x[D]) = f(E'(S[b])).$$

$g$ is $D + b$-bicontrolled as well, since for any $S \subseteq K_y$,

$$g(\mathcal{E}(S)) = g\left(\sum_{x \in K_y \cap S} E_X(x[D])\right) \subseteq g(\mathcal{E}) \cap g(E_X(S[D])) \subseteq \mathcal{E}'' \cap E''_X(S[D + b]) = \mathcal{E}''(S[D + b])$$
and

\[
g(\mathcal{E}) \cap \mathcal{E}''(S) = g(\mathcal{E}) \cap \mathcal{E}''(S)
\]
\[
\subseteq g(\mathcal{E}) \cap g(E(S[b]))
\]
\[
\subseteq \left( \sum_{x \in K_y} g(E(x[D])) \right) \cap \left( \sum_{x \in S[b]} g(E(x[D])) \right)
\]
\[
= g \left( \sum_{x \in K_y \cap S[b]} E(x[D]) \right)
\]
\[
= g(E(S[b])).
\]

Since \(\{y[\delta + \rho(b)]\}\) is a uniformly bounded metric family, \(\{K_y\}_{y \in Y} = \{F^{-1}(y[\delta + \rho(b)])\}_{y \in Y}\) are coarsely coherent as a metric family (by assumption), and \(\mathcal{E}'\) is \(\delta'\)-scattered.

\[
E'_X(F^{-1}(y[\delta])) \subseteq \mathcal{E}' \subseteq E'_X(F^{-1}(y[\delta + \rho(b)])[D + b]) = E'_X(K_y[D + b]),
\]

by design. Therefore,

\[
E'_X(F^{-1}(y[\delta])) \subseteq \sum_{x \in K_y[D + b]} \mathcal{E}'(x[\delta']) = \sum_{x \in K_y[D + b]} f^{-1}(\mathcal{E}) \cap E'_X(x[\delta']) \subseteq \sum_{x \in K_y[D + b]} E'_X(x[\delta'])
\]

Consequently,

\[
E'_X(X) \subseteq \sum_{y \in Y} \sum_{x \in K_y[D + b]} E'_X(x[\delta']),
\]

and \(X\) is coarsely coherent. \(\square\)
Theorem 108 (Fibering Permanence)

If $F : X \to M$ is a uniformly expansive map, $M$ is coarsely coherent, and for any $r \geq 0$, 
$\{F^{-1}(m[r]) : m \in M\}$ is uniformly coarsely coherent, then $X$ is coarsely coherent.

**Proof.** This is just a special case of Theorem 107. □

We next consider direct product permanence. This theorem will be referenced to demonstrate that, since $\mathbb{R}$ is coarsely coherent, $\mathbb{R}^n$ is coarsely coherent for finite $n \geq 1$ (Corollary 135). We will also use it when arguing that a particular space that lacks asymptotic property A is coarsely coherent (Theorem 158).

Theorem 109 (Product Permanence)

If $X, Z$ are coarsely coherent, then $X \times Z$ equipped with any $\ell^p$-norm induced metric ($1 \leq p \leq \infty$) is coarsely coherent.

**Proof.** The projections $\pi_X : X \times Z \to X$, $\pi_Z : X \times Z \to Z$ are uniformly expansive with control functions the identity $\text{id}_{[0, \infty)}$. Pre-images of the form $\pi_X^{-1}(x[B]) = x[B] \times Z$ and $\pi_Z^{-1}(z[B]) = X \times z[B]$ are coarsely coherent because they are coarsely equivalent to $Z, X$, respectively. Therefore, apply the proof for general fibering permanence (Theorem 107) first to the first non-trivial term of an appropriate short exact sequence of $X \times Z$-filtered modules where the $\pi_X, \pi_Z$-induced filtrations have scattering constant less than or equal to $\delta \geq 0$, 

$$E'(X \times Z) \subseteq \sum_{x \in X} E'(\pi_X^{-1}(x[\delta])) = \sum_{x \in X} E'(x[\delta] \times Z),$$

then apply that $\{x[\delta] \times Z\}_{x \in X}$ are coarsely equivalent to one another and to $Z$, and thus are coarsely coherent as a family with common scattering constant $\delta$. Thus, via the technique
involving the construction of a short exact sequence in Theorem 107, each \( E'(x[\delta] \times Z) \) may be rewritten

\[
E'(x[\delta] \times Z) \subseteq \sum_{(x,z) \in x[\delta+2b+D] \times Z} E'((x,z)[\delta])
\]

and the result follows. □

**Corollary 110**

By induction, coarse coherence is closed under finite direct products. Further, finite direct products of coarsely coherent families are coarsely coherent.

Another permanence property, described by Kasprowski, Nicas, and Rosenthal in [27], is finite quotient permanence.

**Theorem 111 (Finite Quotient Permanence)**

Coarse coherence has finite quotient permanence.

**Proof.** Let \( \mathcal{X} \in \mathcal{C} \) with finite group \( F \) acting on each \( X_\alpha \in \mathcal{X} \) be given, let \( X \) be one such \( X_\alpha \in \mathcal{X} \), and let

\[
0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0
\]

be a short exact sequence of \( F\backslash X \)-filtered \( R \) modules with \( f, g \) \( b \)-bicontrolled, \( E \) \( D \)-lean, \( E'' \) \( d \)-insular for fixed \( b, D, d \geq 0 \) as in the definition of coarse coherence of a metric family.

We demonstrate that \( E' \) is scattered. Define a \( X \)-filtration on the exact sequence by

\[
E'_X(T) := E'(q(T)),
\]

for any subset of \( X \), where \( q : X \to F\backslash X \) is the canonical quotient map sending \( x \) to \( Fx \).

Observe that \( q \) is an onto map, and contracting, and so is coarse with control function...
Therefore, we know

\[ E'(U) \subseteq E'_X(q^{-1}(U)) = E'(qq^{-1}(U)) = E'(U), \]

for all \( U \subseteq F \setminus X \). We also note that

\[ E'_X(X) = E'(q(X)) = E'(F \setminus X). \]

\( X \) is coarsely coherent and comes from a coarsely coherent metric family with scattering constant \( \delta \geq 0 \). We compute:

\[ E'(F \setminus X) = E'_X(X) \subseteq \sum_{x \in X} E'_X(x[\delta]) \]

since \( X \) is a member of a coarsely coherent family with scattering constant \( \delta \). Thus, by the definition of the \( X \)-filtration,

\[ E'(F \setminus X) \subseteq \sum_{x \in X} E'(q(x)[\delta]) \subseteq \sum_{x \in X} E'(q(x)[\delta]) = \sum_{Fx \in F \setminus X} E'(Fx[\delta]). \]

Since \( X \) was arbitrarily chosen and the resulting scattering has the same constant \( \delta \) as the metric family, \( F \setminus X' \) is coarsely coherent as a metric family. \( \square \)
3.4 Relative Coarse Coherence Results

The final coherence notion we consider is relative coarse coherence. Included in this section are the results that total spaces of relatively coarsely coherent metric families are relatively coarsely coherent, that relative coarse coherence is a coarse invariant, that subsets of relatively coarsely coherent metric spaces are relatively coarsely coherent (with respect to the internal standard filtration), that families of subspaces of relatively coarsely coherent families are relatively coarsely coherent, that coarse embeddability into a relatively coarsely coherent space implies relative coarse coherence, that the union of relatively coarsely coherent metric families is relatively coarsely coherent, and that the finite direct produce of relatively coarsely coherent metric spaces is relatively coarsely coherent.

Again, since we ultimately intend to rely on some of the consequences from Guentner’s paper [23], it is necessary to argue that our framework and his framework are compatible (and, in fact, equivalent). The following result yields exactly this compatibility.

Recall that we denote by $\mathcal{C}_{rel}$ the collection of metric spaces that are relatively coarsely coherent.

**Theorem 112 (Equivalence of $\mathcal{C}_{rel}$ and Guentner Family Criteria)**

$\mathcal{X} = \{X_\alpha\}$ is relatively coarsely coherent as a metric family if and only if it is relatively coarsely coherent in the sense of Guentner [23]: all total spaces $X$ of $\mathcal{X}$ are relatively coarsely coherent.

**Proof.** Let $\mathcal{X} = \{X_\alpha\}$ be a family of metric spaces, and let $X$ be the total space of $\mathcal{X}$ with respect to some arbitrary structure map $\sigma : \gamma \to \alpha(\gamma)$. Recall that the total space of $\mathcal{X}$ is
the disjoint union

\[ X = \bigsqcup_{\gamma} X_{\alpha(\gamma)} = \{(x, \gamma) : x \in X_{\alpha(\gamma)}\} \]

equipped with the metric

\[
 d_X(x_{\gamma}, y_{\gamma'}) = \begin{cases} 
 d_{X_{\alpha(\gamma)}}(x, y) & \gamma = \gamma' \\
 \infty & \text{else},
\end{cases}
\]

where \( x_{\gamma} = (x, \gamma) \) when \( x \in X_{\alpha(\gamma)} \).

\[
\iff \quad \text{Let } Y \text{ be a metric space equipped with a uniformly expansive map } \pi : Y \to X,
\]

and let

\[ 0 \to E' \to E \to E'' \to 0 \]

be a short exact sequence of lean, insular, \( Y \)-filtered \( R \)-modules as usual. We want to show that for any \( S \subseteq Y \),

\[ E'(S) \subseteq \sum_{x \in X} E'(\pi^{-1}(x[D]) \cap S[D]) \]

Define \( Y_{\alpha} := \pi^{-1}(X_{\alpha}) \) and \( S_{\alpha} := S \cap \pi^{-1}(X_{\alpha}) \). Then \( Y = \bigsqcup Y_{\alpha}, S = \bigsqcup S_{\alpha} \) and \( \pi(S_{\alpha}) \subseteq X_{\alpha} \).

By 88,

\[ E'(S) \subseteq \sum_{\alpha} E'(S_{\alpha}[D]). \]

Each \( E'(S_{\alpha}) = E'_{\alpha}(S_{\alpha}) \), where we give the submodules the standard filtration \( E'_{\alpha}(T) := \)
\[ E'(X_\alpha) \cap E'(T) \]. Since each \( X_\alpha \) is relatively coarsely coherent, we have that

\[
E'_\alpha(S_\alpha) \subseteq \sum_{x \in X_\alpha} E'_\alpha(\pi_\alpha^{-1}(x[\partial_\alpha]) \cap S_\alpha[\partial'_\alpha]).
\]

Here \( \pi_\alpha : Y_\alpha \to X_\alpha \) is both the obvious restriction of \( \pi \) to \( Y_\alpha \) and, by Guentner’s statements in [23] that

\[
\pi((y, \alpha)) = (\pi_\alpha(y), \alpha)
\]

defines an ordinary function \( \pi : Y \to X \) and that every function between total spaces arises in this way from the functions of families.

\[
E'(S) \subseteq \sum_\alpha E'(S_\alpha[D])
\]

\[= \sum_\alpha E'(S_\alpha[D])\]

\[\subseteq \sum_\alpha \sum_{x \in X_\alpha} E'_\alpha(\pi_\alpha^{-1}(x[\partial_\alpha]) \cap S_\alpha[D + \partial'_\alpha])\]

\[\subseteq \sum_\alpha \sum_{x \in X_\alpha} E'_\alpha(\pi^{-1}(x[\partial_\alpha]) \cap S[D + \partial'_\alpha])\]

\[= \sum_{x \in X} E'(\pi^{-1}(x[\partial_\alpha]) \cap S[D + \partial'_\alpha])\]

\[\implies \text{Let } \mathcal{Y} = \{Y_\delta\} \text{ be a family of metric spaces equipped with a uniformly expansive function of families } \pi = \{\pi_\gamma : Y_{\delta(\gamma)} \to X_{\alpha(\gamma)}\}. \text{ Further, let } \gamma \text{ be given and let}\]

\[
0 \to E'_{\delta(\gamma)} \xrightarrow{f_{\delta(\gamma)}} E_{\delta(\gamma)} \xrightarrow{g_{\delta(\gamma)}} E''_{\delta(\gamma)} \to 0
\]

be an exact sequence of \( Y_{\delta(\gamma)} \)-filtered lean, insular \( R \)-modules where \( f_{\delta(\gamma)}, g_{\delta(\gamma)} \) are bicon-
trolled with the same control bounds. We must demonstrate that for any \( S \subseteq Y_{\delta(\gamma)} \),

\[
E'_{\delta(\gamma)}(S) \subseteq \sum_{x \in X_{\alpha(\gamma)}} E'_{\delta(\gamma)}(\pi_{\gamma}^{-1}(x[\partial_{\alpha(\gamma)}]) \cap S[\partial'_{\alpha(\gamma)}]).
\]

Consider the family of maps \( \{i_{\gamma} : X_{\alpha(\gamma)} \to X\} \) the inclusion maps of each component of \( \mathcal{X} \) into the total space \( X \), clearly uniformly expansive with control function the identity map \( \text{id}(t) = t \). Define then \( \hat{\pi}_{\gamma} := i_{\gamma} \circ \pi_{\gamma} \).

\[
\hat{\pi}_{\gamma} : Y_{\delta(\gamma)} \xrightarrow{\pi_{\gamma}} X_{\alpha(\gamma)} \xrightarrow{i_{\gamma}} X,
\]

where \( \hat{\pi}_{\gamma}(Y_{\delta(\gamma)}) \subseteq X_{\alpha} \subseteq X \). Since \( X \) is relatively coarsely coherent, \( X_{\alpha} \) is a subspace of \( X \), and we have a uniformly expansive map from \( Y_{\delta(\gamma)} \to X \), by Lemma 116, \( X_{\alpha} \) is relatively coarsely coherent and we have the result. \( \square \)

**Corollary 113**

The (possibly infinite) coarse disjoint union of relatively coarsely coherent metric spaces \( \{X_{\alpha}\} \) is relatively coarsely coherent if \( \{X_{\alpha}\} \) is relatively coarsely coherent as a metric family.

Now that we’ve demonstrated that the notion of a relatively coarsely coherent metric family that we’ve developed and the version consistent with Guentner’s paper [23] are equivalent, the following permanence results can be understood to have the same consequences that the equivalent statements from [23] imply.

**Theorem 114** (Coarse Invariance for Spaces)

Relative coarse coherence is a coarse invariant of metric spaces.

**Proof.** Let \( X \) be relatively coarsely coherent, let \( Z \) be a metric space that is coarse equiv-
alent to $X$ with coarse equivalence $\psi : Z \to X$ between them, with controls $\rho_-, \rho_+$. Let $Y$ and $\pi_Z : Y \to Z$ uniformly expansive be given, and define $\pi_X := \psi \circ \pi_Z$. Finally, let

$$0 \to E' \to E \to E'' \to 0$$

be a short exact sequence of $Y$-filtered $R$-modules as usual. We want to show that for any $S \subseteq Y$,

$$E'(S) \subseteq \sum_{z \in Z} E'(\pi_Z^{-1}(z[\partial_Z]) \cap S[\partial'_Z])$$

for some $\partial_Z, \partial'_Z \geq 0$. Since $X$ is relatively coarsely coherent, we know that

$$E'(S) \subseteq \sum_{x \in X} E'(\pi_X^{-1}(x[\partial_X]) \cap S[\partial'_X])$$

$$= \sum E'(\psi \circ \pi_Z)^{-1}(x[\partial_X]) \cap S[\partial'_X])$$

$$= \sum E'(\pi_Z^{-1}(\psi^{-1}(x[\partial_X])) \cap S[\partial'_X]).$$

$\psi$ is a coarse equivalence, so it has at least one coarse inverse, say $\phi : X \to Z$, with controls $\delta_-, \delta_+$, such that $\phi \circ \psi$ is $C$-close to the identity on $Z$. Let $\bar{z}, \hat{z} \in \psi^{-1}(x[\partial_X])$. Then, there exist $\bar{x}, \hat{x} \in x[\partial_X]$ such that $\psi(\bar{z}) = \bar{x}$ and $\psi(\hat{z}) = \hat{x}$. Composing with $\phi$ yields that $\phi \circ \psi(\bar{z}) = \phi(\bar{x})$ and $\phi \circ \psi(\hat{z}) = \phi(\hat{x})$. $C$-closeness implies that $d_Z(\phi \circ \psi(\bar{z}), \bar{z}) \leq C$ and $d_Z(\phi \circ \psi(\hat{z}), \hat{z}) \leq C$, and thus that $d_Z(\phi(\bar{x}), \bar{z}) \leq C$ and $d_Z(\phi(\hat{x}), \hat{z}) \leq C$. $d_Z$ is a metric and
therefore satisfies the triangle inequality, yielding

\[
d_Z(z, \hat{z}) \leq d_Z(z, \phi(\overrightarrow{x})) + d_Z(\phi(\overrightarrow{x}), \phi(\overrightarrow{x})) + d_Z(\phi(\overrightarrow{x}), \hat{z}) \\
\leq C + \delta_+(d_X(\overrightarrow{x}, \overrightarrow{x})) + C \\
\leq 2C + \delta_+(2\partial_X).
\]

Thus, \( \psi^{-1}(x[\partial_X]) \subseteq z[2C + \delta_+(2\partial_X)] \) for any \( z \in \psi^{-1}(x[\partial_X]) \), and we have that

\[
E'(S) \subseteq \sum_{z \in \mathcal{Z}} E'(\pi_Z^{-1}(z[2C + \delta_+(2\partial_X)]) \cap S[\partial'_X]),
\]

the desired result. \( \square \)

**Theorem 115 (Coarse Invariance for Families)**

Relative coarse coherence is a coarse invariant of metric families.

**Proof.** Let \( \mathcal{X} = \{X_\alpha\} \) be a relatively coarsely coherent metric family that is coarsely equivalent to \( \mathcal{Z} = \{Z_\beta\} \). Let \( \mathcal{Y} = \{Y_\delta\} \) be a family of metric spaces and \( \pi: \mathcal{Y} \to \mathcal{Z} \), \( \pi = \{\pi_\gamma: Y_\delta(\gamma) \to Z_\beta(\gamma)\} \) a uniformly expansive function of families. Let \( \psi: \mathcal{Z} \to \mathcal{X} \), \( \psi = \{\psi_\tau\} \) be the coarse equivalence, and let \( \phi = \{\phi_\nu\} \) be a coarse inverse of \( \psi \). Define \( \hat{\pi}: \mathcal{Y} \to \mathcal{Z} \) be the composition \( \psi \circ \pi \), clearly uniformly expansive. Finally, let \( \gamma \) be given and let

\[
0 \to E'_{\delta(\gamma)} \xrightarrow{f_{\delta(\gamma)}} E_{\delta(\gamma)} \xrightarrow{g_{\delta(\gamma)}} E''_{\delta(\gamma)} \to 0
\]

be an exact sequence of \( Y_{\delta(\gamma)} \)-filtered lean, insular \( R \)-modules where \( f_{\delta(\gamma)}, g_{\delta(\gamma)} \) are bicontrolled with the same control bounds. We assert that \( E'_{\delta(\gamma)} \) is uniformly coarsely scattered.
Since $X$ is relatively coarsely coherent, for any $S \subseteq \mathcal{Y}_{\delta(\gamma)}$,

$$E_{\delta(\gamma)}(S) \subseteq \sum_{x \in X_{\alpha(\gamma)}} E'_{\delta(\gamma)}(\pi^{-1}(x[\partial_{\alpha(\gamma)}]) \cap S[\partial'_{\alpha(\gamma)}]) = \sum_{x \in X_{\alpha(\gamma)}} E'_{\delta(\gamma)}(\pi^{-1}(\psi_{\tau(\gamma)}(x[\partial_{\alpha(\gamma)}])) \cap S[\partial'_{\alpha(\gamma)}]).$$

The preceding argument yields that

$$E_{\delta(\gamma)}(S) \subseteq \sum_{x \in X_{\alpha(\gamma)}} E'_{\delta(\gamma)}(\pi^{-1}(z[2C + \rho_{\phi}(2\partial_{\alpha(\gamma)})]) \cap S[\partial'_{\alpha(\gamma)}]),$$

as desired, where $\rho_{\phi}$ is the (uniform) control function from $\phi$ a uniformly expansive function of families. \hfill \Box

**Lemma 116** (Subspace Permanence for Spaces)

Subspaces of relatively coarsely coherent metric spaces are relatively coarsely coherent.

---

**Proof.** Let $X$ be a relatively coarsely coherent space with $A \subseteq X$ a subspace of $X$ with the inherited metric. Let $Y$ a metric space and $\pi_A : Y \to A$ a uniformly expansive map be given. Define $\pi_X$ as the composition of $\pi_A$ and the inclusion of $A$ into $X$:

$$\pi_X : Y \xrightarrow{\pi_A} A \xrightarrow{\iota} X.$$  

$\pi_X$ is uniformly expansive via the same non-decreasing function $\rho_+ : [0, \infty) \to [0, \infty)$ since the metric on $A$ agrees with the metric on $X$. 

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Since $X$ is relatively coarsely coherent, for any $S \subseteq Y$ we have that

$$E'(S) \subseteq \sum_{x \in X} E'\left(\pi_X^{-1}(x[\partial]) \cap S[\partial']\right) \tag{3.2}$$

$$= \sum_{x \in X} E'\left(\pi_A^{-1}(\iota(x[\partial])) \cap S[\partial']\right) \tag{3.3}$$

$$= \sum_{x \in A} E'\left(\pi_A^{-1}(x[\partial]) \cap S[\partial']\right) \tag{3.4}$$

which is the desired result. \qed

**Corollary 117**

If $Z$ is a family of subspaces of a relatively coarsely coherent metric family $X$, then $Z$ is relatively coarsely coherent.

A remarkable consequence of the preceding two theorems regarding spaces is the following theorem.

**Theorem 118**

Any space $X$ that can be coarsely embedded into a relatively coarsely coherent space $Z$ is relatively coarsely coherent.

**Proof.** Let $X$ be a metric space such that $\psi : X \to Z$ is a coarse embedding of $X$ into a relatively coarsely coherent space $Z$. Since relative coarse coherence is inherited by subspaces (Lemma 116), $\psi(X) \subseteq Z$ is coarsely coherent. Denote by $\bar{\psi}$ the map given by $\psi$ whose codomain is restricted to its image. That is, $\bar{\psi} : X \to \psi(X), x \mapsto \psi(x)$. This is again a coarse embedding, and is in fact now a coarse equivalence, since $\psi(X)$ is a net in itself. Since $X$ is therefore coarsely equivalent to a relatively coarsely coherent space and coarse coherence is a coarse invariant by Theorem 114, $X$ is relatively coarsely coherent. \qed
Theorem 119 (Union Permanence)

The union of any two relatively coarsely coherent families is relatively coarsely coherent as a family.

Proof. Let $\mathcal{X} = \{X_\alpha\}$, $\mathcal{Z} = \{Z_\beta\}$ be two relatively coarsely coherent families. Let $\{Y_\delta\}$ be an arbitrary metric family with associated uniformly expansive maps $\pi_\gamma : Y_{\delta(\gamma)} \to W_{\omega(\gamma)}$ where $W_{\omega(\gamma)} \in \{X_\alpha\} \cup \{Z_\beta\}$.

Now, let a particular $Y_{\delta(\gamma)}$ be given, and let

$$0 \to E_{\delta(\gamma)}' \xrightarrow{f_{\delta(\gamma)}} E_{\delta(\gamma)} \xrightarrow{g_{\delta(\gamma)}} E_{\delta(\gamma)}'' \to 0$$

be an exact sequence of $Y_{\delta(\gamma)}$-filtered lean, insular $R$-modules where $f_{\delta(\gamma)}$, $g_{\delta(\gamma)}$ are bicontrolled with the same control bounds. We assert that $E_{\delta(\gamma)}'$ is uniformly coarsely scattered. $Y_{\delta(\gamma)}$ is associated to some $W_{\omega(\gamma)}$ that comes from one of $\{X_\alpha\}$ and $\{Z_\beta\}$. Here, we are considering $\{Y_\delta\}$ from the point of view that if a specific $Y_\delta$ is associated to more than one element of $\{X_\alpha\} \cup \{Z_\beta\}$, say $W_\omega, W'_\omega, \ldots$, there are actually multiple copies of that $Y_\delta$ in $\{Y_\delta\}$, say $Y_\delta = Y'_\delta = \ldots$ so that each $Y_\delta$ is only associated to one $W_\omega$. In other words, $Y_\delta$ can be “reused” in mapping to the union, just as they can be “reused” in mapping to either of the original families. Without loss of generality, say that $W_\omega \in \{X_\alpha\}$. Then $W_\omega = X_\alpha$ for some $\alpha$. We already know that for any $S \subseteq Y_\delta$,

$$E_{\alpha}'(S) \subseteq \sum_{x \in X_\alpha} E_{\alpha}'(\pi_\alpha^{-1}(x[\partial]) \cap S[\partial'],)$$

so we have the result. \qed
**Corollary 120 (Finite Union Permanence)**

Finite unions of relatively coarsely coherent families are relatively coarsely coherent families.

**Theorem 121**

The product of two relatively coarsely coherent spaces, equipped with an $\ell^p$-norm induced metric $(1 \leq p \leq \infty)$ is relatively coarsely coherent.

**Proof.** Let $X$, $Z$ be two relatively coarsely coherent spaces. Let

$$0 \to E' \to E \to E'' \to 0$$

be a short exact sequence of $Y$-filtered $R$-modules for some given $Y$, and let $\pi : Y \to X \times Z$ be a uniformly expansive map in the sense that there exists a nondecreasing function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d_{X \times Z}(\pi(y), \pi(y')) \leq \ell(d_Y(y, y'))$ for all $y, y' \in Y$. We want that for any $S \subseteq Y$,

$$E'(S) \subseteq \sum_{(x, z) \in X \times Z} E'(\pi^{-1}((x, z)[\partial]) \cap S[\partial']).$$

Since $X$ and $Z$ are relatively coarsely coherent, we know that

$$E'(S) \subseteq \sum_{x \in X} E'(\pi_X^{-1}(x[\partial_X]) \cap S[\partial_X])$$

and

$$E'(S) \subseteq \sum_{z \in Z} E'(\pi_Z^{-1}(z[\partial_Z]) \cap S[\partial_Z]).$$

where $\pi_X$ is the composition of $\pi$ followed by the projection onto the $X$ coordinate, and
similarly for $\pi_Z$. Consider an arbitrary $S \subseteq Y$. These maps are uniformly expansive with respect to the same function $\ell$ since for $(x, z) = \pi(y), (x', z') = \pi(y'),$

$$d_X(x, x') \leq \max\{d_X(x, x'), d_Z(z, z')\} = d_{X \times Z}((x, z), (x', z')) \leq \ell(d_Y(y, y')).$$

We know that $E'(S)$ can be written as belonging to a sum,

$$E'(S) \subseteq \sum_{z \in Z} E'(\pi_Z^{-1}(z[\partial_Z]) \cap S[\partial'_Z]).$$

Define $S_z = \pi_Z^{-1}(z[\partial_Z]) \cap S[\partial'_Z]$. Thus, we can write $E'(S) \subseteq \sum_{z \in Z} E'(S_z)$. Since $X$ is relatively coarsely coherent, we can further write each $E'(S_z)$ as belonging to a sum,

$$E'(S_z) \subseteq \sum_{x \in X} E'(\pi_X^{-1}(x[\partial_X]) \cap S_z[\partial'_Z]).$$

Substituting in the definition of $S_z$ yields

$$E'(S_z) \subseteq \sum_x E'(\pi_X^{-1}(x[\partial_X]) \cap (\pi_Z^{-1}(z[\partial_Z]) \cap S[\partial'_Z])[\partial'_X])$$

$$\subseteq \sum_x E'(\pi_X^{-1}(x[\partial_X]) \cap \pi_Z^{-1}(z[\partial_Z] + \ell(\partial'_X)) \cap S[\partial'_Z + \partial'_X]).$$

Combining this information with our expression for $E'(S)$ yields

$$E'(S) \subseteq \sum_{x \in X} \sum_{z \in Z} E'(\pi_X^{-1}(x[\partial_X]) \cap \pi_Z^{-1}(z[\partial_Z + \ell(\partial'_X)]) \cap S[\partial'_Z + \partial'_X])$$

$$\subseteq \sum_{(x, z) \in X \times Z} E'(\pi^{-1}((x, z)[\partial_X + \partial_Z + \ell(\partial'_X)]) \cap S[\partial'_Z + \partial'_X]).$$
Therefore, \( X \times Z \) is \( \pi \)-scattered with \( \partial = \partial_X + \partial_Z + \ell(\partial'_X), \partial' = \partial'_Z + \partial'_X \), and products of relatively coarsely coherent spaces are relatively coarsely coherent. \( \square \)

**Corollary 122**

Finite direct products of relatively coarsely coherent spaces are relatively coarsely coherent.

**Theorem 123** (Direct Product Permanence)

The direct product of two relatively coarsely coherent families is relatively coarsely coherent.

**Proof.** Let \( \mathcal{X} = \{ X_\alpha \}, \mathcal{Z} = \{ Z_\beta \} \) be relatively coarsely coherent metric families. Let \( \mathcal{Y} = \{ Y_\delta \} \) be a family of metric spaces and \( \pi_\gamma : Y_\delta(\gamma) \to X_\alpha(\gamma) \times Z_\beta(\gamma) \) be uniformly expansive maps with common control function \( \rho_\gamma \), where \( X_\alpha(\gamma) \in \mathcal{X} \) and the \( \gamma \) argument in the subscript is just to indicate that it is being paired with some particular (given) \( Y_\delta(\gamma) \), and similarly for \( Z_\beta(\gamma) \). Since products of relatively coarsely coherent spaces are relatively coarsely coherent, \( X_\alpha(\gamma) \times Z_\beta(\gamma) \) is relatively coarsely coherent, so for any particular \( Y_\delta \) and

\[
0 \to E'_\delta \xrightarrow{f_\delta} E_\delta \xrightarrow{g_\delta} E''_\delta \to 0
\]

any exact sequence of \( Y_\delta \)-filtered lean, insular \( R \)-modules where \( f_\delta, g_\delta \) are bicontrolled with the same control bounds, we have that

\[
E'_\delta(S) \subseteq \sum_{(x,z) \in X_\alpha(\delta) \times Z_\beta(\delta)} E'_\delta(\pi_\delta^{-1}((x,z)[\partial]) \cap S[\partial']'),
\]

where \( \partial, \partial' \) come from the proof for direct products of relatively coarsely coherent metric spaces. \( \square \)
Corollary 124

Finite direct products of relatively coarsely coherent families are relatively coarsely coherent as families.
4.1 The Coherences

4.1.1 Comparing Coherences

We are now interested in determining the relationships between the three competing coherence notions. We consider these relationships one implication at a time, and make reference to the arguments, specifically the resulting leanness/scattering/$\pi$-scattering constants throughout the thesis.

**Theorem 125**

Relative coarse coherence implies coarse coherence.

**Proof.** Let $X \in \mathcal{C}^{\text{rel}}$. Since the definition of relatively coarsely coherent must hold for any $Y, \pi : Y \rightarrow X$ uniformly expansive, we can take $Y = X$ and $\pi = \text{id}_X$. The definition yields that, given a short exact sequence

$$0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0$$
of $X$-filtered $R$-modules where $f$ and $g$ are bicontrolled maps, $E$ is lean, and $E''$ is insular, we have that for any $S \subseteq X$,

$$E'(S) \subseteq \sum_{x \in X} E'(\text{id}_X^{-1}(x[\partial]) \cap S[\partial'])$$

$$= \sum E'(x[\partial] \cap S[\partial'])$$

$$\subseteq \sum E'(x[\partial]).$$

Clearly then, since $X$ is a subset of itself,

$$E'(X) \subseteq \sum_{x \in X} E'(x[\partial]),$$

as desired. □

**Theorem 126**

**Relative coarse coherence implies coherence.**

**Proof.** Let $X \in \mathcal{C}^{\text{rel}}$. We can again take $Y = X$, $\pi = \text{id}_X$. The definition yields that, given a short exact sequence

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

of $X$-filtered $R$-modules where $f$ and $g$ are bicontrolled maps, $E$ is lean, and $E''$ is insular,
we have that for any \( S \subseteq X \),

\[
E'(S) \subseteq \sum_{x \in X} E'(\text{id}_X^{-1}(x[\partial]) \cap S[\partial']) = \sum_{x \in X} E'(x[\partial] \cap S[\partial']) \subseteq \sum_{x \in S} E'(x[\partial + \partial']).
\]

The last inclusion holds since for any \( x' \in x[\partial] \cap S[\partial'] \), \( x' \) is at most \( \partial' \) from some element of \( S \), say \( s \in S \), and at most \( \partial \) from \( x \). \( \square \)

**Theorem 127**

Coherence implies coarse coherence.

**Proof.** Let \( X \) be a coherent space. The definition yields that, given a short exact sequence

\[
0 \rightarrow E' \overset{f}{\rightarrow} E \overset{g}{\rightarrow} E'' \rightarrow 0
\]

of \( X \)-filtered \( R \)-modules where \( f \) and \( g \) are bicontrolled maps, \( E \) is lean, and \( E'' \) is insular, we have that for any \( S \subseteq X \),

\[
E'(S) \subseteq \sum_{x \in S} E'(x[\delta]).
\]

Clearly, \( S = X \) implies the result. \( \square \)

Consequently, at this stage, the relationship between these three properties can be visualized as in the triangle below.

Relative Coarse Coherence \( \longrightarrow \) Coherence \( \longrightarrow \) Coarse Coherence
As you may recall, our definition of coherence was derived from the “missing statement” from Section 3.1 of [12], and was the impetus for the definitions of coarse and relative coarse coherence. Both coarse and relative coarse coherence were developed as more computable comparable alternatives to coherence. Interestingly, in terms of generality, coherence is sandwiched between these two competing offshoots. Relative coarse coherence is the most restrictive of the three notions, while coarse coherence is the most general.

It is a result of [20] that

**Theorem 128**

*Coarse coherence implies weak coherence.*

We can therefore recast our diagram as a series of implications:

Relative Coarse Coherence $\implies$ Coherence $\implies$ Coarse Coherence $\implies$ Weak Coherence

Thus, all groups and spaces found to be coherent, coarsely coherent, or relatively coarsely coherent in this text are also weakly coherent.

### 4.1.2 The Coherences and $\mathbb{R}$

To demonstrate the computational similarity of coherence and relative coarse coherence, full proofs are given below for the coherence and relative coarse coherence of $\mathbb{R}$. However, rather than belabor the point, the coarse coherence of $\mathbb{R}$ is given as a corollary with no proof.
Proposition 129

The real number line with the usual Euclidean metric $\mathbb{R}$ is coherent.

Proof. Let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of $\mathbb{R}$-filtered $R$-modules with $f$, $g$ bicontrolled with $\text{fil}(f)$, $\text{fil}(g) \leq b$ for some $b \geq 0$, $E$ $D$-lean, and $E''$ $d$-insular. We must show that $E'$ is $D'$-lean for some fixed $D' > 0$. We can take $E' = \ker g$ since the sequence above is exact. Let $S \subseteq \mathbb{R}$ be given. We need to demonstrate that

$$E'(S) \subseteq \sum_{x \in S} E'(x[D']).$$

(4.1)

Let $k \in E'(S) \subseteq E' = \ker g$. We can write $k$ as a sum $k_1 + k_2$ with $k_1 \in E'((-\infty, 0))$ and $k_2 \in E'((0, \infty))$. Since $k \in \ker g$,

$$g(k) = g(k_1 + k_2) = g(k_1) + g(k_2) = 0 \Rightarrow g(k_1) = -g(k_2).$$

Further, $g(k_1) \in E''((-\infty, b])$ and $g(k_2) \in E''([-b, \infty))$, since $g$ is bicontrolled with $\text{fil}(g) \leq b$. By the insularity of $E''$, the intersection of these submodules is contained in $E''([-b - d, b + d])$. $g$ is bicontrolled, so we can find $\overline{k} \in E([-2b - d, 2b + d])$ satisfying $g(\overline{k}) = g(k_1) = -g(k_2)$. Therefore, $k_1 - \overline{k}, k_2 + \overline{k} \in \ker g = E'$.

Emulating this procedure of "chopping up" the support of $k$, we can write $k = \sum_{x \in \mathbb{R}} k_x$,
\( k_x \in E(x[D]) \), since \( E \) is \( D \)-lean. Define

\[
k_i = \sum_{x \in [(5b+3d)i,(5b+3d)(i+1))} k_x.
\]

Now, \( k = \sum_{i \in \mathbb{Z}} k_i \), each \( k_i \in E((5b+3d)i - D, (5b+3d)(i + 1) + D) \). Since \( k = \sum_{i \in \mathbb{Z}} k_i = k_i + \sum_{j \neq i} k_j \in \ker g = E' \), we define \( s_{i,l} \) and \( s_{i,r} \) in \( E'' \) satisfying \( g(k_i) = s_{i,l} + s_{i,r} \) by

\[
s_{i,l} = -g\left(\sum_{j < i} k_j\right) \in E''((5b+3d)i - D - b, (5b+3d)i + D + b)) \quad \text{and}
\]

\[
s_{i,r} = -g\left(\sum_{j > i} k_j\right) \in E''((5b+3d)(i + 1) - D - b, (5b+3d)(i + 1) + D + b)).
\]

These definitions yield that \( s_{i,r} = -s_{i+1,l} \) for all \( i \in \mathbb{Z} \). By the \( d \)-insularity of \( E'' \), the intersection of these modules is contained in \( E''((5b+3d)i - D - b - d, (5b+3d)(i + 1) + D + b + d]\). Because \( g \) is bicontrolled, we can find \( \overline{k_i} \in E((5b+3d)i - D - 2b - d, (5b+3d)(i + 1) + D + 2b + d) \) satisfying \( g(\overline{k_i}) = s_{i,r} = -s_{i+1,l} \). Thus, \( k_i - \overline{k_i} + \overline{k_{i+1}} \in E' = \ker g \) for all \( i \).

Define \( \hat{k}_i = k_i - \overline{k_i} + \overline{k_{i+1}} \in \ker g = E' \). We can therefore write \( k = \sum_{i \in \mathbb{Z}} \hat{k}_i \), a sum of kernel elements. In fact,

\[
\hat{k}_i \in E' \cap E((5b+3d)i - D - 2b - d, (5b+3d)(i + 1) + D + 2b + d)). \tag{4.2}
\]

Since \( f \) is bicontrolled with \( \text{fil}(f) \leq b \), each \( \hat{k}_i \) can be viewed as an element of \( E'(i(5b+3d)i - D - 3b - d, (5b+3d)(i + 1) + D + 3b + d) \).

Suppose \( i \in \mathbb{Z} \) is an index such that \([(5b+3d)i, (5b+3d)(i + 1)] \cap S[b] \neq \emptyset \). Let \( x \) be an
element of this intersection. Then \( k_i \in E(x[2(5b+3d)]) = E(10b+6d) \), by the above. By our argument, \( \hat{k}_i \in E(x[10b+6d+D+3b+d]) \) implies \( \hat{k}_i \in E'(x[10b+6d+D+3b+d+b]) \), where the last \( b \) is a result of taking \( x \) an element of \( S \) instead of \( S[b] \). Adding these terms yields that \( \hat{k}_i \in E'(x[14b+7d+D]) \). So each element of \( E'(S) \) can be written as a sum of elements \( \hat{k}_i \), each in some \( E'(x[14b+7d+D]) \), \( x \in S \), as desired. Therefore, \( E' \) is \( (14b+7d+D) \)-lean and \( \mathbb{R} \) is coherent. \( \square \)

Corollary 130

The integers with the usual Euclidean metric \( \mathbb{Z} \) is coherent.

Proof. By Lemma 95, subspaces of coherent spaces are coherent. \( \square \)

Proposition 131

The real number line with the usual Euclidean metric \( \mathbb{R} \) is relatively coarsely coherent.

Proof. Consider the following argument that \( \mathbb{R} \) is relatively coarsely coherent. Let \( X = \mathbb{R} \), let \( Y \) be a metric space, and let \( \pi : Y \to X \) uniformly expansive with control function \( \rho \) be given. Let

\[
0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0
\]

be a short exact sequence of \( Y \)-filtered \( R \)-modules, where \( f, g \) are bicontrolled maps with \( \text{fil}(f), \text{fil}(g) \leq b \) for some \( b \geq 0 \), and \( E \) and \( E'' \) are both \( D \)-lean and \( d \)-insular.

We proceed analogously to the proof for coherence. Apply the \( \pi \)-induced \( X \)-filtration to the short exact sequence. Denote the modules as filtered by \( X \) with subscript \( X \). Let \( S \subseteq Y \) be given, and let \( k \in E'(S) \subseteq E' = \ker g \). Define the interval \( I_i \) as follows:

\[
I_i = [(5b+3d)i, (5b+3d)(i+1)].
\]
The \( \{ I_i \} \) cover \( X \), so \( \{ \pi^{-1}(I_i) \} \) cover \( Y \). Therefore, we can write \( S = \bigcup S_i \) where

\[
S_i = \pi^{-1}(I_i) \cap S.
\]

Since \( E \) is \( D \)-lean,

\[
E(S) \subseteq \sum_{y \in S} E(x[D]).
\]

Therefore, \( k = \sum_{y \in S} k_y, k_y \in E(y[D]) \). Define

\[
k_i = \sum_{x \in \pi^{-1}(I_i) \cap S = S_i} k_x.
\]

Then \( k = \sum k_i \), where each \( k_i \in E(S_i[D]) \).

Since \( k = \sum_{i \in \mathbb{Z}} k_i = k_i + \sum_{j \neq i} k_j \in \ker g = E' \), we define \( s_{i,l} \) and \( s_{i,r} \) in \( E'' \) satisfying \( g(k_i) = s_{i,l} + s_{i,r} \) by

\[
s_{i,l} = -g\left( \sum_{j < i} k_j \right) \in E''(S_i[D + b]) \quad \text{and} \quad s_{i,r} = -g\left( \sum_{j > i} k_j \right) \in E''(S_{i+1}[D + b]).
\]

These definitions yield that \( s_{i,r} = -s_{i+1,l} \) for all \( i \in \mathbb{Z} \). By the \( d \)-insularity of \( E'' \), the intersection of these modules is contained in \( E''(S_i[5b + 3d + D + b + d]) \). Because \( g \) is bicontrolled, we can find \( \overline{k}_i \in E(S_i[5b + 3d + D + 2b + d]) \) satisfying \( g(\overline{k}_i) = s_{i,r} = -s_{i+1,l} \).

Thus, \( k_i - \overline{k}_i + \overline{k}_{i+1} \in E' = \ker g \) for all \( i \).

Define \( \hat{k}_i = k_i - \overline{k}_i + \overline{k}_{i+1} \in \ker g = E' \). We can therefore write \( k = \sum_{i \in \mathbb{Z}} \hat{k}_i \), a sum of
kernel elements. In fact,
\[
\tilde{k}_i \in E' \cap E(S_i[5b + 3d + D + 2b + d]).
\]

Since \( f \) is bicontrolled with \( \text{fil}(f) \leq b \), each \( \tilde{k}_i \) can be viewed as an element of \( E'(S_i[5b + 3d + D + 3b + d]) = E'(S_i[8b + 4d + D]) \). Recall now the definition of \( S_i \), so
\[
E'(S_i[8b + 4d + D]) = E'(\pi^{-1}(I_i[8b + 4d + D] \cap S[8b + 4d + D])) \subseteq E'(\pi^{-1}(I_i[\rho(8b + 4d + D)] \cap S[8b + 4d + D])).
\]

Further recall that \( I_i = [(5b + 3d)i, (5b + 3d)(i + 1)] = (5b + 3d)(i + \frac{1}{2}) \left[ \frac{5b + 3d}{2} \right] \) and each \( (5b + 3d)(i + \frac{1}{2}) \in X = \mathbb{R} \). Therefore, we have that
\[
E'(S_i[8b + 4d + D]) \subseteq E'(\pi^{-1}((5b + 3d)(i + \frac{1}{2}) \left[ \frac{5b + 3d}{2} \right] \rho(8b + 4d + D)] \cap S[8b + 4d + D])) \subseteq E'(\pi^{-1}(5b + 3d) \left( i + \frac{1}{2} \right) \left[ \frac{5b + 3d + \rho(8b + 4d + D)}{2} \right] \cap S[8b + 4d + D]).
\]

Denote \( x_i := (5b + 3d) \left( i + \frac{1}{2} \right) \). Then
\[
E'(S_i[8b + 4b + D]) \subseteq E'(\pi^{-1}(x_i \left[ \frac{5b + 3d}{2} + \rho(8b + 4d + D) \right] \cap S[8b + 4d + D]).
\]
As a result of this computation, we have that for each \( i \in \mathbb{Z} \),

\[
\tilde{k}_i \in E'(\pi^{-1}(x_i \left[ \frac{5b + 3d}{2} + \rho(8b + 4d + D) \right]) \cap S[8b + 4d + D]),
\]

so

\[
E'(S) \subseteq \sum_{i \in \mathbb{Z}} E'(\pi^{-1}(x_i \left[ \frac{5b + 3d}{2} + \rho(8b + 4d + D) \right]) \cap S[8b + 4d + D]) \subseteq \sum_{x \in \mathbb{R}} E'(\pi^{-1}(x \left[ \frac{5b + 3d}{2} + \rho(8b + 4d + D) \right]) \cap S[8b + 4d + D]),
\]

as desired.

\[\Box\]

**Corollary 132**

The integers with the usual Euclidean metric \( \mathbb{Z} \) is relatively coarsely coherent.

**Proof.** \( \mathbb{Z} \) is relatively coarsely equivalent to \( \mathbb{R} \) and by Theorem 114, relative coarse coherence is a coarse invariant. \[\Box\]

**Corollary 133**

Since \( \mathbb{R} \) is relatively coarsely coherent and, by Theorem 121, direct products of relatively coarsely coherent spaces are relatively coarsely coherent, we now know that for any finite \( n \geq 1 \), \( \mathbb{R}^n \) is relatively coarsely coherent.

**Corollary 134**

Since \( \mathbb{R}^n \) is relatively coarsely coherent and relative coarse coherence implies coherence, \( \mathbb{R}^n \) is coherent.
Corollary 135

Since $\mathbb{R}^n$ is relatively coarsely coherent, $\mathbb{R}^n$ is coarsely coherent.

Definition 136 A polyhedron $P \subseteq \mathbb{R}^n$ is the set of solutions to $m$ linear inequalities in $n$ variables:

$$a_{1,1}x_1 + \ldots + a_{1,n}x_n \leq b_1,$$

$$a_{2,1}x_1 + \ldots + a_{2,n}x_n \leq b_2,$$

$$\vdots$$

$$a_{m,1}x_1 + \ldots + a_{m,n}x_n \leq b_m.$$

That is, $P$ is the set of $\vec{x} \in \mathbb{R}^n$ such that $(A\vec{x})_i \leq \vec{b}_i$ for $A = (a_{i,j})$ a real $(m \times n)$-matrix and $\vec{b} = (b_i)$ a real $m$-vector.

Corollary 137

All polyhedra are (relatively) coarsely coherent.

4.1.3 Outside Results and Permanence Properties

From Guentner’s paper [23], we have that:

Theorem 138 (Adapted from Guentner, [23])

A property satisfying subspace, finite union, and [general] fibering permanence is closed under group extensions.

Consequently,
Corollary 139
\[ \mathcal{C} \text{ is closed under group extensions.} \]

We will see in Section 4.2.1 that any space with finite asymptotic dimension satisfies the coherence notions we’ve discussed.

One of the main theorems of Kasprowski, Nicas, and Rosenthal in their 2016 paper states that

**Theorem 140** (Kasprowski-Nicas-Rosenthal, Thm. 1.1 [27])

Let \( \mathcal{P} \) be a collection of metric families that satisfies [general] fibering permanence and contains all metric families with finite asymptotic dimension. Then \( \mathcal{P} \) satisfies coarse permanence, finite amalgamation permanence, finite union permanence, union permanence, and limit permanence.

This result is propitious! It yields the following, highly significant corollary, when combined with the preceding results:

**Corollary 141**

Coarse coherence satisfies coarse permanence, finite amalgamation permanence, finite union permanence, union permanence, and limit permanence.

Kasprowski, Nicas, and Rosenthal follow the preceding theorem with the corollary:
Corollary 142 (Kasprowski-Nicas-Rosenthal, Cor. 1.2 [27])

Let \( P \) be a property of metric families that is satisfied by all metric families with finite asymptotic dimension and that is closed under fibering. Then the class of (countable) groups with \( P \) is closed under extensions, direct unions, free products (with amalgam), and relative hyperbolicity. Furthermore, all elementary amenable, all linear, and all subgroups of virtually connected Lie groups have \( P \).

This corollary also applies to coarse coherence!

Corollary 143

The class of (countable) groups with coarse coherence is closed under extensions, direct unions, free products (with amalgam), and relative hyperbolicity. Furthermore, all elementary amenable, all linear, and all subgroups of virtually connected Lie groups are coarsely coherent.

4.2 Relationship with Other Coarse Properties

4.2.1 Finite Asymptotic Dimension

In [9], Carlsson and Goldfarb have constructed a proof that spaces with finite asymptotic dimension are weakly coherent. They make use of a key theorem of Dranishnikov’s:

Theorem 144 (Dranishnikov, Thm. A [16])

A group \( \Gamma \) has finite asymptotic dimension if and only if there is a uniform embedding of \( \Gamma \) in a finite product of locally finite simplicial trees.

We leverage the above property of spaces with finite asymptotic dimension to prove
that finite asymptotic dimension implies relative coarse coherence. Given all of the results comprising the preceding chapter, the argument is straightforward.

Theorem 145

If a group $\Gamma$ with Cayley graph $X$ has finite asymptotic dimension then $X$ (and therefore $\Gamma$) is relatively coarsely coherent.

Proof. Let $X$ have finite asymptotic dimension. Then $X$ uniformly embeds into a finite product of locally finite simplicial trees $P$. Locally finite simplicial trees can be embedded into $\mathbb{R}^2$, and thus $P$ can be embedded into $\mathbb{R}^n$ for some $n \geq 0$. Since $\mathbb{R}^n$ is relatively coarsely coherent and $P$ can be viewed as a subspace of $\mathbb{R}^n$, $P$ is relatively coarsely coherent (Theorem Lemma 116). Since $X$ coarsely embeds into a relatively coarsely coherent space, $X$ is relatively coarsely coherent (Theorem 118). □

Corollary 146

Finite asymptotic dimension implies coherence, coarse coherence.

4.2.2 (s)FDC

Lemma 147

Set $D' = 2D + 2b + 2d$. Let $X = \sqcup_{D'} \text{disjoint } X_\alpha, \{X_\alpha\}_\alpha$ a $B$-bounded metric family. Then $X$ is coherent.

Proof. Let

\[ 0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0 \]

be a short exact sequence of $X$-filtered $R$-modules with $f, g$ bicontrolled, $E D$-lean, $E''$
Let $S \subseteq X$ be given. Define $S_\alpha = S \cap X_\alpha$. By Corollary 88,

$$E'(S) = E'(\sqcup_{D'} S_\alpha) \subseteq \sum_{\alpha} E'(S_\alpha[D]) \subseteq \sum_{\alpha} E'(x_\alpha[2B + D]) \subseteq \sum_{x \in S} E'(x[2B + D]),$$

as desired. □

**Theorem 148**

Set $D' = 2D + 2b + 2d$. Let $X = (\sqcup_{D'-\text{disjoint}} X_\alpha) \cup (\sqcup_{D'-\text{disjoint}} X_\beta), \{X_\alpha\}, \{X_\beta\}$ $B$-bounded.

Then $X$ is coarsely coherent.

**Proof.** Let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of $X$-filtered $R$-modules with $f, g$ $b$-bicontrolled, $E$ $D$-lean, $E''$ $d$-insular. Let $S \subseteq X$ be given, and define $S_\alpha = S \cap X_\alpha, S_\beta = S \cap X_\beta$. By Corollary 86,

$$E'(S) = E'((\sqcup_{D'} S_\alpha) \cup (\sqcup_{D'} S_\beta)) \subseteq E'((\sqcup_{D'} S_\alpha)[D + 2b + d]) + E'((\sqcup_{D'} S_\beta)[D + 2b + d]).$$

By Theorem 152,

$$E'((\sqcup_{D'} S_\alpha)[D + 2b + d]) + E'((\sqcup_{D'} S_\beta)[D + 2b + d]) \subseteq \sum_{x \in S} E'(x[2B + 2D + 2b + d]),$$

as desired. □
Corollary 149

Let $X$ have straight finite decomposition complexity. Then $X$ is coherent.

Proof. Let $X$ be a metric space with finite decomposition complexity. Let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of $X$-filtered $R$-modules with $f, g$ $b$-bicontrolled, $E$ $D$-lean, $E''$ $d$-insular. Define $D' = 2D + 2C + 2b + 2d$. We argue inductively. At every stage of the FDC “game,” take $R = D'$. Since $X$ has finite decomposition complexity, we know after $n \geq 1$ rounds a metric family $\mathcal{Y}_{n-1}$ will $R$-decompose over a bounded metric family $\mathcal{Y}_n$.

By Theorem 153, if $n = 1$, $X$ is coherent. If $n = 2$,

$$X = (\bigsqcup R X_\alpha) \cup (\bigsqcup R X_\beta) = ((\bigsqcup R X'_\alpha) \cup (\bigsqcup R X''_\alpha)) \cup ((\bigsqcup R X'_\beta) \cup (\bigsqcup R X''_\beta)),$$

where, by Theorem 148 and Corollary 86, $X$ is coherent. We can continue at each stage writing $X$ as a union of $R$-disjoint unions of metric spaces that are unions of $R$-disjoint unions of metric spaces that are $R$-disjoint unions of metric spaces... $n$ times, that we know to be coarsely coherent by Theorem 148 and Corollary 86. □

Corollary 150

Finite decomposition complexity implies coherence.

Remark 151 Since coherence implies coarse coherence, we now have that straight finite decomposition complexity implies coarse coherence. However, the statements about dis-
joint unions of bounded families that cover $X$ are utilized later. Therefore, the arguments are repeated here for reference, though they may be skipped for the sake of concise prose.

**Lemma 152**

Set $D' = 2D + 2b + 2d$. Let $X = \sqcup_{D'\text{-disjoint}} X_\alpha$, $\{X_\alpha\}_\alpha$ a $B$-bounded metric family. Then $X$ is coarsely coherent.

**Proof.** Let

$$0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0$$

be a short exact sequence of $X$-filtered $R$-modules with $f, g$ $b$-bicontrolled, $E$ $D$-lean, $E''$ $d$-insular. By Corollary 88,

$$E'(X) = E'(\sqcup_{D'} X_\alpha) \subseteq \sum_{\alpha} E'(X_\alpha[D]) \subseteq \sum_{\alpha} E'(x_\alpha[2B + D]) \subseteq \sum_{x \in X} E'(x[2B + D]),$$

as desired. □

**Theorem 153**

Set $D' = 2D + 2b + 2d$. Let $X = (\sqcup_{D'\text{-disjoint}} X_\alpha) \cup (\sqcup_{D'\text{-disjoint}} X_\beta)$, $\{X_\alpha\}, \{X_\beta\}$ $B$-bounded. Then $X$ is coarsely coherent.

**Proof.** Let

$$0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0$$
be a short exact sequence of $X$-filtered $R$-modules with $f$, $g$ $b$-bicontrolled, $E D$-lean, $E''$ $d$-insular. By Corollary 86,

$$E'(X) = E'((\sqcup D'X_\alpha) \cup (\sqcup D'X_\beta)) \subseteq E'((\sqcup D'X_\alpha)[D + 2b + d]) + E'((\sqcup D'X_\beta)[D + 2b + d]).$$

By Theorem 152,

$$E'((\sqcup D'X_\alpha)[D + 2b + d]) + E'((\sqcup D'X_\beta)[D + 2b + d]) \subseteq \sum_{x \in X} E'(x[2B + 2D + 2b + d]),$$

as desired. \hfill \Box

**Corollary 154**

Let $X$ have straight finite decomposition complexity. Then $X$ is coarsely coherent.

**Proof.** Let $X$ be a metric space with finite decomposition complexity. Let

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

be a short exact sequence of $X$-filtered $R$-modules with $f$, $g$ $b$-bicontrolled, $E D$-lean, $E''$ $d$-insular. Define $D' = 2D + 2C + 2b + 2d$. We argue inductively. At every stage of the FDC "game," take $R = D'$. Since $X$ has finite decomposition complexity, we know after $n \geq 1$ rounds a metric family $\mathcal{Y}_{n-1}$ will $R$-decompose over a bounded metric family $\mathcal{Y}_n$. By Theorem 153, if $n = 1$, $X$ is coarsely coherent. If $n = 2$,

$$X = (\sqcup R X_\alpha) \cup (\sqcup R X_\beta) = (\sqcup R X'_\alpha) \cup (\sqcup R X'_\beta) \cup (\sqcup R X''_\alpha) \cup (\sqcup R X''_\beta),$$
where, by Theorem 153 and Corollary 86, $X$ is coarsely coherent. We can continue at each stage writing $X$ as a union of $R$-disjoint unions of metric spaces that are unions of $R$-disjoint unions of metric spaces that are $R$-disjoint unions of metric spaces... $n$ times, that we know to be coarsely coherent by Theorem 153 and Corollary 86.

\[ \square \]

**Corollary 155**

*Finite decomposition complexity implies coarse coherence.*

The work of Kasprowski, Nicas, and Rosenthal shows that regular finite decomposition complexity, a particular subset of finite decomposition complexity, has finite quotient permanence [27]. It is not known whether or not finite decomposition complexity satisfies finite quotient permanence. Theorem 111 demonstrates that coarse coherence also has finite quotient permanence. Therefore, while finite quotient permanence for FDC is unknown, we can summarize the situation as in the following diagram.

\[
\begin{array}{ccc}
\text{regular FDC} & \longrightarrow & \text{FDC} \\
\uparrow & \uparrow & \uparrow \\
\text{FQP} & \text{FQP} & \text{FQP} \\
\text{FQP = Finite Quotient Permanence} & & \\
\end{array}
\]

We can further then state that

**Corollary 156**

*The quotient of an FDC group by a finite group is coarsely coherent.*

---

### 4.2.3 Property A

It is known that
Theorem 157 (P.W. Nowak, Thm. 5.1 [34])

Let $G$ be a finite group of order at least 2. The coarse disjoint union of $\{G^n\}_{n \in \mathbb{N}}$ does not have Property A.

In particular, any coarse disjoint union $\bigsqcup (\mathbb{Z}/2\mathbb{Z})^n$ lacks Property A. However, this space is coarsely coherent.

Theorem 158

Any total space of the family $\{(\mathbb{Z}/2\mathbb{Z})^n\}_{n \in \mathbb{N}}$ is coarsely coherent.

Proof. $\mathbb{Z}/2\mathbb{Z}$ is coarsely coherent since it is bounded. Hence, $(\mathbb{Z}/2\mathbb{Z})^n$ is a finite product of copies of $\mathbb{Z}/2\mathbb{Z}$, and is by Corollary 110 coarsely coherent. Moreover, the proof of Theorem 109 indicates that for any short exact sequence of $(\mathbb{Z}/2\mathbb{Z})^n$-filtered $R$-modules

$$0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0$$

of the type the definition of coarse coherence employs, a scattering constant for $E'$ is given by the maximum of the scattering constants of each factor in the product of $(\mathbb{Z}/2\mathbb{Z})^n$. Since each copy of $\mathbb{Z}/2\mathbb{Z}$ has the same scattering constant, say $\delta \geq 0$, the product $(\mathbb{Z}/2\mathbb{Z})^n$ has scattering constant $\delta$ as well. Consequently, $\{(\mathbb{Z}/2\mathbb{Z})^n\}_{n \in \mathbb{N}}$ is coarsely coherent as a family. By the equivalence of the notion of coarse coherence of a family and coarse coherence of the family’s total spaces (Theorem 100), any total space of the family is coarsely coherent.

Thus, we have an example of a space that lacks Property A, but possesses coarse coherence. At a minimum, we have eliminated the possibility that coarse coherence necessarily
implies Property A. Further, we have eliminated the possibility that weak coherence necessarily implies Property A.

**Corollary 159**

Any total space of the family \( \{ (\mathbb{Z} / 2\mathbb{Z})^n \} \) is weakly coherent.
Chapter 5

Conclusion

5.1 Applications to Weak Coherence

Recall that weak coherence is a property of group rings, defined by Carlsson and Goldfarb in [9], that is leveraged in $K$- and $L$-theory computations, specifically the Borel conjecture. As may be found in [20], coarse coherence implies weak coherence, and all known weakly coherent groups are coarsely coherent. However, weak coherence is a predominantly algebraic property, and much of the structure of the class of weakly coherent groups remains unknown. For example, it is not known if the direct product of two weakly coherent groups is weakly coherent. The work of this dissertation has illuminated much previously unknown structure in the collection of known members of the class of weakly coherent groups. It has been demonstrated that finite direct products of coarsely coherent groups are coarsely coherent, as are free products (with amalgam), semi-direct products, group extensions, etc. We again summarize the results of this inquiry in the following table, and suggest they now be viewed as establishing newly discovered properties of the
known portion of the class of weakly coherent groups, and as a confirmation of coarse invariants known to imply weak coherence.

<table>
<thead>
<tr>
<th>Permanence Property</th>
<th>Geometric Coherence</th>
<th>Coarse Geometric Coherence</th>
<th>Relative Coarse Coherence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse Invariance</td>
<td>✓ [98]</td>
<td>✓ [102]</td>
<td>✓ [115]</td>
</tr>
<tr>
<td>Subspace Perm.</td>
<td>✓ [97]</td>
<td>✓ [104]</td>
<td>✓ [117]</td>
</tr>
<tr>
<td>Union Permanence</td>
<td>✓ [99]</td>
<td>✓ [105]</td>
<td>✓ [119]</td>
</tr>
<tr>
<td>Fibering Permanence</td>
<td>✓ [107]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Limit Permanence</td>
<td>✓ [141]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Direct Product Perm.</td>
<td>✓ [109]</td>
<td>✓ [123]</td>
<td></td>
</tr>
<tr>
<td>Free Product Perm.</td>
<td>✓ [141]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Group Extension Perm.</td>
<td>✓ [139]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Finite Quotient Perm.</td>
<td>✓ [111]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

✓ = Coherence Possesses Permanence Property

<table>
<thead>
<tr>
<th>Coarse Invariant</th>
<th>Geometric Coherence</th>
<th>Coarse Geometric Coherence</th>
<th>Relative Coarse Coherence</th>
</tr>
</thead>
<tbody>
<tr>
<td>FAD</td>
<td>⇐[146]</td>
<td>⇐[146]</td>
<td>⇐[145]</td>
</tr>
<tr>
<td>sFDC</td>
<td>⇐[149]</td>
<td>⇐[154]</td>
<td></td>
</tr>
</tbody>
</table>

⇒ = Coarse Invariant Implies Coherence Type
5.2 Future Avenues of Inquiry

As shown in [20] and reiterated in the preceding section, coarse coherence implies the weak coherence of Carlsson and Goldfarb. Therefore, the results of their work can be leveraged to make statements regarding assembly maps in algebraic $K$-theory (see [10], [11], [12]), though the explicit applications to $K$-theory are beyond the purview of this dissertation. Further avenues of inquiry for this project include exploring those applications, as well as expanding the collection of coarsely coherent groups and metric spaces to which those results pertain.

Following this project, study of the relationships between coarse coherence and other established coarse invariants, including asymptotic property A and coarse embeddability into Hilbert space will be pursued. Additionally, newer coarse invariants (for example, consider Chen, Wang, and Yu’s fibred coarse embeddability into Hilbert space [13]) will be explored.

Also to be investigated are constructions of geometric examples of groups or metric spaces which do not possess coarse coherence. While algebraic constructions exist (see [12]), they are not intelligible or intuitive in the realm of coarse geometry. It is the hope of the author that a more straightforward and geometric non-example may be constructed. In keeping with the Borel conjecture’s roots in manifold topology, the aspherical manifolds of Sapir [37] whose fundamental groups lack property A are of interest and will be considered in this venture.

Finally, characterizing the groups that act appropriately on these coarsely coherent
metric spaces is a topic that I intend to pursue. Analogously to relatively hyperbolic groups (groups with hyperbolic subgroups of finite index that act in a particular given manner on a hyperbolic metric space), the properties of groups that act similarly on coarsely coherent metric spaces will be assessed.


