Essays on public policy analysis using macroeconomic models

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ESSAYS ON PUBLIC POLICY ANALYSIS
USING MACROECONOMIC MODELS

by

Akihiro Nomura

A Dissertation
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Dedication

To Tomomi, for your continued support, encouragement and understanding.
Abstract

This thesis is composed of three separate essays.

The first essay constructs a model with heterogeneous agents and money. In the model, households differ in age and productivity. These households hold two types of assets: capital and money. Money is used to smooth consumption across periods and increase contemporaneous utility. We calibrate the model to match the Japanese economy and find that inflation has different effects across age groups and productivity levels. Under the assumption that the government maintains a balanced budget by adjusting the labor income tax, inflation has positive effects on aggregate capital, labor and welfare. This is mainly because positive inflation transfers resources from money-rich to money-poor households that are typically young and have low productivity.

The second essay investigates effects of a consumption tax hike on aggregate variables, welfare and wealth distribution using a large-scale OLG model calibrated to match the Japanese economy. The model successfully replicates earnings and wealth distributions. Our simulation results suggest that the consumption tax hike has a small impact on the Japanese economy if this hike is used to fund a labor tax reduction and a pension benefits increase. We found, however, that this result highly depends on the assumption regarding how the government reallocates the additional consumption taxes. It is much better to use the additional revenue for the labor tax reduction than for the pension benefits increase because the intertemporal transfers from the young to the old reduce expected lifetime utility. Another main finding is that the effects of the consumption tax hike differ by household income levels. In our benchmark simulation, while wealthy households accumulate more assets, the poor young and the poor old reduce their savings sharply. These asymmetric responses widen wealth inequality, especially within very young ages and retirees.

The last essay constructs an endogenous cash-in-advance constraint model by assuming that
an economy has a financial sector offering services that can facilitate credit transactions. Our steady-state analysis finds that with the endogenous CIA constraint, inflation have a positive effect on capital accumulation (Tobin effect) and the Friedman rule does not hold (i.e., a zero nominal interest rate is not optimal). Then we investigate short-run dynamics. Our stochastic model exhibits an effect to stabilize the economy against transitory money growth shocks. When the representative household does not internalize the transaction technology in its decision making, the Tobin effect disappears in the long run but the short-run stabilization effect still remains.
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Chapter 1

A Life-cycle Model with Heterogeneous Agents and Money

1.1 Introduction

One of the most fundamental questions in monetary economics is how money and inflation affect an economy. It is obvious that policy-makers around the world are pursuing mild inflation, not deflation. For example, the central banks in the U.S., the Euro area and Japan have essentially set an inflation target of 2%. There are two main reasons for setting this sort of positive-rate target: (i) price indices such as the CPI have a positive bias due to measurement error, and (ii) mild but positive inflation rates give policy-makers room for economic stimulus measures when negative shocks hit the economy. These reasons are based on practical aspects of economic policies (Oda, 2016).

In the literature, there is no consensus on the level of optimal inflation. On one hand, a number of economic models indicate that a positive inflation rate is undesirable. In his seminal paper, Friedman (1969) offered the robust theoretical conclusion (known as the Friedman rule) which states that the optimal level of the nominal interest rate should be zero to make the opportunity cost of holding money equal to zero. If the real interest rate is positive, then this famous theorem implies that the optimal inflation rate is negative, which is totally contrary to the policies implemented in many advanced countries. In later works, Lucas and Stokey (1983) and Cooley and Hansen
argued the “distortion of the inflation tax” using cash/credit goods models. If cash goods (e.g., consumption) are subject to a cash-in-advance constraint but credit goods (e.g., leisure) are not, inflation is a kind of tax on cash goods as a household needs money to purchase these goods and inflation increases the opportunity cost of holding money. Inflation makes cash goods more expensive, causing demand shifts to credit goods, and this distortion reduces welfare. Also, in the New Keynesian models in which prices are sticky, the optimal inflation rate should be zero since price dispersion itself (i.e., price changes including not only inflation but also deflation) decreases welfare. Schmitt-Grohé and Uribe (2010) surveyed the previous literature and concluded that the long-term optimal inflation rate is likely to be around zero.

On the other hand, some other researchers have argued that inflation has effects to improve welfare or output. There are two main reasons why positive inflation is desirable. First, constructed a model with capital stock. Inflation reduces the real value of money holdings and relatively increases the value of physical assets, which promotes capital accumulation and leads to an increase in output eventually (Tobin effect). Second, another important effect, called the “distribution effect”, is shown in models such as those of Ireland (2005), Bhattacharya et al. (2005) and Bhattacharya et al. (2008). These models combine a life-cycle structure with inflation tax redistribution schemes for households. If there are households facing a liquidity constraint, then lump-sum transfers through inflation taxes cause redistribution from rich to poor, helping consumption smoothing of the poor households. This effect thus suggests that a positive inflation rate can be optimal. These studies indicate that heterogeneity across agents created by the life-cycle structure is important in discussions of optimal monetary policy.

To study the question, we extend the model to incorporate the demand for money in a life-cycle framework. Households in an economy demand money primarily because its liquidity service facilitates transactions as a medium of exchange, as captured by a money-in-utility function. The model is calibrated to match the Japanese economy, which is still struggling to exit prolonged deflation over two decades. This model differs from the previous liter-

---

1 When prices are sticky and each firm does not adjust its price every period, inflation causes a dispersion of relative prices. Suppose that the representative household’s utility depends on a composite of differentiated goods with different prices. When the relative prices change, the household buys more of the relatively cheaper goods and less of the relatively more expensive goods. Because of diminishing marginal utility, the increase in utility due to consuming more of some goods is always less than the decrease in utility due to consuming less of other goods. Hence, price dispersion reduces utility. (See Aiyagari, 2001.)
ature in that it considers both heterogeneity across generations and heterogeneity within each age group. We think it is very important to introduce heterogeneity in order to study the effects of inflation as inflation has different effects on different types of people. Inflation is similar to a tax on money, and amounts of money holdings are different across households depending on their types. Reflecting the two kinds of heterogeneity (inter-generational and intra-generational heterogeneity), inflation has quantitatively significant implications in our results. We found that inflation can promote capital accumulation and improve welfare, mainly due to redistribution between households through the inflation tax.

The rest of the paper is organized as follows. Section 1.2 reviews the literature on heterogeneous agent models with money. Section 1.3 presents our model and Section 1.4 calibrates it to match the Japanese economy. In Section 1.5, we report the benchmark result with zero inflation and then implement counter-factual experiments that raise the inflation rate exogenously. Section 1.6 briefly concludes.

1.2 Literature Review

Most previous studies have used models that included only a few periods with which to examine the role of money in a heterogeneous agent framework. Bhattacharya et al. (2005) incorporated money into a two-period OLG model in two ways. The first one is a reserve requirement specification and the second one is a transaction cost specification. The study found that money growth with lump-sum transfers has inter-temporal redistribution effects from old to young, which implies the sub-optimality of the Friedman rule. Palivos (2005) constructed a two-period OLG model involving a money-in-utility function and two types of agents with different degrees of altruism towards their descendants. The model shows significant distributional effects and a small but positive optimal inflation rate. Bhattacharya et al. (2009) provided a two-period OLG model with capital and agents who are located on two separate islands and randomly relocated between them. The model exhibits the Tobin effect and a positive optimal inflation rate.

Two-period models can give us good insights about the models’ theoretical aspects, but it is useful to extend them to multi-period OLG models in order to make them more suitable for quantitative analyses. Oda (2016) developed a large-scale OLG model that includes (i) money and
a cash-in-advance constraint, (ii) capital stock, (iii) endogenous labor supply and (iv) a zero lower bound on the nominal interest rate. By calibrating the model to match the Japanese Economy, he found that mild inflation is optimal, suggesting that the Tobin effect and the distribution effect dominate. The paper also suggested that the welfare cost of deflation can be very large due to the Tobin effect, if the economy is close to the zero lower bound on the nominal interest rate. This result is consistent with the fact that the authorities in many countries are desperate to avoid deflation.

Another strand of literature does not include life-cycle heterogeneity. For instance, İmrohoroglu (1992) is an early work which quantitatively examined the welfare cost in an economy with money and uninsurable unemployment risk. In her model, money is a pure saving instrument and thus inflation simply reduces the role of money to smooth consumption. Erosa and Ventura (2002) interpreted money in the context of costly credit transactions. They showed that inflation substantially increases wealth inequality because poorer households tend to have more cash than richer ones, and inflation is a non-linear tax on monetary transactions (they call it a “regressive consumption tax”) with an assumption that credit technology exhibits economies of scale. Akyol (2004) introduced interest bearing bonds as illiquid assets, as well as money as liquid assets. In his model, inflation reduces the self-insurance role of money, as shown in previous literature, but it also has a redistribution effect—from “lucky” agents who received a substantial endowment to agents who were not so “lucky”—which dominates at moderate rates of inflation. His results indicate that the optimal inflation rate is 10%. While the above three studies assume endowment economies, Algan and Ragot (2010) incorporated capital as a production factor and an endogenous labor choice into a money-in-utility model. They calibrated the U.S. economy and found that a combination of money, uninsured income risk and the presence of a borrowing constraint exhibits a new precautionary savings motive, which implies that inflation promotes capital accumulation. As more recent works, Kaplan et al. (2018) constructed the HANK (Heterogeneous Agent New Keynesian) model by introducing the Aiyagari (1994)-type income risk into monopolistic competition.

Among these earlier studies, Huggett (1992), Algan and Ragot (2011) and Oda (2016) are the most relevant to this paper. Our model has a structure that incorporates a many-period OLG model like that of Huggett (1992) and Oda (2016) with the incomplete market monetary model provided by Algan and Ragot (2011). To the best of my knowledge, there is no model combining the heterogeneous agent OLG model with the presence of money. In Oda (2016), heterogeneity
comes only from the life-cycle structure, and heterogeneity within each generation is not taken into account. The model in Algan and Ragot (2010) has heterogeneity from uninsured idiosyncratic risk, but does not have a life-cycle structure. Additionally, no welfare implications are described.

1.3 Model

This section provides a large-scale OLG model like that of Huggett (1996) in which agents face uninsured idiosyncratic productivity risk. Major additional components are the presence of money and an endogenous labor choice.

1.3.1 Demographics

We assume that there is no aggregate uncertainty in this economy. Each household enters the economy at age 20 (denoted by \( j = 1 \)) with no asset, and lives up to the maximum age of 95 (\( J = 16 \)). One period of the model corresponds to five years. All households receive pension benefits after age 65 (\( R = 10 \)). Note that they can continue to work after the age \( R \) if they want.

Households face life-span uncertainty, and \( s_j \) denotes the probability of surviving up to age \( j \) conditional on surviving up to age \( j - 1 \). All of them die at age \( J + 1 \) with probability one, i.e., \( s_{J+1} = 0 \). We assume that demographic patterns are stable. In this case, age-\( j \) households make up a fraction \( \mu_j \in [0,1] \) of the population, with

\[
\mu_{j+1} = \frac{s_{j+1}}{1 + n} \mu_j
\]

where \( \mu_j \) is constant over time and \( n \) is the constant growth rate of new cohort.

Households that died before the maximum age \( J \) may leave accidental bequests, which are collected by the government and equally distributed to all surviving households as lump-sum transfers.

1.3.2 Households

Households maximize their expected lifetime utility.

\[
E_{1,t} \left[ \sum_{j=1}^{J} \beta^{j-1} \left( \prod_{j=1}^{j} s_j \right) u \left( c_{j,t+j-1}^i, m_{j+1,t+j}^i, l_{j,t+j-1}^i \right) \right]
\]
where $c_{j,t}$ is a household $i$’s real consumption at age $j$ in time $t$, $m_{j,t}$ is real money balances, $l_{j,t} \in [0,1]$ is labor supply and $\beta$ is the subjective discount factor. We assume that flow utility $u(\cdot)$ is strictly concave and twice continuously differentiable and satisfies the Inada conditions for each argument, $c$, $m$ and $1 - l$, respectively. According to Walsh (2010), it is standard in the money-in-utility approach to assume that the end-of-period money holdings $(m_{j+1,t+1})$ yield utility.\footnote{This timing assumption is criticized by Carlstrom and Fuerst (2001) arguing it is appropriate to assume that the beginning-of-period money holdings available before going to purchase consumption goods yield utility. In our model, however, the Carlstrom and Fuerst’s assumption technically conflicts with the other assumptions that a household is born with no asset and $\lim_{m \to 0} u_m(\cdot) = \infty$ where $u_m(\cdot)$ is the first partial derivative of $u(\cdot)$ with respect to $m$.
}

Households that are working receive labor income $q_{j,t}^i$, which includes a stochastic part and a deterministic age-specific part. That is,

$$q_{j,t}^i = z_{j,t}^i \eta_j w_{jt}$$

where $z_{j,t}^i$ is the idiosyncratic productivity shock, $\eta_j$ is the deterministic and age-specific mean wage profile and $w_t$ is the average real wage. As we will see soon, $w_t$ is determined at a level that equals marginal productivity of labor in the firm sector. Therefore, $\eta_j$ can be interpreted as a ratio of mean productivity of age-$j$ households relative to overall mean productivity $w_t$.

Let $~z_{j,t}^i$ denote log of $z_{j,t}^i$. We assume that $~z_{j,t}^i$ follows an AR(1) process.

$$~z_{j,t}^i = \rho z_{j-1,t-1} + \epsilon_{j,t}$$

where $\epsilon_{j,t} \sim N(0, \sigma^2)$. The differences across households in their productivity histories generate endogenous cross-sectional distributions of capital and money even within the same age group.

Households’ financial resources consist of the labor income discussed above, capital stock and money held at the previous period, accidental bequests, and social security benefits (only after the age $R$). Households allocate the resources between consumption, next-period capital, and next-period money. The budget constraint in nominal terms is

$$P_t c_{j,t}^i + P_t k_{j+1,t+1} + P_t m_{j+1,t+1} = (1 - \tau_t^i - \phi_t) P_t q_{j,t}^i + (1 + r_t) P_t k_{j,t}^i + P_{-1} m_{j,t} + P_t b_t + P_t ss_j + P_t \tau_t$$

where $P_t$ is the price level, $k_{j,t}^i$ is capital stock, $r_t$ is the real interest rate, $\tau_t$ is the labor income tax.
rate, $b_t$ is accidental bequest transfers from deceased, $\phi_t$ is the pension contribution rate, and $ss_{j,t}$ is social security benefits. $\tau_t$ in the last term is lump-sum transfers financed by the inflation tax, which later will be used in the alternative experiment. For a moment, we set $\tau_t = 0$ and can ignore it. Dividing the above equation by $P_t$, we have

$$c_{j,t}^i + k_{j+1,t+1}^i + m_{j+1,t+1}^i = (1 - \tau_t^i - \phi_t) q_{j,t}^i + (1 + r_t) k_{j,t}^i + \frac{m_{j,t}^i}{1 + \pi_t} + b_t + ss_{j,t} + \tau_t$$

where $\pi_t \equiv \frac{P_t}{P_{t-1}} - 1$ is the inflation rate. We also define the beginning-of-period total assets $a_{j,t}^i$ as

$$a_{j,t}^i \equiv (1 + r_t) k_{j,t}^i + \frac{m_{j,t}^i}{1 + \pi_t}$$

(1.2)

We will use $a_{j,t}$ as a state variable for numerical computation (see Appendix A.1.4). Now the budget constraint is rewritten as

$$c_{j,t}^i + k_{j+1,t+1}^i + m_{j+1,t+1}^i = a_{j,t}^i + (1 - \tau_t^i - \phi_t) q_{j,t}^i + b_t + ss_{j,t} + \tau_t.$$  

(1.3)

Additionally, in this economy households are not allowed to borrow.

$$k_{j+1,t+1}^i \geq 0.$$  

(1.4)

Note that money is always positive because marginal utility of money diverges to infinity as $m_{j,t}^i$ goes to zero by the Inada condition. As long as $1 + r_t > \frac{1}{1 + \pi_t}$, money is strictly dominated by capital as a saving instrument. However, nonetheless money is in demand for its liquidity services.

We will focus on the stationary equilibrium. Then our state variables are age $j$, productivity $z$ and total assets $a$. (Total assets $a$ consist of capital stock $k$ and money $m$.) We denote a state vector as $(x, j)$ where $x \equiv (a, z)$. Let any variable with a prime symbol ($'$) denote a one-period-ahead variable, for example $x' = x_{j+1}$ if $x = x_j$. The recursive formulation of the households’ problem (ignoring individual and time subscripts) is
\[ V(x, j) = \max_{c, k', m', l} u(c, m', l) + \beta s_{j+1} E \left[ V(x', j + 1) \right] z \]  
\text{(1.5)}

s.t. \[ c = a + (1 - \tau^l - \phi) z_j \eta_j w_l + b + ss_j w + \tau - k' - m', \]
\[ a' = (1 + r)k' + \frac{m'}{1 + \pi}, \]
\[ k' \geq 0 \text{ and } l \geq 0. \]

1.3.3 Firm

A representative firm produces consumption goods using a constant returns to scale technology.

\[ Y_t = \zeta K_t^\alpha L_t^{1-\alpha} \]  
\text{(1.6)}

where \( Y_t \) is aggregate real output, \( \zeta \) is the TFP, \( K_t \) is aggregate real capital, \( L_t \) is aggregate effective labor. Given the above production function and factor prices \( r_t \) and \( w_t \), the firm maximizes profits under perfect competition.

\[ \max_{K_t, L_t} Y_t - (r_t + \delta) K_t - w_t L_t \]  
\text{(1.7)}

where \( \alpha \) is the capital share and \( \delta \) is the rate of capital depreciation.

1.3.4 Government

The government conducts monetary and fiscal policies. The fiscal policy is separated further into three: (i) social security system, (ii) collection and redistribution of accidental bequests, and (iii) other government expenditures and taxes including the inflation tax (or seigniorage). Each system is assumed to be balanced and independently operated, which means the government must satisfy three budget constraints. (See the equations (1.13) to (1.15) in the definition of equilibrium.)

Money Supply Rule  Let \( M_t^N \) be aggregate nominal money supply provided by the end of time \( t - 1 \). Then the real quantity of money, \( M_t \), is given by
\[ M_t \equiv \frac{M_t^N}{P_{t-1}}. \]

Suppose that the economy becomes active at time \( t = 0 \) and there exists a given amount of nominal money supply \( (M_1^N) \) at the beginning of time \( t = 0 \).

At each period \( t \), the government issues new money, which equals a constant fraction \( \theta \) of the nominal money supply in the previous period, \( M_t^N \). Then the total quantity of money newly provided at time \( t \) is \( M_{t+1}^N - M_t^N \equiv \theta M_t^N \) (hence \( \theta \) is the nominal money growth rate). In real terms, this is expressed as

\[
\frac{M_{t+1}^N - M_t^N}{P_t} = \frac{\theta M_t^N}{P_t} = \frac{\theta}{1 + \pi_t} M_t
\]

where \( \frac{\theta}{1 + \pi_t} M_t \) represents the seigniorage that the government can obtain at time \( t \). The second equality uses the relationship that \( M_t^N = M_t P_{t-1} \) by the above definition. Also, the money growth path in real terms is expressed as

\[ M_{t+1} = \frac{1 + \theta}{1 + \pi_t} M_t. \quad (1.8) \]

**Pension System** The government operates a pay-as-you-go pension system. All workers must pay pension contributions at the rate \( \phi_t \) out of their labor income as long as they work. All households start to receive pension benefits at the age \( R \), which equal a fraction \( ss \) of the workers’ average wage \( w_t \). Hence \( ss \) is interpreted as a wage replacement rate.

\[
ss_{j,t} = ss_j w_t, \quad ss_j = \begin{cases} 
ss > 0 & \text{if } j \geq R \\
0 & \text{otherwise}.
\end{cases}
\]

Pension spending to retirees is funded only by the pension contributions from workers.
Accidental Bequests The government collects accidental bequests of deceased and equally distributes them to the living households as lump-sum transfers $b_t$.

Other Government Expenditures and Taxes The government purchases $G_t$ are funded by the remaining resources, i.e., the labor income taxes and the seigniorage. (In the alternative experiment, the lump-sum transfers will be added in the expenditure side, while it is just zero in the benchmark model.) We assume that $G_t$ is fixed at some positive level, while any purchase is completely wasteful in the model.

1.3.5 Equilibrium

Our analysis focuses on stationary equilibria and now we ignore the time subscripts $t$. The probability space $(X, \mathcal{X}, \psi_j)$ is defined as follows.

Let $\psi_j$ be a probability measure on a measurable space $(\mathbb{X}, \mathcal{X})$. $\mathbb{X}$ is the state space of $x$, i.e., $\mathbb{X} \equiv A \times Z$ where $A$ and $Z$ are the state space of total assets $a$ and productivity $z$ respectively. $A$ can be further separated into $A = K \times M$ where $K$ and $M$ are the state space of capital $k$ and money $m$ respectively. $\mathcal{X}$ is the Borel $\sigma$-algebra on $\mathbb{X}$. Then, for any subset of $\mathcal{X}$ (denoted by $\mathcal{X}_s$), the measure $\psi_j(\mathcal{X}_s)$ represents a fraction of age-$j$ households whose individual state vector $x$ lies in $\mathcal{X}_s$ as a proportion of all age-$j$ households. In this case, $\mu_j \psi_j(\mathcal{X}_s)$ equals a fraction of such households to total population in the economy.

Since we assume that households have no asset when they enter the economy, a distribution across age-1 households is exogenously determined by an initial distribution of idiosyncratic productivity $z_1$. Then, distributions across households age $j = 2, \ldots, J$ are recursively given by

$$\psi_j(\mathcal{X}_s) = \int_X P(x, j-1, \mathcal{X}_s) d\psi_{j-1} \quad \forall \mathcal{X}_s \in \mathcal{X}$$

where $P(\cdot)$ is a transition function, i.e., $P(x, j, \mathcal{X}_s) \equiv Pr(x_{j+1} \in \mathcal{X}_s | x_j = x)$.

Definition: Given the initial conditions $k_1 = m_1 = 0$ and $M_1^N$, the population growth rate $n$, the conditional survival probabilities $\{s_j\}$ and the money growth rate $\theta$, a stationary equilibrium
consists of

1. decision rules \( c_j = c(x, j), \ k_j+1 = k'(x, j), \ m_j+1 = m'(x, j) \) and \( l_j = l(x, j) \),

2. prices \( r \) and \( w \) and an inflation rate \( \pi \),

3. quantities \( K \) and \( L \),

4. fiscal policies \( \{ G, M, \tau^l, b, \phi, ss \} \) and

5. distributions \( \{ \psi_j \}_{j=1}^J \),

such that

1. given prices and fiscal policies, \( c(x, j), \ k'(x, j), \ m'(x, j) \) and \( l(x, j) \) solve the households' problem (1.5),

2. given prices, \( K \) and \( L \) solve the firm's problem (1.6),

3. markets clear, i.e.,

\[
[\text{capital}] \quad \sum_j \mu_j \int_X k'(x, j) d\psi_j = K', \quad (1.9)
\]
\[
[\text{money}] \quad \sum_j \mu_j \int_X m'(x, j) d\psi_j = M', \quad (1.10)
\]
\[
[\text{labor}] \quad \sum_j \mu_j \int_X z_j \eta_j l_j(x, j) d\psi_j = L, \quad (1.11)
\]
\[
[\text{good}] \quad \sum_j \mu_j \int_X \left[ c(x, j) + k'(x, j) \right] d\psi_j + G = Y + (1 - \delta)K, \quad (1.12)
\]

4. the government satisfies the money growth rule (1.8) and the following budget constraints, i.e.,

(a) pension benefits equal sum of pension contributions:

\[
\sum_{j=R}^J \mu_{jss} \cdot w = \sum_j \mu_j \int_X \phi z_j \eta_j w l(x, j) d\psi_j, \quad (1.13)
\]
(b) bequest transfers equal sum of accidental bequests:

\[ b' = \frac{1}{1+n} \sum_j \mu_j (1 - s_{j+1}) \int_{X} \left[ (1 + r)k'(x, j) + \frac{m'(x, j)}{1 + \pi} \right] d\psi_j, \quad (1.14) \]

(c) government purchases (and lump-sum transfers if doing the alternative experiment) equal sum of labor income taxes and seigniorage:

\[ G + \tau = \sum_j \mu_j \int_X \tau' z_j \eta_j w l(x, j) d\psi_j + \frac{\theta}{1 + \pi} M, \quad (1.15) \]

5. distributions are stationary and consistent with individual behavior, i.e.,

\[ \psi_{j+1}(\mathcal{X}_s) = \int_{X} P(x, j, \mathcal{X}_s) d\psi_j \quad \forall j = \{1, \ldots, J - 1\} \text{ and } \forall \mathcal{X}_s \in \mathcal{X} \]

where \( P(x, j, \mathcal{X}_s) = Pr(x' \in \mathcal{X}_s \text{ s.t. } a' = a'(x, j)|x_j = x) \) and \( a'(x, j) \equiv (1 + r)k'(x, j) + \frac{m'(x, j)}{1 + \pi} \).

1.4 Calibration

In this section the model is calibrated to match the Japanese economy, where how to exit deflation is one of the biggest policy issues over the last two decades. In Japan, the average rate of inflation between 2000-2016 was zero, measured by the Consumer Price Index (CPI, all items). Therefore, we set the money growth rate \( \theta \) at zero as our benchmark calculation, which means the inflation rate \( \pi \) equals zero as well in the steady state.

We assume the flow utility function has a general CES specification, following Chari et al. (2000) and Algan and Ragot (2010).

\[ u(c_{j,t}, m_{j+1,t+1}, l_{j,t}) = \left[ \left( \frac{\omega e^{\frac{\nu-1}{\nu}}}{\nu} + (1 - \omega) m_{j+1,t+1}^{\frac{\nu-1}{\nu}} \right)^{\frac{\nu}{\nu-1}} (1 - l_{j,t})^\gamma \right]^{1-\sigma} \quad (1.16) \]

where \( \sigma \) is the parameter of relative risk aversion, or the inverse of inter-temporal elasticity, \( \omega \) is the weight on consumption relative to money, \( \nu \) is the elasticity of money demand with respect to the interest rate, and \( \gamma \) is the weight on leisure.

The idiosyncratic productivity process (1.1) is approximated by a first-order Markov chain with \( S \) possible states of \( \tilde{z} \) by using the method provided by Tauchen (1986). We set \( S = 15 \).
The following two tables are lists of the exogenous parameters (Table 1.1) and the calibrated parameters (Table 1.2). Note that each value in the tables is expressed in annual terms. Therefore, for some parameters or calibration targets such as the depreciation rate $\delta$ and the capital-output ratio $K/Y$, we have to convert them into a five-year frequency when we compute the equilibrium.\footnote{For example $\delta = 1 - (1 - 0.089)^5 \approx 0.37$ and $\frac{K}{Y} = \frac{2.5}{5} = 0.5$. For the idiosyncratic productivity process, first we generated the Markov chain transition matrix based on the annual parameters and then multiplied it five times.}

### 1.4.1 Exogenous Parameters

Table 1.1 summarizes the exogenous parameters which are from literature or data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Demographics, etc.</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Population growth rate $n$</td>
<td>0.0</td>
<td>Population Estimates etc., MIC.</td>
</tr>
<tr>
<td>Conditional survival probability ${s_j}_{j=1}^J$</td>
<td>-</td>
<td>Japanese Mortality Database, IPSS</td>
</tr>
<tr>
<td><strong>Preference</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk aversion $\sigma$</td>
<td>2.0</td>
<td>Yamada (2012) etc.</td>
</tr>
<tr>
<td>Interest rate elasticity $\nu$</td>
<td>0.5</td>
<td>Algan and Ragot (2010)</td>
</tr>
<tr>
<td><strong>Technology</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capital share $\alpha$</td>
<td>0.362</td>
<td>Hayashi and Prescott (2002)</td>
</tr>
<tr>
<td>Depreciation rate $\delta$</td>
<td>0.089</td>
<td></td>
</tr>
<tr>
<td>TFP $\zeta$</td>
<td>1</td>
<td>(Normalization)</td>
</tr>
<tr>
<td><strong>Labor income</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean age-wage profile ${\eta_j}_{j=0}^J$</td>
<td>-</td>
<td>Lise et al. (2013)</td>
</tr>
<tr>
<td>Persistency of shocks $\rho$</td>
<td>0.97</td>
<td>Yamada (2012)</td>
</tr>
<tr>
<td>S.D. of $z$ at age 20 $\sigma_z$</td>
<td>0.24</td>
<td>Lise et al. (2014)</td>
</tr>
<tr>
<td><strong>Government policy</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expenditure-output ratio $G/Y$</td>
<td>0.1938</td>
<td>National Accounts, Cabinet Office</td>
</tr>
<tr>
<td>Pension contribution rate $\phi$</td>
<td>0.18</td>
<td>MHLW</td>
</tr>
</tbody>
</table>

Note: MIC: Ministry of Internal Affairs and Communications.  
IPSS: National Institute of Population and Social Security Research  

**Demographics**  The population growth rate $n$ is a simple average of annual rates from 2000 to 2016, which are surveyed in the Population Census every five years and are estimated for intercensal years. The conditional survival probability is calculated from the “Death Rates” published by the National Institute of Population and Social Security Research, which is the public organization supervised by the Ministry of Health, Labour and Welfare (MHLW). We took an average from 2005 to 2014 for each age group.
Technology and Preferences  The technology parameters $\alpha$ and $\delta$ are borrowed from a study by Hayashi and Prescott (2002), which calibrates a dynamic general equilibrium model to match the Japanese economy in the 1990’s. The level of total factor productivity ($\zeta$) is normalized at one.

The preference parameters are borrowed from literature as well. The risk aversion parameter $\sigma$ is set to 2.0. This value is used in many studies on the Japanese economy, such as Yamada (2012), as with in many other countries. The interest rate elasticity with respect to money demand, $\nu$, is from Algan and Ragot (2010). Since they calibrate their model to match the U.S. economy, our implicit assumption is that Japanese households’ preference about interest rates and money demand is the same as the American’s one. Later we will discuss the sensitivity to $\nu$ in Section 1.5.3.4.

Labor Income Process  For the age-wage profile $\eta_{j,j=1}^J$, we use data for male workers provided by Lise et al. (2014), who document main features of distributions of wages, earnings and consumption in Japan since the early 1980’s (Figure 1.1). Since the wage data reported in the paper is limited to ages 25-59, we obtained values of wages for workers under age 25 and aged 60 and over by extrapolation using approximation with third degree polynomials.

![Figure 1.1: Life-cycle Wage Profile](image)

We also use the Lise et al. (2014)’s data to calculate the standard deviation of wage at age 20, $\sigma_{z_1}$. The persistency parameter $\rho$ is borrowed from Yamada (2012) which analyzes the increases in earnings and consumption inequality in the 1990’s in Japan by using a large-scale OLG model. While Yamada (2012) assumes that his standard error of $\epsilon_t$ in an AR(1) process gradually increases

---

5 The extrapolated wage profile reaches to zero at age 72. However, seeing data on employment rates by age group, employment rates over age 70 were about 13% in the 2000’s (Labour Force Survey, MHLW), which is not so close to zero. Hence, our wage-profile approximation for over age 70 may be a bit underestimation.
as age increases, in our model we adopt the manner which Huggett (1996) did in his calibration. That is, we choose the value of $\sigma_\epsilon$ so that the model matches the actual earnings Gini for the overall economy, as shown in Table 1.2.

**Fiscal Policies** The government expenditure per real GDP ($G/Y$) is an average of the national account data between 2000-2015. The pension contribution rate $\phi$ is set to 18%, which is the earnings-proportional rate for employed workers in the private sector. The wage replacement rate $ss$ and the labor income tax rate $\tau^l$ are endogenously determined to balance the government budget constraints (1.13) and (1.15), respectively.

### 1.4.2 Calibration Targets

The remainings are the calibrated parameters which are set to match data moments (Table 1.2).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Target</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount factor</td>
<td>$\beta$</td>
<td>$K/Y = 2.5$ ($\beta = 0.992$)</td>
</tr>
<tr>
<td>Share parameter</td>
<td>$\omega$</td>
<td>$M/Y = 0.26$ ($\omega = 0.998$)</td>
</tr>
<tr>
<td>Weight on leisure</td>
<td>$\gamma$ Working time below age 60 = 0.375</td>
<td>Labour Force Survey, MHLW.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($\gamma = 1.531$)</td>
</tr>
<tr>
<td>S.D. of $\epsilon$</td>
<td>$\sigma_\epsilon$ Earnings Gini = 0.235</td>
<td>Lise et al. (2013)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($\sigma_\epsilon = 0.0483$)</td>
</tr>
</tbody>
</table>

For the subjective discount factor $\beta$, Hansen and Imrohoroglu (2010) show the capital-output ratio in the 2000’s in Japan was 2.3 to 2.8. Therefore we adjust the value of $\beta$ so that the calculated $K/Y$ equals to 2.5. In the benchmark case, we set $\beta = 0.963$, which corresponds to $\beta = 0.992$ on an annual basis.

The preference weight on consumption relative to money, $\omega$, is set so that $M/Y = 0.26$. This target is the average ratio of M1 to GNP until 1994, which is used in Oda (2016) and referred as the standard value of the M1-GNP ratio that represents Japanese household’s preference for holding money.\(^6\) In the benchmark, we have $\omega = 0.998$.

\(^6\)He argues that after 1995, the M1-GNP ratio increased sharply due to low interest rates and deflation, while it was relatively stable before that year.
The preference weight on leisure $\gamma$ is set to match the calculated average working time below age 60. In 2016, the average weekly-working hours for males of ages 20-59 were about 45 hours (Figure 1.2). During ages 25-59, working hours are mostly unchanged and start to decrease after 60, which is the age that many companies set as their employees’ retirement age. Also, the male workers’ average weekly days were 5.1 days in recent years (2013, 2014 and 2015). Based on these observations, we set the target for working time per time endowment at 0.375 ($\approx 45$ working hours/($24 \text{hours} \times 5\text{days}$)). In the benchmark, we have $\gamma = 1.531$.

![Figure 1.2: Life-cycle Work Hours Profile](image)

Source: the Labour Force Survey, MHLW.

As we already discussed above, the standard error $\sigma_\epsilon$ is set to match the earnings Gini, which is the value provided in Lise et al. (2014). We took an average between 1981-2008 for earnings of household’s heads. We have $\sigma_\epsilon = 0.0483$ in the benchmark.

### 1.5 Results

This section provides our benchmark results and then move on to counter-factual experiments to see how inflation affects heterogeneous households and aggregate variables.

---

7In Japan, the minimum retirement age is determined at 60 by law, but the actual retirement age is set by each firm. A survey shows that 80.7% of firms set their retirement age at 60, and 15.2% set at age 65 (General Survey on Working Conditions 2016, MHLW). Since retirees become eligible for public pension benefits after age 65, many companies (94.1%) have a system of “extended employment” or “re-hiring” or both of them.
1.5.1 Benchmark Model

First we assess how well the model replicates the actual economy. Table 1.3 compares the calibrated model with data.

<table>
<thead>
<tr>
<th></th>
<th>Model</th>
<th>Data</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K/Y$</td>
<td>2.5</td>
<td>2.5</td>
<td>(Target)</td>
</tr>
<tr>
<td>$M/Y$</td>
<td>0.26</td>
<td>0.26</td>
<td>(Target)</td>
</tr>
<tr>
<td>$C/Y$</td>
<td>0.62</td>
<td>0.58</td>
<td>National Accounts, Cabinet Office</td>
</tr>
<tr>
<td>$r$</td>
<td>6.2%</td>
<td>4.5%</td>
<td>Kitao (2015)</td>
</tr>
<tr>
<td>Asset Gini</td>
<td>0.478</td>
<td>0.587</td>
<td></td>
</tr>
<tr>
<td>Asset share: Top 1%</td>
<td>3.5%</td>
<td>11.1%</td>
<td>Yamada (2008)</td>
</tr>
<tr>
<td>Top 5%</td>
<td>14.9%</td>
<td>27.6%</td>
<td></td>
</tr>
<tr>
<td>Top 20%</td>
<td>46.5%</td>
<td>58.9%</td>
<td></td>
</tr>
<tr>
<td>Earnings Gini</td>
<td>0.235</td>
<td>0.235</td>
<td></td>
</tr>
<tr>
<td>Labor income tax ($\tau_l$)</td>
<td>30.4%</td>
<td>8.9%</td>
<td>Oda (2016)</td>
</tr>
<tr>
<td>Earnings replacement rate</td>
<td>43.2%</td>
<td>62.3%</td>
<td>MHLW</td>
</tr>
</tbody>
</table>

The calculated consumption-output ratio $C/Y$ is close to the data. The real return on capital $r$ is higher than the Kitao (2015)’s baseline calibration for the Japanese economy in 2010, in which the model does not include the capital income tax as with our model.

For the wealth distribution, the asset Gini in our model is 0.478, which is fairly lower than the data as of 2004 provided by Yamada (2008). The model also fails to replicate the asset share even for top 20%. In the data, the asset share of top 20% wealthy households is about 60%, but it is only 40% in the model. Possibly we need another source of heterogeneity to generate reasonable wealth inequality, for example uncertainty for capital income. In recent works, Benhabib et al. (2015) empirically show that key factors to well approximate the wealth distribution are capital income risk and differential saving rates across wealth levels, as well as earnings risk.

The fiscal parameters do not exhibit a good fit as well. The implied labor tax rate $\tau_l$ is much higher than the data. The possible reason is that the tax structure in our model is too simple. In the benchmark result, labor taxes are only revenue to fund government expenditures as the government has no seigniorage when an inflation rate is zero. There is neither a consumption tax nor a capital income tax.

---

8Recent nominal interest rates on Japanese government bonds are much lower than the calibrated real rates of return on capital. The 10-year JGB yield broke through 1% in 2012 and dived into negative territory in 2016, now fluctuating around zero. However, other macroeconomic models also report that the calibrated real return on capital is much higher than the nominal JGB yields. In the model by Hansen and Imrohoroglu (2016), the before-tax rate of return was roughly in [6.0, 8.0] in 2005-2010 and the after-tax rate was roughly in [4.0, 5.0].
The earnings replacement rate is lower than the data. The wage replacement rate $ss$ is endogenously computed so as to balance the social security system under the fixed pension contribution rate $\phi$, but the public pension system in Japan is not annually balanced only by the worker’s contributions. The pension benefits are financed also by other taxes, government bond issuing or withdrawal from the sovereign pension fund, which are abstracted in this model.

Before moving into the next, it is useful to check the households’ asset portfolio as money has an important role in this model. Assets held by each household largely vary reflecting the intra-generational and inter-generational heterogeneity. Figure 1.3 plots the composition of capital and money by household’s productivity and age. To simplify pictures, we separate the fifteen productivity states into three bins ─ high, middle and low productivity, respectively.

![Figure 1.3: Capital and Money by Productivity and Age](image)

Note that the lines are plotted also for retirees but productivity does not affect their behavior. The “productivities” for retirees just represent average histories of idiosyncratic shocks that were realized before retirement. We can see that higher productivity households have more capital and

---

9 The earnings replacement rate is defined as a ratio of pension benefits to average earnings of households below the age of $R$, which equals $w \cdot ss / \sum_{j=1}^{R-1} \mu_j \int_z \eta_j w(x, j) d\tilde{\epsilon}_j$.

10 Most of households retire at age 65, but there are a limited number of households that continue to work at that age. This is because (i) in our model, workers are allowed to keep their job even after the pension eligible age if they want, and (ii) the wage-age profile $\eta_j$ at age 65 is relatively small but not zero. All people exit the labor market by age 70.
money and the graphs are roughly hump shaped. Since the productivity process has persistency, high-productivity people are more likely to be asset-rich ones in our model.

1.5.2 Counter-factual Experiment: Increase in Inflation Rate

This subsection shows the results of counter-factual experiments, in which the inflation rate is raised from 0% to 10% (in annual terms) and we compare the steady states. In order to capture what factors are affecting the results, we examine the following three models.

- Case 1: exogenous labor supply + lump-sum transfers financed by inflation tax.
- Case 2: exogenous labor supply + labor tax reduction financed by inflation tax.
- Benchmark case: endogenous labor + labor tax reduction financed by inflation tax.

In the benchmark case, the labor-leisure choice is endogenous as before. With inflation, the government uses seigniorage to finance reduction in the labor taxes, which means the labor tax rate $\tau_l$ declines as the inflation rate $\pi$ rises.

Case 1 and Case 2 are more simplified models. In Case 1, the labor choice is exogenous (we use the resulting values of aggregate labor supply by age in the benchmark calculation), and the government redistributes the inflation tax to households in the form of lump-sum transfers. The terms $\tau$ in the households’ budget constraint (1.3) and the GBC (1.15) can be positive with inflation, while the labor tax rate $\tau_l$ is fixed at its benchmark value.

In Case 2, the labor choice is exogenous as with Case 1, but the government’s tax policy is the same as the benchmark case. In Case 2, the labor tax is not distortionary as the labor decision is exogenous. In the benchmark case, however, both inflation and labor taxes are distortionary, which means inflation increases distortion on an individual monetary choice, while reduction in labor taxes lowers distortion on a labor choice.

Changes in welfare compared with the benchmark economy is evaluated by consumption equivalent variation (CEV), which is computed as
\[
\sum_{j=1}^{J} \beta^{j-1} \left( \prod_{j=1}^{J} s_j \right) \int_X u \left[ (1 + CEV)c^0_j, m_{j+1}^0, l^0_j \right] d\psi^0_j = \sum_{j=1}^{J} \beta^j \left( \prod_{j=1}^{J} s_j \right) \int_X u \left( c^\pi_j, m_{j+1}^\pi, l^\pi_j \right) d\psi^\pi_j
\]

(1.17)

where the superscripts “0” in the left-hand side of the equation represent the zero-inflation steady-state values and the superscripts “\(\pi\)” in the right-hand side represent the steady-state values with any inflation rate. The right-hand side is the ex-ante social welfare for the economy (or, expected lifetime utility for a newly born household) with different levels of inflation. Thus the CEV means that how much total consumption is needed to increase in the benchmark such that households are indifferent between living in the benchmark economy and in the counterfactual economy. Positive CEV implies an increase in welfare and negative CEV implies a decrease in welfare.

In this experiment, first we will focus on heterogeneity of agents, in other words, how inflation differently affects households depending on their types. We can utilize the CEV measure to see the effects of inflation for households of different productivity and age. Then, we will see changes in aggregate variables.

1.5.2.1 Effects on Different Productivity and Age Groups

In the following three experiments, we will see that redistribution between households has important implications.

**Case 1: Exogenous Labor & Lump-sum Transfers** Figure 1.4 shows the result of Case 1 experiment, in which labor is exogenous and the inflation tax is redistributed in the form of lump-sum transfers.
The left graph displays changes in net transfers when inflation rises from 0% to 10%. Note that net transfers are defined as “transfers − taxes”, which are expressed as a percentage to the benchmark wage rate. We can see that the change in net transfers differs in productivity and age, and therefore this graph shows the redistribution between different types of households. Now transfers are lump-sum and equal for all households, but the burden of the inflation tax is not even. Since the inflation tax is heavier for money-rich households, there is redistribution from money-rich people to money-poor people.

As Figure 1.3 described before, in our model, middle to old generations have more money than young and very old. Hence the net transfers are positive in young and very old and negative in middle to old (except for middle ages with low productivity). As a result, in the right graph we can see that the young generations are better off (positive CEV) and the remaining population is worse off (negative CEV).

Productivity also affects net transfers and CEV. In our model, high-income households have more money than low-productivity ones (Figure 1.3). Thus, the net transfers are larger for the low-productivity households, which benefit more than the high-productivity ones.

Case 2: Exogenous Labor & Labor Tax Reduction Figure 1.5 shows the result of Case 2 experiment, in which the labor supply choice is still exogenous, but the lump-sum transfers are
replaced by labor tax reduction, i.e., higher inflation reduces the labor income tax rate $\tau^l$.

Compared with Case 1, transfers are unequally distributed as now they depend on individual productivity. Retirees, who do not work, can not receive any benefit from labor tax reduction. Therefore the net transfers to young are larger than in Case 1, which implies young people benefit more and old people lose more. Also we can see that the welfare implication is not monotonic in productivity. Seeing the CEV at age 20 or 25, high-productivity households and low-productivity households have almost the same welfare gain, but middle-productivity households are a bit lower than other two types. More precisely, for age 20, low-productivity households benefit more than high-productivity ones. The reason is there are two offsetting forces. The size of tax reduction is larger for high-productivity households as the amount of tax reduction is proportional to productivity level, while marginal utility of consumption is greater for poor households, which are typically young.

**Benchmark Case: Endogenous Labor & Labor Tax Reduction**  
Figure 1.6 shows the result of the benchmark case, in which households choose their working time endogenously.
Compared with Case 2, transfers are endogenous to labor supply decisions. Higher productivity households can get more benefits from the tax reduction if they work more, and thus net transfers to high-productivity people are even larger than Case 2. Young people benefit more compared with Case 2, although losses in old are smaller than in Case 2. Now labor is endogenous, thus workers can adjust their working hours and save for retirement.

1.5.2.2 Effects at Aggregate Level

Next we see changes in aggregate variables. Figure shows the steady-state values corresponding to different rates of inflation. Each value is normalized so that it equals one when an inflation rate is zero.
Figure 1.7: Effect of Inflation on Aggregate Variables (1)

In the benchmark case, aggregate effective labor supply ($L$) increases as inflation rises mainly because of the reduction in labor taxes. Aggregate consumption ($C$) increases in all three cases.

For money and capital, we can see that inflation decreases aggregate money demand ($M$) and increases aggregate capital stock ($K$). In Case 1, with exogenous labor supply and lump-sum transfers, capital stock increases with inflation. This effect is amplified by endogenous labor and by substituting lump-sum transfers with labor tax reduction. In Section 1.5.3.2, we will discuss the mechanism why aggregate capital increases with inflation in the steady state. In our model, money is not superneutral unlike representative agent models, and inflation causes substitution from money to capital.

Figure 1.8 shows changes in a $K/L$ ratio and factor prices. Since capital increases more than
labor supply, in the general equilibrium the real interest rate \((r)\) falls and the average real wage rate \((w)\) rises. These changes in prices induce an additional redistribution effect because the decrease in the interest rate harms capital-rich people, while the increase in the wage benefits workers. In Section 1.5.3.1, we will see more details about this general equilibrium effect.

![Figure 1.8: Effect of Inflation on Aggregate Variables (2)](image)

Figure 1.8 shows that inflation improves welfare. CEV is the smallest in Case 1, in which the inflation taxes are redistributed via lump-sum transfers. The welfare gain in Case 2 is higher than in Case 1, this is because transfers are concentrated to young people who are more likely to be borrowing constrained. With an endogenous labor choice, labor tax reduction decreases distortion on an individual labor choice, and thus the welfare gain in the benchmark case is the highest among the three cases.
1.5.3 Discussions

In this section, we discuss more details about what mechanisms in our model generate the results. We will discuss some shortcomings of this paper as well.

1.5.3.1 Decomposition of Welfare Gain: Partial Equilibrium Analysis

First we investigate how inflation affects welfare by utilizing the benchmark model. In this case, the effect of inflation on welfare can be separated to (i) the effect from labor tax reduction (tax policy effect) and (ii) the effect from price changes in the general equilibrium (GE effect). The tax policy effect involves (i-1) the effect of redistribution from retirees to workers and (i-2) the effect of the decrease in distortion on an individual labor choice. For the GE effect, in Section 1.5.2.2 we saw that with 10% inflation the real interest rate falls and the wage rate rises in the general equilibrium. This change provides another redistribution effect.

To do this decomposition, the partial equilibrium analysis is useful. The idea is to fix the general equilibrium values obtained from the benchmark experiment. More precisely, we fix the labor tax rate $\tau^l$, the real interest rate $r$, the real wage rate $w$ and the social security benefits $w \cdot ss$ at the initial benchmark level, and then raise only the value of the inflation rate $\pi$ from 0% to 10%. This means we allow the government budget constraints (1.13) and (1.15) and first order
conditions for the firm (equations (A.1) and (A.2) in Appendix A.1) not to hold with equality. In the general equilibrium, prices and fiscal policies are affected by inflation as we discussed above, but in this experiment the economy does not receive any feedback from changes in price and tax policies (except for inflation taxes). Hence, this experiment can isolate the effect of inflation from other confounding factors. Then, we move to the next experiment that allows only the inflation rate $\pi$ and the labor tax rate $\tau_l$ to change while keeping $r, w$ and $w \cdot ss$ fixed. The difference between this experiment and the previous experiment can isolate the effect of the labor tax reduction from other confounding factors. Likewise, we can do further experiments that change $r, w$ and $w \cdot ss$, respectively, to isolate the effect of the price adjustments through the general equilibrium from other confounding factors. Figure 1.10 displays the result of decomposition.

We start from the benchmark experiment (Experiment 1). This is the general equilibrium shown in the previous sections, and the welfare gain (CEV) is about 3.6% with 10% inflation. In Experiment 2, all of $\tau_l, r, w$ and $w \cdot ss$ are fixed at the initial benchmark level. Revenue from the inflation tax is now thrown to the ocean. Since inflation is purely distortionary, welfare decreases (negative CEV).
Next, in Experiment 3, based on Experiment 2, we allow only the labor tax rate $\tau^l$ to change (the amount of the change in $\tau^l$ is the same as the benchmark experiment) and the others are still fixed. This experiment shows us the quantity of the tax policy effect, and now the welfare gain turns from negative to positive due to the labor tax reduction. Here, the difference between Experiments 1 and 3 represents the GE effect. We can see that the tax policy effect on welfare is positive and large, while the GE effect is negative and relatively small in our quantitative result.

The remaining experiments from No.4 to No.6 are further decomposition of the GE effect. In Experiment 4, we allow only $\tau^l$ and $r$ to change. The difference between Experiments 3 and 4 represents the effect of the fall in $r$. The welfare gain becomes smaller because the fall in the interest rate harms people who have a lot of capital. In Experiment 5, we allow only $\tau^l$ and $w$ to change. CEV increases further compared with Experiment 3 because the wage increase benefits workers. Finally, in Experiment 6, we allow $\tau^l$, $w$ and $ss$ to change. The welfare gain slightly decreases compared with Experiment 5. This difference represents transfers from workers to retirees due to the increase in social security benefits. Since the negative effect of the fall in $r$ is larger than the positive effect of the wage increase, the whole of GE effect is negative.

1.5.3.2 Why Does Capital Increase with Inflation?

In standard monetary models such as Sidrauski (1967), money is superneutral in the long-run, and steady-state capital stock is not affected by inflation. Why does capital stock increase as inflation rises in our model?

First, precautionary saving is not a primary reason. We implemented an additional experiment that reduces the uncertainty parameter (i.e., the standard deviation of idiosyncratic productivity shocks, $\sigma_\epsilon$) by 10%. If higher capital stock is due to the precautionary saving motive, the low uncertainty would promote capital accumulation further. However, the result of the experiment was almost the same as the benchmark model.

Next, we applied the partial equilibrium analysis in Section 1.5.3.1 to assets. Figure 1.11 shows a decomposition of changes in aggregate capital $K$ and money $M$ in the same manner as Figure 1.11.
In Experiment 2, in which all of $\tau^l$, $r$, $w$ and $w \cdot ss$ are fixed at the initial benchmark level and only the inflation rate $\pi$ increases from 0% to 10%, capital increases while money decreases. The increase in capital is largely amplified by the tax policy effect (Experiment 3), but eventually it is restrained by the GE effect (difference between Experiments 1 and 3). Experiments 4 to 6 suggest that the negative impact of the GE effect is mainly due to the fall in the real interest rate.

Returning to Experiment 2, the amounts of the decrease in money and the increase in capital are almost the same. From this fact, we can guess there is a substitution from money to capital through inflation. To check the idea, Figure 1.12 shows changes in $K$ and $M$ with 10% inflation. The left graph gives us a pure effect of inflation by implementing the partial equilibrium analysis, in which all of $\tau^l$, $r$, $w$ and $w \cdot ss$ are fixed. The four curves in the upper side represent the changes in capital, and the four curves in the lower side represent the changes in money. The right graph is the general equilibrium case, which involves the tax policy effect and the GE effect.

In the left graph, the decrease in money is almost parallel to the increase in capital, which

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11 For Experiment 2, we found that the amounts of changes in capital and money are the same in the absolute terms regardless of the model setting about the labor choice and the government tax policy, i.e., the results of Experiment 2 are the same in Case 1, Case 2 and the benchmark case.
strongly suggests there exists a substitution from money to capital via inflation. In the right graph, the changes in money are very similar to the ones in the left graph, but the changes in capital become much larger. This is mainly from the effect of labor tax reduction.

Figure 1.12: Changes in Capital and Money with 10% Inflation

This mechanism can be interpreted as follows. In general, OLG models do not exhibit the superneutrality of money. To understand this, it is useful to recall that the classical dichotomy in the Sidrauski model is from the Euler equation for consumption. In an infinite-lived representative agent model without uncertainty, the Euler equation is typically expressed as

$$u_c(c_t, m_{t+1}) = \beta(1 + r_{t+1})u(c_{t+1}, m_{t+2}).$$

The steady-state equilibrium requires that $c_t = c_{t+1}$ and $m_t = m_{t+1} \forall t$ and thus the $u'(\cdot)$ terms in both sides are canceled out. Hence, the gross real interest rate $1 + r_t$ becomes constant at $\frac{1}{\beta}$, which implies that steady-state capital is also constant as the firm’s first order condition requires $f'(k_t) = r_t + \delta$. In other words, to be consistent with the steady-state equilibrium, the gross real interest rate must be equal to the constant time preference rate. If not so, for example if $1 + r > \frac{1}{\beta}$, then an increase in capital continues until $r$ declines to satisfy the equality.

In OLG models, the Euler equation is expressed such as

$$u_c(c_j, m_{j+1,t+1}) = \beta(1 + r_{t+1})u_c(c_{j+1,t+1}, m_{j+2,t+2}).$$
Now the subscripts include not only the time index $t$ but also the age index $j$. Therefore the $u'(\cdot)$ terms cannot be eliminated because in general $c_{j,t}$ does not equal $c_{j+1,t+1}$ even if $c_{j,t} = c_{j,t+1}$ in the steady state (the same is true for $m_{j,t}$). The steady-state real interest rate $1 + r_{ss}$ is determined by a complicated function $\frac{u_c(c_j,m_{j+1})}{\beta u_c(c_{j+1},m_{j+2})}$ and other equilibrium conditions, and thus it is not neutral to inflation.

In summary, the Euler equation in the Sidrauski model holds both at the individual level and at the aggregate level. In OLG models, however, the individual’s Euler equation cannot be directly applied to the aggregate level due to the difference in age. The equilibrium interest rate is affected by the pattern of intertemporal substitution of consumption. Since the value of money is decreased by inflation, households substitute money to capital, and it affects aggregate capital stock as well.\footnote{Considering a simplified monetary OLG model, we can analytically show that capital increases as inflation rises. See an example in Appendix A.2.}

With policies like labor tax reduction, the increases in young generations’ resources promote the capital accumulation more, hence the Tobin effect is amplified.

1.5.3.3 Optimal Inflation Rate

This subsection briefly discusses an optimal inflation rate. In our numerical results, welfare is monotonically increasing in inflation at least in a realistic range (the bold black line in Figure 1.13). For very high rates of inflation, the curve of CEV becomes almost flat and existence of the optimal inflation rate is ambiguous.

\textbf{Figure 1.13: Welfare Gain for Large $\pi$}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{welfare_gain.png}
\caption{Welfare Gain for Large $\pi$}
\end{figure}
This means that for new born households, higher inflation is always preferred, but such a reaction is fairly counter-intuitive. Considering the reason for this phenomenon, we have a few possible hypotheses.

The first one is the possibility that the perfect foresight for inflation may affect the results. Our counter-factual experiments are based on the unrealistic assumption that future inflation rates have no uncertainty and are fixed at a constant rate over time. If the future inflation rates fluctuate stochastically, people might no longer prefer such volatility of inflation. Then, the optimal inflation rate might come to more realistic range.

The second possible reason is that our experiments are simple comparisons of steady states for different inflation rates. There is no transitional dynamics. During transition, however, inflation is likely to harm existing generations, especially old generations.

The third reason is the possibility of over-saving due to the low earnings replacement rate, which means people may save too much for retirement. However, our additional experiment already rejected this idea. We tried modifying the model so that the replacement rate is fixed at an exogenous constant rate set from actual data and the social security system is allowed to be not balanced. This change did not make a big difference from the benchmark result.

1.5.3.4 Sensitivity Analysis: Change in Elasticity of Money Demand

This subsection discusses sensitivity of the model to the interest elasticity of money demand ($\nu$) which is an important parameter determining how money demand responses to inflation. We increase $\nu$ from 0.5 to 1.0 and adjust the share parameter $\omega$ to keep $M/Y$ at the initial benchmark target. In this case, when an inflation rate is zero, the profile of capital and money is very similar to the benchmark case, except for money held by very old (top graphs in Figure 1.14).
Seeing the changes (bottom-left graph in Figure 1.14), however, the amount of the increase in capital shrinks compared with the benchmark calibration, while the amount of the decrease in money widens because now money demand is more elastic with respect to inflation. The reason why the increase in capital declines is a decrease in seigniorage (bottom-right graph in Figure 1.14). Since seigniorage equals $\frac{\pi}{1+\pi} M$ in steady states, transfers to young (in the form of labor tax reduction) decrease as $M$ decreases given inflation rates $\pi$, which lowers the amount of the increase in capital.

Welfare is also affected by interest elasticity (Figure 1.13). When $\nu = 0.5$, CEV increases monotonically in a realistic range and becomes almost flat after an inflation rate exceeds about 100%. When $\nu = 1.0$, CEV becomes almost flat around at 30% of inflation. If $\nu = 1.0$ and the labor choice is exogenous, we can see that a peak of CEV is at 18% of inflation and it slightly turns
to decline after that point. Conversely, if we set values of $\nu$ less than 0.5, the welfare gain from inflation becomes even larger.

In this paper, we assumed that the interest rate elasticity for Japanese individuals is the same as the one for U.S. individuals, and that value is borrowed from the U.S. literature, Algan and Ragot (2010). Although to find an appropriate value of $\nu$ is beyond the scope of this paper, we would like to mention that the estimation of $\nu$ seems to be a difficult problem both in the United States and Japan. For the U.S., Walsh (2010) surveyed papers that estimate money demand functions and showed that the range of estimates is very wide. Many researchers report that the estimated interest rate elasticity of money demand is less than 1.0. On the other hand, Holman (1998) directly estimated the parameters of the utility function and obtained estimates of elasticity of 10, although she reported that the data failed to reject an elasticity of 1.0, the case of Cobb-Douglas preferences. Under these circumstances, Algan and Ragot (2010) examined two cases of the sensitivity analysis with $\nu = 0.25$ and 1.0 in addition to the benchmark value of 0.5.

For Japan, several researchers report the interest rate elasticity lower than 0.5. For example, Bae et al. (2006) estimated money demand functions consistent with a money-in-utility model, and reported that the estimated value of elasticity is about 0.1. There exist several other papers estimating Japanese interest rate elasticity, and they are in the range between 0.08 to 0.18. (See Miyao, 2002 and Fujiki and Watanabe, 2004.) As discussed above, however, if we set the lower values of $\nu$ from the empirical literature, the welfare gain from inflation rises further and enters an even less realistic region. With 0.1 of $\nu$ and 10% inflation, virtually CEV cannot be calculated since the welfare gain is too large relative to the benchmark economy. Therefore we reported only the $\nu = 1.0$ case in the above graphs.

\footnote{This situation may look like the problem concerning the elasticity of labor supply in macroeconomic models. In DSGE models, researchers often set values of wage elasticity of labor supply at a level more than 1.0 so that their models can well replicate business cycles. In empirical studies, however, the elasticity of labor supply is often reported as much lower values close to zero. This gap between DSGE models and empirical studies is still remained as a puzzle.}

\footnote{This means that when $\nu = 0.1$ and $\pi = 0.1$, the value of LHS in the equation (\ref{eq:1.17}) cannot reach the value of RHS even if CEV in LHS is extremely large because marginal utility of consumption relatively quickly goes to zero. For lower inflation rates, CEV is 10.3% if $\pi = 0.01$ and 53.7% if $\pi = 0.03$. For the inflation rates that equal or are higher than 0.04, we could not compute the value of CEV.}
1.6 Conclusion

This paper constructed the model with heterogeneous agents and money. Our model involves two kinds of heterogeneity, the first one is from a life-cycle structure with 80 years and another is from uninsured idiosyncratic risk in an incomplete market. Therefore each household in the model differs in age and productivity, which generates ex-post heterogeneity in asset levels. These households hold two types of assets, capital and money, and money is assumed to yield utility. We calibrated the model to match the Japanese economy and found that inflation has different effects on individual households depending on their age and productivity. At the aggregate level, inflation has positive effects on capital stock and welfare, even assuming exogenous labor. With endogenous labor and a combination of an increase in the inflation tax and a reduction in the labor income tax, the positive effects of inflation are amplified. This is mainly because positive inflation transfers resources from money-rich to money-poor households, which are typically from old to young generations and from high-productivity to low-productivity households.
Chapter 2

Effects of a Consumption Tax Hike on Aggregate Variables, Welfare and Wealth Distribution in Japan

2.1 Introduction

In 2012, the Japanese government decided to raise the consumption tax rate by two steps. It was raised from 5% to 8% in 2014 as planned, but the second hike to 10%, which was originally for 2016, has been postponed twice so far and is currently scheduled for October 2019. The consumption tax is controversial in Japan. The two-step hikes were agreed upon by ruling and opposition parties and a law to enact them passed the National Diet in late 2012. After the first raising, however, the economy has slipped and another law was introduced to amend the timing of the second one.

This paper studies effects of the consumption tax hike using a large-scale OLG model with incomplete markets. While there are some papers that calculate required increases in the consumption tax rate to reduce or stabilize fiscal deficit and debt (for example, see Kitao, 2015), we focus on macroeconomic and distributional impacts rather than how to balance a budget. We acknowledge the importance of fiscal sustainability, but as some market economists and many politicians have expressed their concerns, if the consumption tax hikes have a large negative impact on economic activities, it would matter especially for poor people even if it is just a temporary effect. In partic-
ular, compared to economies with a representative agent, it is known that welfare implications are sometimes more complicated in economies with heterogeneous agents. In reviewing the U.S. literature using OLG models with incomplete markets, it seems to be unclear whether the consumption tax is truly a good policy tool or not in terms of social welfare (for example, see Nishiyama and Smetters, 2005). We think that distributional effects are also important, while seemingly they are not so much emphasized in recent studies regarding the Japanese fiscal policy. Since the Japan’s social security system is virtually set up as pay-as-you-go (notional defined contribution) and the government promised that it uses part of the increase in consumption taxes to fund pension benefits, the consumption tax hikes primarily induce transfers from working generations to retirees. In the context of inter-generational and intra-generational heterogeneity, this might have significant implication on welfare via effects on life-cycle and precautionary savings.

To answer these questions, we extend a model constructed by Huggett (1996), which is the seminal paper that well replicates the U.S. wealth distribution by using a large-scale OLG model with heterogeneous agents and uninsurable earnings risk. His model is widely utilized in papers studying fiscal policies in incomplete markets, including Kitao (2015) and Nishiyama and Smetters (2005) mentioned above. In this paper, first we calibrate our model to match the Japanese economy before the first tax hike, and then examine the effects of the consumption tax increase from 5% to 10%.

We found that our model can successfully reproduce Japan’s earnings and wealth distributions, although model-implied fiscal parameters such as the labor tax rate were far from data probably due to the balanced-budget assumption. The simulation results show that impacts on aggregate variables such as labor supply, capital stock and consumption are very small if we assume that the additional consumption tax revenue is used to fund both a labor tax reduction and an expansion of pension benefits. Our sensitivity analyses, however, suggest that this result strongly depends on the assumption. If all of consumption taxes are used to finance pension spending, then the economy shrinks and welfare deteriorates because of the transfers from the young to the old. In contrast, if all of the additional revenue is used to reduce labor taxes, then the economy expands and

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1They calibrate their OLG model to match the U.S. economy and compare a consumption tax with a labor income tax. They conclude that if idiosyncratic wage shocks are uninsurable, the flat consumption tax is more costly in terms of welfare than the progressive labor income tax, which is in contrast to the previous literature using deterministic OLG models.
welfare improves. Another finding is that the consumption tax hike widens wealth inequality. In our benchmark model, while wealthy households accumulate more assets in response to the labor tax cut, the poor young and the poor old reduce their savings sharply in response to the consumption tax increase. These asymmetric responses widen wealth inequality, especially within very young ages and retirees.

The rest of the paper is organized as follows. Section 2.2 presents our model and Section 2.3 calibrates it to match the Japanese economy. In Section 2.4, we report benchmark results with the 5% consumption tax rate as the statutory rate before 2014, and then implement counterfactual experiments that raise the rate to 10% or more exogenously. This section also discusses the robustness of the benchmark results. Section 2.5 briefly concludes.

### 2.2 Model

This section provides a large-scale OLG model as a natural extension of Huggett (1996) in which households face uninsured idiosyncratic productivity risk. Main differences from the Huggett model are: (i) there is a consumption tax, (ii) labor and capital income taxes are separated, and (iii) labor supply is endogenous.

#### 2.2.1 Demographics

We assume that there is no aggregate uncertainty in this economy. Each household enters the economy at age 20 (denoted by $j = 1$) with no asset, and lives up to the maximum age of 95 ($J = 16$). One period of the model corresponds to five years. All households receive pension benefits after age 65 ($R = 10$). Note that they can continue to work after the age $R$ if they want.

Households face life-span uncertainty, and $s_j$ denotes the probability of surviving up to age $j$ conditional on surviving up to age $j - 1$. All of them die at age $J + 1$ with probability one, i.e., $s_{J+1} = 0$. For simplicity we assume that demographic patterns are stable.\(^2\) In this case, age-$j$ households make up a fraction $\mu_j \in [0, 1]$ of the population, with

$$\mu_{j+1} = \frac{s_{j+1}}{1 + n} \mu_j$$

where

\(^2\)This assumption is necessary to compute stationary equilibria, although we acknowledge that it is not ideal as Japan faces unprecedented population aging.
where $\mu_j$ is constant over time and $n$ is the constant growth rate of new cohort.

Households that died before the maximum age $J$ may leave accidental bequests, which are collected by the government and equally distributed to all surviving households as lump-sum transfers.

### 2.2.2 Households

Households maximize their expected lifetime utility.

$$E_{1,t} \left[ \sum_{j=1}^{J} \beta^{j-1} \left( \prod_{j=1}^{J} s_j \right) u \left( c_{j,t}^{i} + l_{j,t}^{i} \right) \right]$$

where $c_{j,t}^{i}$ is a household $i$’s real consumption at age $j$ in time $t$, $l_{j,t} \in [0, 1]$ is labor supply and $\beta$ is the subjective discount factor. We assume that flow utility $u(\cdot)$ is strictly concave and twice continuously differentiable and satisfies the Inada conditions for each argument, $c$ and $1-l$, respectively.

Households that are working receive labor income $q_{j,t}^{i}$, which includes a stochastic part and a deterministic age-specific part. That is,

$$q_{j,t}^{i} = z_{j,t}^{i} \eta_j w_{j,t}^{i}$$

where $z_{j,t}^{i}$ is the idiosyncratic productivity shock, $\eta_j$ is the deterministic and age-specific mean wage profile and $w_t$ is the average real wage. As we will see later, $w_t$ is determined at a level that equals marginal productivity of labor in the firm sector. Thus, $\eta_j$ can be interpreted as a ratio of mean productivity of age-$j$ households relative to overall mean productivity $w_t$.\(^3\) Let $\tilde{z}_{j,t}^{i}$ denote log of $z_{j,t}^{i}$. We assume that $\tilde{z}_{j,t}^{i}$ follows an AR(1) process.

$$\tilde{z}_{j,t}^{i} = \rho \tilde{z}_{j-1,t-1}^{i} + \epsilon_{j,t}^{i} \quad (2.1)$$

where $\epsilon_{j,t}^{i} \sim N(0, \sigma^2_{\epsilon}).$ The differences across households in their productivity histories generate endogenous cross-sectional distributions of assets even within the same age group.

Households’ financial resources consist of the labor income discussed above, financial assets held at the previous period, accidental bequests, and social security benefits (only after the age $R$). Households allocate the resources between consumption and next-period assets. The budget constraint is

\(^3\)Therefore, $\eta_j$ can be also defined as an average wage within age $j$ ($w_{j}$) to an average wage over all age groups ($w$), i.e., $\eta_j \equiv w_{j} / w$. In that case, $z_{j,t}^{i}$ is defined as a ratio of an individual wage ($w_{j,t}^{i}$) to an average wage within age $j$, i.e., $z_{j,t}^{i} \equiv w_{j,t}^{i} / w_{j}$. Then, log of $z_{j,t}^{i}$ equals a deviation of $w_{j,t}^{i}$ from $w_{j}$, i.e., $\tilde{z}_{j,t}^{i} \equiv \ln z_{j,t}^{i} = \ln(w_{j,t}^{i} / w_{j}) = \ln w_{j,t}^{i} - \ln w_{j}$.  

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\[(1 + \tau_t^c)c_{j,t}^i + a_{j+1,t+1}^i = (1 - \tau_t^l - \phi_t)q_{j,t}^i + [1 + (1 - \tau_t^a)r_t]a_{j,t}^i + b_t + ss_{j,t}\]  \quad (2.2)

where \(a_{j,t}^i\) is financial assets, \(r_t\) is the real interest rate, \(\tau_t^c\) is the consumption tax rate, \(\tau_t^l\) is the labor income tax rate, \(\tau_t^a\) is the capital income tax rate, \(b_t\) is accidental bequest transfers from deceased, \(\phi_t\) is the pension contribution rate, and \(ss_{j,t}\) is social security benefits.

We allow the government to use consumption taxes to finance both pension benefits and the other government purchases. As we will discuss in Section 2.3.1, when the Japanese government decided the consumption tax hikes in 2012, it also promised to use a large part of additional revenues from the hikes in order to finance pension payments.

\[\tau_t^c = \tau_t^{css} + \tau_t^{cq}\]

where \(\tau_t^{css}\) is the rate of the consumption tax used to fund pension benefits and \(\tau_t^{cq}\) is the rate of the consumption tax used to fund the other government expenditures. An entire structure of the government budget will be discussed in Section 2.2.4.

Additionally, in this economy households are not allowed to borrow.

\[a_{j+1,t+1}^i \geq 0.\]  \quad (2.3)

We will focus on stationary equilibria. Then our state variables are age \(j\), productivity \(z\) and assets \(a\). We denote a state vector as \((x,j)\) where \(x \equiv (a,z)\). Let any variable with a prime symbol (’) denote a one-period-ahead variable, for example \(x' = x_{j+1}\) if \(x = x_j\). The recursive formulation of the households’ problem (ignoring individual and time subscripts) is

\[V(x,j) = \max_{c,a',l} u(c,l) + \beta s_{j+1} E \left[ V(x', j + 1) \mid z \right] \quad (2.4)\]

s.t. \( (1 + \tau^c)c = [1 + (1 - \tau^a)r]a + (1 - \tau^l - \phi)z_j\eta_jwl + b + ss_{j}w - a' \),

\( a' \geq 0 \) and \( l \geq 0.\)
2.2.3 Firm

A representative firm produces consumption goods using a constant returns to scale technology.

\[ Y_t = \zeta K_t^\alpha L_t^{1-\alpha} \] (2.5)

where \( Y_t \) is aggregate real output, \( \zeta \) is the TFP, \( K_t \) is aggregate real capital, \( L_t \) is aggregate effective labor. Given the above production function and factor prices \( r_t \) and \( w_t \), the firm maximizes profits under perfect competition.

\[
\max_{K_t, L_t} Y_t - (r_t + \delta)K_t - w_tL_t
\] (2.6)

where \( \alpha \) is the capital share and \( \delta \) is the rate of capital depreciation.

2.2.4 Government

The roles of government are: (i) operating a social security system, (ii) collecting and redistributing accidental bequests, and (iii) operating the other government expenditures and taxes. Each system is assumed to be balanced, and thus the government must satisfy three budget constraints. (See the equations (2.10) to (2.12) in the definition of equilibrium.)

**Pension System** The government operates a pay-as-you-go pension system. All workers must pay pension contributions at the rate \( \phi_t \) out of their labor income as long as they work. All households start to receive pension benefits at the age \( R \), which equals the fraction \( ss \) of the workers’ average wage \( w_t \). Hence \( ss \) is interpreted as a wage replacement rate.

\[
ss_{j,t} = ss_j w_t, \\
ss_j = \begin{cases} 
ss > 0 & \text{if } j \geq R \\
0 & \text{otherwise.}
\end{cases}
\]
Pension spending to retirees is funded by pension contributions from workers and part of consumption taxes. Suppose that a fraction $p_c \in [0, 1]$ of consumption tax revenue is used to finance pension expenditures. Then the consumption tax rates $\tau^{c_{ss}}_t$ and $\tau^{c_g}_t$ mentioned in Section 2.2.2 can be expressed as

$$\tau^{c_{ss}}_t = p_c \tau^{c}_t,$$
$$\tau^{c_g}_t = (1 - p_c) \tau^{c}_t.$$ 

As we will see in Section 2.4.4, effects of a consumption tax hike are deeply influenced by assumptions regarding how to allocate the additional revenue, in terms of both the directions and the amplitudes.

**Accidental Bequests** The government collects accidental bequests of deceased and equally distributes them to the living households as lump-sum transfers $b_t$.

**Other Government Expenditures and Taxes** The government purchases $G_t$ are funded by the remaining resources, i.e., taxes on consumption and labor and capital income. By the above assumption, the remained fraction $(1 - p_c)$ of consumption taxes is used as a resource of government expenditures. We also assume that $G_t$ is fixed at some positive level, while any purchase is completely wasteful in the model.

### 2.2.5 Equilibrium

Our analysis focuses on stationary equilibria and now we ignore the time subscripts $t$. The probability space $(\mathcal{X}, \mathcal{X}, \psi_j)$ is defined as follows.

Let $\psi_j$ be a probability measure on a measurable space $(\mathcal{X}, \mathcal{X})$. $\mathcal{X}$ is the state space of $x$, i.e., $\mathcal{X} \equiv A \times Z$ where $A$ and $Z$ are the state space of assets $a$ and productivity $z$ respectively. $\mathcal{X}$ is the Borel $\sigma$-algebra on $\mathcal{X}$. Then, for any subset of $\mathcal{X}$ (denoted by $\mathcal{X}_s$), the measure $\psi_j(\mathcal{X}_s)$ represents a fraction of age-$j$ households whose individual state vector $x$ lies in $\mathcal{X}_s$ as a proportion of all age-$j$ households. In this case, $\mu_j \psi_j(\mathcal{X}_s)$ equals a fraction of such households to total population in the economy.
Since we assume that households have no asset when they enter the economy, a distribution across age-1 households is exogenously determined by an initial distribution of idiosyncratic productivity \( z_1 \). Then, distributions across households of age \( j = 2, \ldots, J \) are recursively given by

\[
\psi_j (\mathcal{X}_s) = \int_{\mathcal{X}} P(x, j - 1, \mathcal{X}_s) d\psi_{j-1} \quad \forall \mathcal{X}_s \in \mathcal{X}
\]

where \( P(\cdot) \) is a transition function, i.e., \( P(x, j, \mathcal{X}_s) \equiv Pr(x_{j+1} \in \mathcal{X}_s | x_j = x) \).

**Definition:** Given the initial condition \( a_1 = 0 \), the population growth rate \( n \) and the conditional survival probabilities \( \{s_j\} \), a stationary equilibrium consists of

1. decision rules \( c_j = c(x, j), a_{j+1} = k'(x, j) \) and \( l_j = l(x, j) \),
2. prices \( r \) and \( w \),
3. quantities \( K \) and \( L \),
4. fiscal policies \( \{G, \tau^c, \tau^l, \tau^a, b, \phi, p_c, ss\} \) and
5. distributions \( \{\psi_j\}_{j=1}^J \),

such that

1. given prices and fiscal policies, \( c(x, j), a'(x, j) \) and \( l(x, j) \) solve the households’ problem (2.4),
2. given prices, \( K \) and \( L \) solve the firm’s problem (2.5),
3. markets clear, i.e.,

\[
\text{[capital]} \quad \sum_j \mu_j \int_{\mathcal{X}} a'(x, j) d\psi_j = K', \quad (2.7)
\]
\[
\text{[labor]} \quad \sum_j \mu_j \int_{\mathcal{X}} z_j \eta_j l_j(x, j) d\psi_j = L, \quad (2.8)
\]
\[
\text{[good]} \quad \sum_j \mu_j \int_{\mathcal{X}} [c(x, j) + a'(x, j)] d\psi_j + G = Y + (1 - \delta)K, \quad (2.9)
\]
4. the government satisfies the following three budget constraints, i.e.,

(a) pension benefits equal sum of pension contributions and some of consumption taxes:

\[
\sum_{j=R}^{J} \mu_j w \cdot \psi_j = \sum_{j} \mu_j \int_{X} [\phi z_j \eta_j w l(x, j) + p_c \tau^c c(x, j)] d\psi_j, \tag{2.10}
\]

(b) bequest transfers equal sum of accidental bequests:

\[
b' = \frac{1}{1 + n} \sum_{j} \mu_j (1 - s_{j+1}) \int_{X} [1 + (1 - \tau^a) r] a'(x, j) d\psi_j, \tag{2.11}
\]

(c) government purchases equal sum of taxes on labor and capital income and some of consumption taxes:

\[
G = \sum_{j} \mu_j \int_{X} \left[ \tau^l z_j \eta_j w l(x, j) + \tau^a r a_{\psi_{j-1}} + (1 - p_c) \tau^c c(x, j) \right] d\psi_j, \tag{2.12}
\]

where \(a_{\psi_{j-1}}\) is beginning-of-period assets.

5. distributions are stationary and consistent with individual behavior, i.e.,

\[
\psi_{j+1}(X_s) = \int_{X} P(x, j, X_s) d\psi_j \quad \forall j = \{1, \ldots, J - 1\} \text{ and } \forall X_s \in \mathcal{X}
\]

where \(P(x, j, X_s) = Pr(x' \in X_s \text{ s.t. } a' = a'(x, j) | x_j = x)\).

### 2.3 Calibration

In this section the model is calibrated to match the Japanese economy before 2010. The reason to choose 2010 is that the tax hikes were agreed across political parties in the middle of 2012, but in the previous year, 2011, Japan experienced the Great East Japan Earthquake and some statistics are not available for that year.

We assume that the flow utility function has a general CES specification, following Kitao (2015), in which her benchmark model is calibrated to the Japanese economy as of 2010 and used to analyze effects of fiscal policies.

\[
u(c, l) = \left[ \frac{c^{1-\gamma} (1 - l)^\gamma}{1 - \sigma} \right]^{1-\sigma} \tag{2.13}
\]
where \( \sigma \) is the parameter of relative risk aversion, or the inverse of elasticity of inter-temporal substitution, and \( \gamma \) is the weight on leisure.

The idiosyncratic productivity process (2.1) is approximated by a first-order Markov chain with \( S \) possible states of \( z \) by using the method provided by Tauchen (1986). We set \( S = 15 \).

The following two tables are lists of the exogenous parameters (Table 2.1) and the calibrated parameters (Table 2.2). Note that each value in the tables is expressed in annual terms. Therefore, for some parameters or calibration targets such as the depreciation rate \( \delta \) and the capital-output ratio \( K/Y \), we have to convert them into a five-year frequency when we compute the equilibrium.\(^4\)

### 2.3.1 Exogenous Parameters

Table 2.1 summarizes the exogenous parameters which are from literature or data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population growth rate ( n )</td>
<td>0.0</td>
<td>Population Estimates etc., MIC.</td>
</tr>
<tr>
<td>Conditional survival probability ( { s_j }_j=1 )</td>
<td>-</td>
<td>Japanese Mortality Database, IPSS</td>
</tr>
<tr>
<td>Risk aversion ( \sigma )</td>
<td>3.0</td>
<td>Kitao (2015)</td>
</tr>
<tr>
<td>Capital share ( \alpha )</td>
<td>0.362</td>
<td>Hayashi and Prescott (2002)</td>
</tr>
<tr>
<td>Depreciation rate ( \delta )</td>
<td>0.089</td>
<td></td>
</tr>
<tr>
<td>TFP ( \zeta )</td>
<td>1</td>
<td>(Normalization)</td>
</tr>
<tr>
<td>Mean age-wage profile ( { \eta_j }_j=1 )</td>
<td>-</td>
<td>Lise et al. (2013)</td>
</tr>
<tr>
<td>Persistency of shocks ( \rho )</td>
<td>0.99</td>
<td>(See Section 2.3.2)</td>
</tr>
<tr>
<td>S.D. of ( z ) at age 20 ( \sigma_{z_1} )</td>
<td>0.24</td>
<td>Lise et al. (2013)</td>
</tr>
<tr>
<td>Expenditure-output ratio ( G/Y )</td>
<td>0.19</td>
<td>National Accounts, Cabinet Office</td>
</tr>
<tr>
<td>Consumption tax rate ( \tau^c )</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>Allocation parameter ( p_c )</td>
<td>0.42</td>
<td>Ministry of Finance</td>
</tr>
<tr>
<td>Capital income tax rate ( \tau^a )</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>Pension contribution rate ( \phi )</td>
<td>0.18</td>
<td>MHLW</td>
</tr>
</tbody>
</table>

Note: MIC: Ministry of Internal Affairs and Communications.


\(^4\)For example \( \delta = 1 - (1 - 0.089)^5 \approx 0.37 \) and \( \frac{K}{Y} = \frac{2.5}{0.5} = 5.0 \). For the idiosyncratic productivity process, first we generated the Markov chain transition matrix based on the annual parameters and then multiplied it five times.
Demographics  The population growth rate $n$ is a simple average of annual rates over 10 years from 2001 to 2010, which are surveyed in the Population Census every five years and are estimated for intercensal years. The conditional survival probability is calculated from the “Death Rates” published by the National Institute of Population and Social Security Research, which is the public organization supervised by the Ministry of Health, Labour and Welfare (MHLW). We took an average from 2001 to 2010 for each age.

Technology and Preferences  The technology parameters $\alpha$ and $\delta$ are borrowed from a study by [Hayashi and Prescott (2002)](1), which calibrates a dynamic general equilibrium model to match the Japanese economy in the 1990’s. The level of total factor productivity ($\zeta$) is normalized at one.

The risk aversion parameter $\sigma$ is set to 3.0 used in [Kitao (2015)](2) as with the specification for the utility function.

Mean age-wage profile  For the mean age-wage profile $\{\eta_j\}_{j=1}^J$, we use data for male workers provided by [Lise et al. (2014)](3), who document main features of distributions of wages, earnings and consumption in Japan since the early 1980’s (Figure 2.1). Since the wage data reported in the paper is limited to ages 25-59, we obtained values of wages for workers under age 25 and aged 60 and over by extrapolation using approximation with third degree polynomials.\footnote{The extrapolated wage profile reaches to zero at age 72. However, seeing data on employment rates by age group, employment rates over age 70 were about 13% in the 2000’s (Labour Force Survey, MHLW), which is not so close to zero. Hence, our wage-profile approximation for over age 70 may be a bit underestimation.}

For parameters to generate the productivity process, we will explain them in the next subsection 2.3.2 as the exogenous parameters ($\rho$ and $\sigma_z$) and the calibrated parameter ($\sigma_\varepsilon$) are closely related each other.
Figure 2.1: Life-cycle Wage Profile

Fiscal Policies  The government expenditure per real GDP ($G/Y$) is an average of the national account data between 2001-2010. The pension contribution rate $\phi$ is set to 18%, which is the earnings-proportional rate for employed workers in the private sector. The capital income tax rate $\tau^a$ is set to 0.2, which equals the statutory rate. The wage replacement rate $ss$ and the labor income tax rate $l$ are endogenously determined to balance the government budget constraints (2.10) and (2.12), respectively.

The allocation parameter $p_c$, which is the fraction of consumption taxes used for pension benefits, cannot be directly observed from data. We know the ratios of each tax/expenditure item relative to total revenue/expenditures, respectively, but there is no data that links each tax item to the corresponding expenditure item for which the tax is used. Although the government promises that after the tax hike, the government uses the increase in consumption taxes only for the social-security related expenditures, this “social-security related expenditures” include not only pension expenditures, but also health care expenditures, long-term care expenditures and expenditures for child care support. To keep the model simple, we assume that the fraction $p_c$ is the same between before and after the tax hike\(^6\) and also assume that the fraction used as pension benefits is proportional to the current share of pension benefits relative to the total social-security related expenditures, although they are somewhat arbitrary assumptions. We set $p_c = 0.42$, which means

\(^{6}\text{Even before 2012, part of consumption taxes have been used to fund pension benefits because the pension system is not balanced in the real world. As of 2011, about a one-fifth of pension expenditures were replenished by transfers from the general account of the central government, most of which are considered to be taxes other than pension contributions.}\)

\(^{7}\text{The social-security related expenditures in the FY 2017 budget (general account) of the central government consists of 12.1 trillion yen of pension related expenditures, 11.5 trillion yen of health care related expenditures, 3.0}\)
about 60% of consumption taxes used to fund government purchases $G$ and the remained 40% finances pension expenditures. In our counter-factual experiments (in Section 2.4.2) that raises the consumption tax rate from 5% to 10%, this means that 60% of the additional consumption tax revenue is used for the labor tax reduction since we assume $G$ is fixed. In Section 2.4.3, we check how the effects of the consumption tax hike change if this assumption is altered.

### 2.3.2 Calibration Targets

The remainings are the calibrated parameters which are set to match data moments (Table 2.2).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Target</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount factor $\beta$</td>
<td>$K/Y = 2.5$ $(\beta = 0.991)$</td>
<td>[Hansen and Imrohoroglu (2010)]</td>
</tr>
<tr>
<td>Weight on leisure $\gamma$</td>
<td>Working time below age 60 = 0.385 $(\gamma = 0.605)$</td>
<td>Labour Force Survey, MHLW.</td>
</tr>
<tr>
<td>S.D. of $\epsilon$ $\sigma_\epsilon$</td>
<td>S.D. of log wages by age $(\sigma_\epsilon = 0.177)$</td>
<td>[Lise et al. (2013)]</td>
</tr>
</tbody>
</table>

For the subjective discount factor $\beta$, [Hansen and Imrohoroglu (2010)] show the capital-output ratio in the 2000’s in Japan was 2.3 to 2.8. Therefore we adjust the value of $\beta$ so that the calculated $K/Y$ equals to 2.5. In the benchmark case, we set $\beta = 0.954$, which corresponds to $\beta = 0.991$ on an annual basis.

The preference weight on leisure $\gamma$ is set to match the calculated average working time below age 60 with data. On an average of 2009 and 2010 (we could not find data before then), the weekly-working hours for males of ages 20-59 were about 46.2 hours (Figure 2.2). During ages 25-59, working hours are mostly unchanged and start to decrease after 60, which is the age that many companies set as their employees’ retirement age. Based on these observations, we set the target for working time per time endowment to 0.385 ($\approx 46.2$ working hours/(24 hours $\times$ 5 days)). In the benchmark, we have $\gamma = 0.605$. This value is similar to 0.63 of the Kitao (2013)’s baseline trillion yen of long-term care related expenditures and 2.1 trillion yen of child care supports. More precisely, after the tax hike to 10%, 2.2% of the consumption taxes are allocated to the local governments. For simplicity we assume that the local governments also use the consumption taxes in the same proportion as the central government.

---

8In Japan, the minimum retirement age is determined at 60 by law, but the actual retirement age is set by each firm. A survey shows that 80.7% of firms set their retirement age at 60, and 15.2% set at age 65 (General Survey on Working Conditions 2016, MHLW). Since retirees become eligible for public pension benefits after age 65, many companies (94.1%) have a system of “extended employment” or “re-hiring” or both of them.
Productivity Process  We set the standard deviation of idiosyncratic shocks ($\sigma_z$) to match an age profile of variance of log wages with data, which is series provided in Lise et al. (2014). We obtained missing values outside of ages 25-59 using third degree polynomials as with the mean profile $\{\eta_j\}$, then the standard deviation of wage at age 20 is $\sigma_{z1} = 0.24$.

The persistency parameter $\rho$ is set to 0.99. This value is a bit higher than the standard value 0.95 broadly used in Japanese literature, but we took the following two things into account; (i) Starting from the observation that an age profile of the variance of log earnings is convex, Abe and Yamada (2009) estimate an earnings process using micro data and report that they do not reject the possibility of $\rho > 1$ in Japan. Following this result, some papers set values of $\rho$ close to 1, for example $\rho = 0.98$ in Yamada (2008) and $\rho = 0.97$ in Yamada (2012). Particularly Yamada (2012) tries to replicate the convexity of variance by assuming the variance of the idiosyncratic shocks ($\sigma_z^2$, not $\sigma^2_z$) gradually increases with age. (ii) Our stochastic process is about wages, not earnings, but a black line in Figure 2.3 shows that the variance of wages for male workers almost linearly increases as they get older (moreover it may look slightly convex especially at younger ages).
This implies that we have to set a value of $\rho$ very close to one to generate the almost linearly increasing variances as long as assuming $\sigma_\epsilon$ is constant. Based on these considerations, we set $\rho = 0.99$.

As a result we have $\sigma_\epsilon = 0.177$. A red dot line in Figure (2.3) shows that our parameter values can very well replicate the age profile of variance of log wages without assuming ad-hoc heteroscedasticity on the unobservable $\sigma_\epsilon$. As we will see immediately below, these parameter values also can successfully replicate not only the wage distribution but also earnings and assets distributions.

### 2.4 Results

This section provides our benchmark results and then moves on to counter-factual experiments to see how the consumption tax increase affects aggregate variables, welfare and wealth distribution. We also implement some sensitivity analyses to understand more about what factors contribute the results.

---

We also found that our approximated AR(1) process computed by the [Tauchen](1986)’s discretization method exhibits a bit less accuracy for $\rho$ than a time series generated by the true process in the equation (2.1). We employed the Monte Carlo simulation and created an artificial series of $\tilde{z}_t$ by using our discrete 15-state first-order Markov process. Our estimate of $\rho$ was about 0.984, which corresponds to about 0.94 on a five year basis.
2.4.1 Benchmark Model

First we assess how well the model replicates the Japanese economy. Table 2.3 compares the calibrated model with data.

<table>
<thead>
<tr>
<th></th>
<th>Model</th>
<th>Data</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K/Y$</td>
<td>2.5</td>
<td>2.5</td>
<td>(Target)</td>
</tr>
<tr>
<td>$C/Y$</td>
<td>0.62</td>
<td>0.58</td>
<td>National Accounts, Cabinet Office</td>
</tr>
<tr>
<td>$r$</td>
<td>6.2%</td>
<td>4.5%</td>
<td>Kitao (2015)</td>
</tr>
<tr>
<td>Asset Gini</td>
<td>0.611</td>
<td>0.587</td>
<td></td>
</tr>
<tr>
<td>Asset share:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top 1%</td>
<td>7.5%</td>
<td>11.1%</td>
<td></td>
</tr>
<tr>
<td>Top 5%</td>
<td>25.0%</td>
<td>27.6%</td>
<td></td>
</tr>
<tr>
<td>Top 10%</td>
<td>39.9%</td>
<td>40.7%</td>
<td></td>
</tr>
<tr>
<td>Top 20%</td>
<td>61.0%</td>
<td>58.9%</td>
<td></td>
</tr>
<tr>
<td>S.D. of log earnings</td>
<td>0.448</td>
<td>0.484</td>
<td></td>
</tr>
<tr>
<td>Labor income tax ($\tau$)</td>
<td>21.4%</td>
<td>8.9%</td>
<td></td>
</tr>
<tr>
<td>Earnings replacement rate</td>
<td>48.9%</td>
<td>62.3%</td>
<td></td>
</tr>
</tbody>
</table>

The calculated consumption-output ratio $C/Y$ is close to the data. However, the real return on capital $r$ does not match data. For example, nominal interest rates on Japanese government bonds are much lower than our calibrated real rates of return on capital. The 10-year JGB yield from 2001 to 2010 was about 1.4% (as a simple average of daily data) and dived into negative territory in 2016, now fluctuating around zero. Although other macroeconomic models also report that their calibrated real returns on capital are much higher than the nominal JGB yields, still those calibrated values are lower than our result. The 4.5% in Table 2.3 is from Kitao (2015)’s baseline calibration for the Japanese economy as of 2010. In a model by Hansen and Imrohoroglu (2016), the before-tax rate of return was roughly in [6.0, 8.0] in 2005-2010 and the after-tax rate was roughly in [4.0, 5.0].

Our model successfully reproduces the wealth distribution except for extremely rich groups. Data for asset distribution in Table 2.3 is from Yamada (2008). He computes asset Gini coefficients and asset shares by utilizing the National Survey of Family Income and Expenditure (NSFIE) conducted by the Statistics Bureau in Japan. According to this, the asset Gini as of 2004 was 0.587, which is similar to our model result of 0.611. Yamada (2008) also shows the Gini coefficients at distant five years and they are distributed around 0.6 (i.e., 0.627 in 1989, 0.596 in 1994 and 0.572 in 1999).\footnote{We do not have data that is more recent and can be easily utilized. In Japan, to the best of our knowledge, the} The model also shows good performances on shares of assets owned by top 20% and 10%
wealth holders.

The precision of model prediction becomes a bit worse for the even more wealthy households. In the model, the top 1% individuals own 7.5% of total assets in the economy, while in the data top 1% own 11.1% of total assets. The model’s asset share of top 5% is also a bit lower than the data. Huggett (1996) reports that his model calibrated to match the U.S. economy also fails to replicate the upper tail of distribution, and our model has this tendency as well. However, compared with the results in his paper, our model’s prediction looks relatively more accurate as the actual wealth distribution in Japan is less concentrated to the richest people than that in the U.S.

Our model generates earnings dispersion with fairly high precision as well. As a result of calibrating $\sigma_e$ and $\gamma$ so that the model-implied wage dispersion by age and work hours match data, the standard deviation of log earnings (before tax, i.e. $z_j^i \eta_j t_j^i$) is similar to the value provided in Lise et al. (2014) (an average of household heads between 1981-2008 that is the whole sample period available in the database).

Figure 2.4 shows the model’s wealth distribution by age for some percentiles of asset holdings. We do not have Japanese data that can be compared with this, but each curve exhibits roughly the same pattern as the one calculated by Huggett (1996). In our model young households gradually accumulate assets as getting older, and peaks come at age 55 or 60. Older people can receive pension benefits from age 65 and then they break into their savings. At age 60, assets held by a 80th percentile household (20% from top) are about 1.8 times higher than those of a median household. 5.1% of households are borrowing constrained, who are typically very young and poor old people.11

data of asset Gini coefficients including both financial and housing assets is not released by public organizations. We need to access micro-level data such as NSFIE to obtain information about asset distribution, but the public use of it is highly restricted by law.

11 Age-95 households are excluded from this calculation since they always choose $a_{J+1} = 0$. 52
On the other hand, the fiscal parameters do not exhibit a good fit, which is another limitation of our model. The implied labor tax rate $\tau^l$ is much higher than the data. This may stem from a gap between the actual fiscal position and our assumption that the government always balances its budget. As mentioned in Section 2.1, Japan has a huge amount of debt and a large deficit, which suggests that in the real world the government do not finance its annual expenditures ($G$ in our model) by its tax revenues.

The earnings replacement rate is lower than the data. The wage replacement rate $ss$ is endogenously computed so as to balance the social security system under the fixed consumption tax rate $\tau^c$ and the fixed pension contribution rate $\phi$, but the public pension system in Japan is not annually balanced only by the consumption taxes and worker’s contributions. The pension benefits are financed also by other taxes, government bond issuing or withdrawal from the sovereign pension fund, which are abstracted in this model.

### 2.4.2 Counter-factual Experiment: Consumption Tax Hike

Next, we implement policy simulations using the calibrated model described in Section 2.3 and 2.4.1. Our main interests are focused on effects of the consumption tax hike on aggregate variables,
welfare and wealth distribution.

2.4.2.1 Assumptions: Balanced Budget and Labor Tax Reduction

As described in Section 2.1, the Japanese government raised the consumption tax rate from 5% to 8% in April 2014, and expects to further increase to 10% in October 2019. Now our benchmark model is calibrated to the Japanese economy before 2010. In this experiment we raise the consumption tax rate $\tau^c$ from 5% to 10% and compare the two steady states.

We assume that the government in the model still keeps the balanced budget and the level of government expenditures $G$ is unchanged from the initial steady state. About 40% of the consumption tax revenues are used to fund pension benefits, and the remained 60% is used to fund the government expenditures $G$ (see Section 2.3.1). This implies that higher consumption taxes reduce the labor income tax rate $\tau^l$.\(^{13}\) Likewise, with the fixed contribution rate $\phi$, the rise in the consumption tax rate leads to an increase in the wage replacement rate $ss$.\(^{14}\) In summary, through this experiment households in the model face the following three changes; (i) an increase in the consumption taxes over the lifetime, (ii) a decrease in the labor income taxes during working ages, and (iii) an increase in pension benefits after age 65.

As we will see immediately below, quantitative impacts of the 5 percentage point (pp) increase in the consumption tax rate are limited, and thus we also calculated the case of $\tau^c = 20\%$ to show up the responses more clearly. 20% of the consumption tax rate (or the value added tax rate) is close to the simple average across the OECD countries.

2.4.2.2 Results

Table 2.4 compares some variables between the two stationary equilibria. The column (1) is our benchmark case with $\tau^c = 5\%$ and the columns (2) and (3) are the counter-factual experiments with $\tau^c = 10\%$ and $\tau^c = 20\%$, respectively. Variables $C$, $K$, $L$, $Y$, $K/L$ and $w$ are normalized as 100 in the benchmark case. Real interest rate is expressed as an annual rate.

\(^{13}\)In Japan, corporate income tax rates were reduced in conjunction with the first consumption tax hike to 8%.

\(^{14}\)More technically speaking, $\tau^l$ and $ss$ are also affected by changes in aggregate variables $K$, $L$ and $C$ and relative prices $r$ and $w$. See the equations (2.10) and (2.12) in Section 2.2.5 or (B.4) and (B.5) in Appendix A.1.3.
Table 2.4: Effects of Consumption Tax Increase

<table>
<thead>
<tr>
<th></th>
<th>( p_c = 0.42 )</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau^c )</td>
<td>(1) 0.05</td>
<td>(2) 0.10</td>
<td>(3) 0.20</td>
<td>(4) 0.10</td>
<td>(5) 0.10</td>
</tr>
<tr>
<td>( L )</td>
<td>Labor</td>
<td>100</td>
<td>100.06</td>
<td>100.21</td>
<td>101.62</td>
</tr>
<tr>
<td>( K )</td>
<td>Capital</td>
<td>100</td>
<td>100.19</td>
<td>100.74</td>
<td>106.60</td>
</tr>
<tr>
<td>( Y )</td>
<td>Output</td>
<td>100</td>
<td>100.11</td>
<td>100.40</td>
<td>103.40</td>
</tr>
<tr>
<td>( C )</td>
<td>Consumption</td>
<td>100</td>
<td>100.12</td>
<td>100.42</td>
<td>103.48</td>
</tr>
<tr>
<td>( K/L )</td>
<td>Capital-labor ratio</td>
<td>100</td>
<td>100.13</td>
<td>100.53</td>
<td>104.90</td>
</tr>
<tr>
<td>( r )</td>
<td>Real interest rate</td>
<td>6.21%</td>
<td>6.20%</td>
<td>6.17%</td>
<td>5.86%</td>
</tr>
<tr>
<td>( w )</td>
<td>Mean real wage</td>
<td>100</td>
<td>100.05</td>
<td>100.19</td>
<td>101.75</td>
</tr>
<tr>
<td>( \tau^l )</td>
<td>Labor tax rate</td>
<td>21.4%</td>
<td>18.6%</td>
<td>12.8%</td>
<td>15.7%</td>
</tr>
<tr>
<td></td>
<td>Earnings replacement rate</td>
<td>48.9%</td>
<td>53.9%</td>
<td>63.9%</td>
<td>48.7%</td>
</tr>
<tr>
<td>( A )</td>
<td>Asset Gini</td>
<td>0.611</td>
<td>0.619</td>
<td>0.633</td>
<td>0.624</td>
</tr>
<tr>
<td>CEV</td>
<td>0</td>
<td>-0.12%</td>
<td>-0.27%</td>
<td>-4.62%</td>
<td>4.62%</td>
</tr>
</tbody>
</table>

The values concerning fiscal policies are moved to the theoretically expected directions. When \( \tau^c = 0.1 \), the labor tax rate \( \tau^l \) falls by about 3 pp and the earnings replacement rate rises by 5 pp. Although the earnings replacement closely matches data when \( \tau^c = 0.2 \), the gap in the labor tax rate discussed in Section 2.4.1 still remains.

On the other hand, impacts on the other aggregate variables seem to be small particularly in the \( \tau^c = 0.1 \) case. The percent changes are within ±0.2% except for the asset Gini. (Those of the asset Gini and the real interest rate are 1.24% and -0.17%, respectively.) In the \( \tau^c = 0.2 \) case, each change is amplified but still looks modest. Exceptionally the percent change in the asset Gini becomes 3.45% relative to the benchmark.

The change in labor supply \( L \) is very small but slightly higher than before the policy changes. The change is still small even in the \( \tau^c = 0.2 \) case, but we confirmed that \( L \) is monotonically increasing in the consumption tax rate \( \tau^c \) at least in the range \( \tau^c \in [0,0.3] \). As discussed in Section 2.4.1, in this experiment households experience the changes in three fiscal parameters. The decrease in the labor income tax rate has a positive effect on labor supply, reflecting that the substitution effect dominates the income effect. However, the increases in the consumption tax rate and pension benefits partially offset the positive effect because the higher consumption tax rate makes leisure relatively cheaper, and enough benefits after retirement may reduce precautionary...
savings in the youth.\footnote{15}

Another reason making the rise in labor supply look smaller is that \( L \) is defined as effective labor (sum of \( z^j_\eta_j l^j \)). The increase in labor in “non-effective” units is 0.11% when \( \tau^c = 0.1 \) and 0.27% when \( \tau^c = 0.2 \). Since the distribution of \( z^j \) and the mean age profile \( \eta_j \) are not affected by the policy changes, this fact suggests that poorer households (with lower productivity in general) become working more than richer households (with higher productivity) after the consumption tax hike. Later we can confirm this by Figure 2.8.

Capital stock \( K \) increases and its amount is a bit larger compared with the other aggregate variables. A household must pay consumption taxes throughout his life, but can benefit from the labor tax reduction only when he is working. If the increase in pension benefits is large enough, it may be possible that households reduce their lifetime assets. In this case, however, the amount of retirees’ income (pension benefits) is still lower than workers’ average earnings. Therefore especially middle-aged households have an incentive to save more while they can work.

Since the increase in \( K \) is larger than the increase in \( L \), the real interest rate \( r \) falls and the average real wage \( w \) rises. Output \( Y \) expands as both \( K \) and \( L \) increase, and therefore aggregate consumption \( C \) expands in spite of the consumption tax increase.

**Changes in Aggregate Variables by Age** Looking at only the changes in the entire economy, seemingly we can conclude that the macroeconomic impact of the consumption tax hike is not big at least in the long run. We found, however, that different groups of households are differently affected by the policy shift. Here we would focus on differences by age. Figure 2.8 shows percent changes in capital \( K \), labor \( L \) and consumption \( C \) when the consumption tax rate \( \tau^c \) is raised from 5% to 10%.

\footnote{15 Later we will see that the precautionary saving is not a main factor to generate our results. See the sensitivity analysis in Section 2.4.4.2.}
In the left graph \((K)\), we can see that households at ages 20 and 25 largely decrease capital. This is because they cannot earn so much relative to the middle ages, and thus with the higher consumption tax rate, they give priority to keep their consumption path smooth as much as possible rather than to save money.\(^\text{16}\) On the other hand, once they enter stages where they can get enough income, the lower labor tax rate enables them to accumulate assets more rapidly. Since a large share of assets are owned by the middle ages (see Figure 2.4\(^\text{17}\)), aggregate capital increases as well. After retirement, dissaving becomes quicker than the benchmark case. As we will see later, the reason is that poor households sharply decrease their assets right after earnings uncertainty disappeared.

In the middle graph \((L)\), we can see that younger households work more in response to the labor tax reduction, but the sizes of the changes are relatively modest. As shown in Figure 2.8 later, this increase is mainly caused by low-productivity workers.

At last, the right graph \((C)\) shows contrastive responses of the young and old. The consumption tax hike modestly reduces young household’s consumption, and the increase in pension benefits pushes up retiree’s consumption.

\(^{16}\)Another possibility is that the increase in future pension benefits may reduce a precautionary saving motive especially for very young people who are more likely to be borrowing constrained. Our sensitivity analysis, however, did not support this hypothesis. (See Section 2.4.4.2.)

\(^{17}\)Note that aggregate capital \(K\) is calculated as the sum of end-of-period individual assets by its definition, while Figure 2.4 is calculated from beginning-of-period assets as resources for consumption and investment in the current period. As the result the peak of \(K\) in Figure 2.4 is at age 55, while the peaks of assets for the median and 80th percentile households are at age 60 in Figure 2.4. The total sum of beginning-of-period assets in time \(t\) equals \(K_{t-1}\) less assets held by people who died by the beginning of time \(t\). Also note that the asset Gini coefficients are measured on a beginning-of-period basis, while the “fraction of borrowing constrained households” is measured on an end-of-period basis.
Ex-ante Social Welfare  As the graph for \( C \) suggests, in our experiment the consumption tax hike affects the life-cycle consumption path via the increase in pension benefits for retirees, which leads to the decrease in ex-ante social welfare in terms of consumption equivalent variation (CEV). We calculated CEV as

\[
\sum_{j=1}^{J} \beta^{j-1} \left( \prod_{j=1}^{J} s_{j} \right) \int_{X} u \left[ (1 + CEV) c_{j}^{0}, l_{j}^{0} \right] d\psi_{j}^{0} = \sum_{j=1}^{J} \beta^{j-1} \left( \prod_{j=1}^{J} s_{j} \right) \int_{X} u \left( c_{j}^{\tau_{c}}, l_{j}^{\tau_{c}} \right) d\psi_{j}^{\tau_{c}} \tag{2.14}
\]

where the superscripts “0” in the left-hand side of the equation represent initial steady-state values with the 5% consumption tax rate and the superscripts “\( \tau_{c} \)” in the right-hand side represent steady-state values with any other rate of \( \tau_{c} \). Hence, this measure compares expected lifetime utility for a newly born household in the counter-factual steady states with that in the initial steady state. Negative CEV means a decrease in welfare.

The last row in Table 2.4 shows that the fiscal policy that we are considering makes households worse off (although it is mild) in the sense that the economy after the consumption tax hike becomes more costly from the perspective of newly-born agents. The decreases in consumption and leisure in the youth push down the CEV, while the increases in consumption in old ages do not so much contribute to improve welfare because utility in old ages is more discounted due to the cumulative discount factor \( \beta^{j-1} \) and the cumulative survival probability \( \prod_{j=1}^{J} s_{j} \).

Wealth Inequality  In contrast to the limited responses in the aggregate variables, wealth distribution is a bit more clearly affected by the change in the consumption tax rate. The asset Gini coefficient increases from 0.611 to 0.619 by 1.24% when \( \tau_{c} = 0.1 \) and to 0.633 by 3.45% when \( \tau_{c} = 0.2 \). The Gini in the 20% tax case is higher than 0.628 in 1989, which is the highest value reported in Yamada (2008) (see Section 2.4.1).

Since part of extra consumption taxes are utilized to finance pension spending for old ages, it has a role of insurance for future low-earnings and longevity risks. Why does such a redistributational

\[18\text{This welfare implication seems to be consistent with a study by [Imrohoroglu et al. (1995), which analyzes the U.S. social security using a heterogeneous-agent OLG model. They report that a pay-as-you-go pension system can raise welfare compared with an economy with no pension system, but an optimal income replacement rate is much lower than the one in the real world. They found that a replacement rate that is higher than 30% increases deviation of life-cycle consumption paths from that implied by the first-best (planner’s) solution, and thus reduces welfare.}

58
policy cause the widening of wealth inequality? The main reason is that the inter-generational transfers induce different reactions across households. Figure 2.6 shows asset Gini coefficients by age, in which each value represents wealth inequality within each generation.

Figure 2.6: Change in Asset Gini by ages

The black solid line is the benchmark case ($\tau^c = 0.05$). We abbreviated age 20 because by assumption they are in perfect equality in terms of beginning-of-period assets. The wealth inequality decreases with age until around 45 but then it gradually increases. The consumption tax hike widens the wealth inequality in each generation (the red dotted line is the case of $\tau^c = 0.1$ and the blue dashed line is the case of $\tau^c = 0.2$). The widening is relatively large in very young generations and old generations, and its magnitude is at peak around age 65.

As seen in Figure 2.7, after the consumption tax hike, capital decreases at very young ages and old ages and increases at middle ages. Seeing more details, we can find that the decrease is mainly caused by poor people and the increase is caused by rich people. Figure 2.7 clearly shows that the reason why the wealth inequality is bigger in old ages. These graphs describe changes in assets held by the 20th, 50th (median) and 90th percentile households from bottom.
Compared with the case of $\tau^c = 0.05$, due to the increase in pension benefits, asset-poor households sharply reduce their assets in the timing of retirement, after which there is no uncertainty about labor earnings (the left graph in Figure 2.7). At the same time, this sharp fall of savings is the reason of the discrete change in consumption seen in Figure 2.5. In contrast, wealthy households keep their assets for a long time even after retirement (the right graph in Figure 2.7).

Figure 2.8 decomposes the changes in $K$, $L$ and $C$ by age and productivity. Since the idiosyncratic productivity shocks are persistent, in this model high-productivity people are likely to be asset-rich and low-productivity people are likely to be asset-poor.

High-productivity households increase assets through their middle ages, while low- and medium-productivity households significantly decrease assets at young and old ages (the left graph). The

19 More precisely, very few households continue to work and face uncertainty even at age 65 since retirement is not mandatory in our model, but their share in aggregate labor is trivial. All people exit the labor market by age 70.

20 Note that changes in $K$ and $C$ are plotted also for retirees but productivity does not affect retiree’s behavior. The “productivities” for retirees just represent average histories of shocks that were realized before retirement.
figure also show that the policy change mainly influences on low-productivity households by affecting their labor supply and consumption (the middle and right graphs). These differences in responses between rich and poor people widen the inequality within each age group, and it is caused by the combination of the labor tax reduction and the expansion of social security benefits.

2.4.3 Alternative Policy Schemes: Ways that Consumption Taxes Are Used

As discussed in Section 2.4.2.1, in our benchmark model the extra revenue from the consumption hike is assumed to be used for two purposes: the labor tax reduction for workers and the expansion of pension benefits for retirees. This combination creates the complicated mechanism described in Section 2.4.2.2, and eventually it leads to the limited impacts on aggregate variables.

To understand the benchmark results more clearly, in this subsection we try additional two experiments with simpler tax schemes, i.e., the cases that all of the additional consumption tax revenue is used to fund: (i) only the labor tax reduction and (ii) only the pension expenditures, respectively. As mentioned in Section 2.4.1, the Japanese government intends to use the revenue from the additional consumption taxes for four purposes: (a) pension benefits, (b) health care, (c) long-term care and (d) child care support. Since the expenditures for child care support is not so large, possibly most of these components are spent for the old generations. If so, effects of the consumption tax hike may become more similar to the one in the case (ii).

The results are shown in the columns (4) and (5) in Table 2.3. For convenience, we call the case (i) as “$p_c = 0$ case” and the case (ii) as “$p_c = 1$ case”. (For comparison purpose, we assume that the initial revenue collected by the government before the consumption tax hike is still used to fund both government expenditures and pension benefits according to the weight $p_c = 0.42$.)

In the $p_c = 0$ case, macroeconomic impacts triggered by the 5 pp increase in consumption tax rate are much greater than the previous case.\footnote{The slight change in the earnings replacement rate reflects the changes in $w$ and $L$ in the general equilibrium. Likewise, in the $p_c = 1$ case the labor tax rate slightly rises without any additional funding from consumption taxes.} In response to the large labor tax reduction, aggregate labor supply increases by more than 1.5%, and the increases in capital, output and consumption are also larger. Since CEV is positive and large, we can conclude that the additional consumption tax revenue should be used for the labor tax reduction if ignoring transition dynamics and any other issues regarding fiscal sustainability at least based on our framework.
Figure 2.9 shows that effects on each age group are totally different from the benchmark case.

Figure 2.9: Changes in $K$, $L$ and $C$ when All Consumption Taxes are Used for Labor Tax Reduction

Under the new policy there is no income shifts from the young to the old, but rather there are transfers from the old to the young. The rise in future after-tax earnings enables young households to drastically decrease their assets (the left graph, $K$) and instead to increase consumption (the right graph, $C$), which significantly improves expected lifetime utility. Households save more over the lifetime since now they cannot receive additional pension benefits despite that consumption is taxed even after retirement. The middle graph ($L$) shows that now households increasingly supply labor as they getting older, while very young households decrease it since they have a priority on consumption smoothing rather than saving. The increase in leisure in the youth also might contribute to the rise in CEV. This experiment suggests that in our model, policies such that reduces tax burdens on young generations seem to be desirable in terms of welfare.

In the $p_c = 1$ case, the economy shrinks and welfare worsens (the column (5) in Table 2.4). Figure 2.10 shows responses opposite to Figure 2.9.

Figure 2.10: Changes in $K$, $L$ and $C$ when Additional Consumption Taxes are Used for Benefits
Young households significantly increase their savings and decrease consumption due to the heavier burden of consumption taxes without support from the labor tax reduction. This policy makes consumption smoothing for the youth more difficult, and thereby severely reduces expected lifetime utility.

At last, implications for wealth distribution are also different from the benchmark. Figure 2.11 plots the asset Gini by age for both experiments. In the $p_c = 0$ case, the widening of asset reduction at retirement age seen in Figure 2.7 no longer occurs, and thus the asset Gini coefficients for old ages are almost the same as the benchmark. For young ages, however, the wealth inequality increases since the sharp decline in capital seen in Figure 2.9 is mainly caused by low-productivity households. In this case, we should notice that utility is improved in spite of the greater wealth disparity.

In contrast, in the $p_c = 1$ case the wealth inequality narrows in young generations, although higher pension benefits widen inequality in old generations more than the benchmark case.

In summary, impacts on aggregate variables, welfare and wealth distribution are strongly affected by assumptions regarding how to use consumption taxes. Particularly, an expansion of pension benefits in exchange for burdens on young generations might harm welfare.
2.4.4 Sensitivity Analyses

The results described in previous sections suggest that household responses to intertemporal transfers play a key role in our model. In this section, we change some parameter values that possibly affect the household’s behavior and check the robustness of the results.

2.4.4.1 Elasticity of Intertemporal Substitution

Based on the results provided in Section 2.4.2, we conclude that the impact of 5 pp increase in the consumption tax rate (with a labor tax reduction and an expansion of pension benefits) seems small at least in the long run. However, this result might depend on the parameter $\sigma$, which equals the inverse of the elasticity of intertemporal substitution (EIS). 0.33 of EIS (i.e., $\sigma = 3.0$) in our benchmark is a bit lower than some other Japanese literature using the CRRA preference specification such as 0.5 of EIS (i.e., $\sigma = 2.0$). Hence, we recalibrated the model using $\sigma = 2.0$ and checked how the results change.

In this experiment, first we set values of $\beta$ and $\gamma$ to satisfy their targets such as $K/Y = 2.5$ under the 5% consumption tax rate. We confirmed that this recalibrated model generates very similar data to the benchmark model except for the asset Gini, which rises from 0.611 in the benchmark to 0.632 in the recalibrated model. Then we conducted the simulation that raises $\tau^c$ from 0.05 to 0.10 again. Table 2.5 shows that this experiment does not affect most of the conclusions in Section 2.4.2, especially for changes in aggregate variables.
Table 2.5: Effects of Consumption Tax Increase when $\sigma = 2.0$

<table>
<thead>
<tr>
<th></th>
<th>(1') $\tau^c = 0.05$</th>
<th>(2') $\tau^c = 0.10$</th>
<th>(3') $\tau^c = 0.20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>Labor</td>
<td>100</td>
<td>100.05</td>
</tr>
<tr>
<td>$K$</td>
<td>Capital</td>
<td>100</td>
<td>100.25</td>
</tr>
<tr>
<td>$Y$</td>
<td>Output</td>
<td>100</td>
<td>100.12</td>
</tr>
<tr>
<td>$C$</td>
<td>Consumption</td>
<td>100</td>
<td>100.12</td>
</tr>
<tr>
<td>$K/L$</td>
<td>Capital-labor ratio</td>
<td>100</td>
<td>100.20</td>
</tr>
<tr>
<td>$r$</td>
<td>Real interest rate</td>
<td>6.21%</td>
<td>6.19%</td>
</tr>
<tr>
<td>$w$</td>
<td>Mean real wage</td>
<td>100</td>
<td>100.07</td>
</tr>
<tr>
<td>$\tau^l$</td>
<td>Labor tax rate</td>
<td>21.4%</td>
<td>18.6%</td>
</tr>
<tr>
<td></td>
<td>Earnings replacement rate</td>
<td>48.9%</td>
<td>53.9%</td>
</tr>
<tr>
<td>Asset Gini</td>
<td></td>
<td>0.632</td>
<td>0.639</td>
</tr>
<tr>
<td>CEV</td>
<td>0</td>
<td>-0.06%</td>
<td>-0.09%</td>
</tr>
</tbody>
</table>

The changes are quite similar to the benchmark. (Percent changes in the asset Gini are 1.22% when $\tau^c = 0.1$ and 3.40% when $\tau^c = 0.2$.) As an exception, however, welfare cost measured by CEV is reduced by half (-0.06%). This is because households strongly prefer to smooth their consumption path when EIS is low ($\sigma$ is high). In general younger households are more likely to be poor, and thus have more or less difficulty to smooth their consumption sufficiently. Hence, policy changes that hinder the smoothing have a larger impact on utility compared with the case that the EIS is high and households allow some extent of intertemporal substitution. Figure 2.12 illustrates this situation, showing the differential responses by age.

Figure 2.12: Changes in $K$, $L$ and $C$ when $\sigma = 2.0$

When the EIS was 0.33, the 5 pp increase in the consumption tax rate decreased capital by 4.5% at age 20 and 2.2% at age 25 (the left graph in Figure 2.12). In the case of $EIS = 0.5$, the decreases
are significantly larger and reaches 11.6% at age 20 and 5.4% at age 25, although changes for other ages are relatively similar to the benchmark case (the left graph in Figure 2.12). The decrease in consumption at age 20 is almost the same in both cases (-0.1%), but in the high EIS case consumption of the other working generations does not respond much to the consumption tax hike. (Recall that the changes in consumption when EIS = 0.33 looks more gradual.) This means that when EIS is high, households can allow the consumption path over life cycle to fluctuate more, mitigating the negative effect of income shift between generations on lifetime utility.

Since young households constitute only a small share of aggregates, the change in EIS does not so much affect the changes in aggregate variables. However, the above discussion suggests that the value of $\sigma$ may have more important implications for responses of specific age groups and welfare.

### 2.4.4.2 Uncertainty in Idiosyncratic Productivity

From the discussions so far, one of main causes generating our results in the counter-factual experiments seems to be a life cycle saving motive via an intertemporal shift of income, not a precautionary one. To check this hypothesis, we implemented an additional experiment that increases the value of the uncertainty parameter $\sigma_e$ by 10%. In this experiment, we did not recalibrate the model, i.e., $\beta$ and $\gamma$ are unchanged from the benchmark model, because $\sigma_e$ itself is one of the calibrated parameters.

Table 2.6 shows that this experiment does not change the results so much.

<table>
<thead>
<tr>
<th></th>
<th>(1') $\tau^c = 0.05$</th>
<th>(2') $\tau^c = 0.10$</th>
<th>(3') $\tau^c = 0.20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$ Labor</td>
<td>100</td>
<td>100.07</td>
<td>100.24</td>
</tr>
<tr>
<td>$K$ Capital</td>
<td>100</td>
<td>100.28</td>
<td>101.02</td>
</tr>
<tr>
<td>$Y$ Output</td>
<td>100</td>
<td>100.15</td>
<td>100.52</td>
</tr>
<tr>
<td>$C$ Consumption</td>
<td>100</td>
<td>100.16</td>
<td>100.53</td>
</tr>
<tr>
<td>$K/L$ Capital-labor ratio</td>
<td>100</td>
<td>100.21</td>
<td>100.78</td>
</tr>
<tr>
<td>$r$ Real interest rate</td>
<td>6.13%</td>
<td>6.11%</td>
<td>6.07%</td>
</tr>
<tr>
<td>$w$ Mean real wage</td>
<td>100</td>
<td>100.08</td>
<td>100.28</td>
</tr>
<tr>
<td>$\tau^l$ Labor tax rate</td>
<td>21.5%</td>
<td>18.6%</td>
<td>12.9%</td>
</tr>
<tr>
<td>Earnings replacement rate</td>
<td>48.9%</td>
<td>53.9%</td>
<td>63.9%</td>
</tr>
<tr>
<td>Asset Gini</td>
<td>0.624</td>
<td>0.631</td>
<td>0.644</td>
</tr>
<tr>
<td>CEV</td>
<td>0</td>
<td>-0.11%</td>
<td>-0.22%</td>
</tr>
</tbody>
</table>
Percent changes in variables except for capital are very similar to the benchmark. (Percent changes
in the asset Gini are 1.19% when $\tau^e = 0.1$ and 3.31% when $\tau^e = 0.2$.) Percent changes in capital are
slightly larger than the benchmark result, but they are still not large especially for the $\tau^e = 0.10$
case.

Figure 2.13 shows changes in $K$, $L$ and $C$ by age. Roughly speaking age profiles are similar to
Figure 2.5 in Section 2.4.2.

Figure 2.13: Changes in $K$, $L$ and $C$ when $\sigma_e$ is higher by 10%

2.5 Conclusions

In this paper, we extended the Huggett-type OLG model to include endogenous labor and a con-
sumption tax, and calibrated the model to match the Japanese economy. We found that the model
well replicates earnings and wealth distributions. Although Huggett (1996) reports that his model
cannot reproduce upper and lower tails of the U.S. wealth distribution, our model more successfully
approximates the upper tail of distribution because in Japan wealth is less concentrated to the
richest people relative to the United States.

Then we investigated effects of consumption tax hike on aggregate variables, welfare and wealth
distribution using the calibrated model. Our simulation result suggests that a combination of a
consumption tax hike, a labor tax reduction and an expansion of pension benefits has limited
impacts on aggregate variables and welfare. However, we also found that this result highly depends
on the assumption regarding how the government reallocate the additional consumption tax revenue.

Note that this statement means that “changes” between the two steady states are not so affected. The “level” of
capital stock (before normalized) in each steady state significantly rises (nearly 3%) due to the increase in uncertainty
for future earnings.
Based on our framework, it is much better in terms of welfare to use the additional consumption taxes for the labor tax reduction than for the expansion of pension benefits. The intertemporal transfers from the young to the old reduce expected lifetime utility by affecting life-cycle consumption paths.

Another main finding is that the consumption tax hike widens wealth inequality within very young ages and retirees. The consumption tax hike differently affects different groups of households. In our simulation, while wealthy households accumulate more assets, the poor young and the poor old reduce their savings sharply. This asymmetric responses between households widen wealth inequality.
Chapter 3

A Model with an Endogenous Cash-in-advance Constraint

3.1 Introduction

One of central issues in monetary economics is discussions about whether money has real effects or not and inflation is desirable or not. Many economists seem to believe that the classical dichotomy holds in the long run and some also may believe it is true even in the short run, but in the real world a number of countries are pursuing mild inflation as a mid- and long-term target. The seminal paper by Friedman (1969) provides a robust theoretical conclusion called the Friedman rule using a money-in-utility model, which states that the optimal nominal interest rate is zero so that the opportunity cost of money holdings equals zero. This famous theorem implies that the optimal inflation rate should be negative, and is known that it holds in various models. In a number of representative agent models with multiple goods, the Friedman rule holds because inflation acts as a distortionary tax, even if money does not appear in the utility function. For example, Lucas and Stokey (1983) constructed a model in which there are cash and credit goods and only the cash good is subject to a cash-in-advance (CIA) constraint. They found that the Friedman rule holds because the positive nominal interest rate can be interpreted as a tax on the cash good, and thus distorts household behavior.

In this paper, we extend a basic CIA constraint model by introducing an endogenous CIA
constraint, and examine whether the model can change conclusions in traditional CIA models such as
monetary supernutrality and the Friedman rule. To do this, we consider that in our model
economy there is a financial sector offering a service that facilitates transactions to purchase goods.
We assume that expansion of economy develops the financial sector and its financial technology,
which relaxes the CIA constraint that a household faces. In general, advanced countries with large
GDP tend to have a well developed financial services industry which provides people with a wide
range of financial services, including payment services such as credit card transactions (Figure 3.1
for the U.S. economy). Therefore, we think that it is a good exercise to see what happens if we
assume correlation between total output growth and growth of financial sector.

Figure 3.1: Real GDP of the U.S. Economy and the U.S. Financial Sector

![Real GDP Chart]

Note: Real estate and rental and leasing industries are excluded from the financial sector.
Source: the Bureau of Economic Analysis.

If development in the financial sector leads to progress of the financial technology, then it may
help households to buy more goods and services without cash. For example, a household’s choice
about credit card usage seems to be determined by how it is convenient and safe to use, and it may
depend on technological development of transaction services, e.g., varieties of payment methods
and the number of shops where people can use the card. If so, the advance in financial technology

\footnote{Another possible reason is people’s preference for cash, but this may also come from the financial technology. For example, Japan is a country that has the strongest cash preference among major countries reported in the BIS statistics, where in 2016 a cash-GDP ratio was the highest at 19.96% across the countries and card payments per GDP was at 10%, which was lower than the average, 12.1%. Some surveys suggest that at least to some extent,}
may give people incentives to buy more with credit. This paper’s motivation is to provide a simple model to analyze effects of such endogenous transaction technology on a real economy, although a study about empirical relationships themselves between economic growth, development of financial sector and transaction technology is beyond the scope.

Our steady-state analysis found that the monetary superneutrality no longer holds even in the long run and inflation can have a positive effect on capital accumulation (Tobin effect). As with standard CIA models, inflation increases the opportunity cost of holding money, but in our model, capital investment has an additional return via the financial sector and a household can partially offset the negative effect of inflation by increasing capital. The Friedman rule also does not hold and the optimal inflation rate may be positive. Then we extended the model into a stochastic one and investigated effects of transitory monetary shocks in the short run. In our stochastic model, the endogenous CIA constraint has an effect to stabilize an economy against money growth shocks.

The rest of paper is organized as follows. Section 3.2 presents our model with an endogenous CIA constraint. Section 3.3 investigates the model properties in the steady state. Section 3.4 explore short-run dynamics using a stochastic version of the model. Section 3.5 conducts sensitivity analyses, thereby discussing about some limitations of our model. Section 3.6 briefly concludes.

3.2 Model

In this section, we construct a model with an endogenous CIA constraint by introducing a transaction technology which is affected by size of a financial sector.

3.2.1 Household’s Problem

We assume that there is no population growth and the population is normalized as one. Labor supply is exogenous, and thus we treat per capita capital stock \( k_t \) as the only production factor.

A representative yeoman household maximizes its expected lifetime utility subject to a budget constraint and an endogenous CIA constraint. The expectation is over monetary shocks.
\[
\max \{c_t, k_t, b_t, m_t\} \quad E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]
\]  
\[\text{s.t.} \quad c_t + k_t + b_t + m_t \leq \tau_t + y_t + (1 - \delta)k_{t-1} + \frac{(1 + \delta_i t - 1)b_{t-1}}{1 + \pi_t} + \frac{m_{t-1}}{1 + \pi_t}, \quad (3.2)\]

where \(c_t\) is consumption at time \(t\), \(k_t\) is capital stock, \(b_t\) is one-period bonds, \(m_t\) is real money balances, \(\tau_t\) is lump-sum transfers, \(y_t\) is output, \(\delta_i t\) is the nominal interest rate, \(\pi_t\) is the inflation rate, \(\delta\) is the capital depreciation rate. \(\tilde{\rho}(\varphi_t)\) represents the transaction technology. The production function is

\[y_t = f(k_{t-1})\]  
\[(3.4)\]

which is assumed to satisfy usual conditions such as \(f'(\cdot) > 0, f''(\cdot) < 0, \lim_{k \to 0} f'(k) = \infty\) and \(\lim_{k \to \infty} f''(k) = 0\).

The above second inequality (3.3) is the endogenous CIA constraint. The only difference from a basic CIA model (e.g. see Chapter 3.3 in [Walsh, 2010]) is the presence of the term \(\tilde{\rho}(\varphi_t)\). The basic CIA model assumes that a household needs money (including government transfers provided in the form of money) for any purchase of consumption goods. In our model, only a part of consumption is subject to the CIA constraint, and the fraction \(\tilde{\rho}\) is endogenously determined reflecting development of a financial services sector, captured by the variable \(\varphi_t\). We assume that the function \(\tilde{\rho}(\cdot)\) satisfies the following conditions.

\[
\tilde{\rho}(0) = 1, \quad (3.5)
\]

\[
\tilde{\rho}'(\cdot) < 0, \quad (3.6)
\]

\[
\tilde{\rho}''(\cdot) > 0, \quad (3.7)
\]

\[
\lim_{\varphi \to 0} \tilde{\rho}'(\varphi) = -\infty, \quad (3.8)
\]

\[
\lim_{\varphi \to \infty} \tilde{\rho}'(\varphi) = 0. \quad (3.9)
\]
is decreasing in $\phi$, which means when the financial sector develops, the CIA constraint is eased via improvement in transaction technology, and as a result a household can buy more without cash. This easing effect is, however, marginally diminishing as $\phi$ increases. We also assume that the development in the financial sector depends on size of the total economy.

$$\phi_t = \chi y_t^\kappa$$

(3.10)

where $\chi \in (0, 1)$ and $\kappa > 0$. Figure 3.2 illustrates an example of a shape of $\hat{\rho}$. In summary, our assumptions imply that output growth ends up relaxing the CIA constraint through financial sector’s development.

![Figure 3.2: Example of $\hat{\rho}$](image)

We included a parameter $\kappa$ to capture possible non-linearity between the financial sector and the overall economy. If $\kappa = 1$, the relationship between $\phi_t$ and $y_t$ would be linear, which implies that $\chi$ represents a constant share of the financial sector relative to GDP ($\chi = \frac{\phi_t}{y_t}$). Using the above assumptions (3.9)-(3.11), we can redefine $\hat{\rho}$ as a function of capital stock $k$.

$$\hat{\rho}(\phi_t) = \hat{\rho}(\chi [f(k_{t-1})]^\kappa)$$

$$\equiv \rho(k_{t-1}).$$

---

Empirically it is not necessarily clear whether the share of financial sector is linear to $y$. In the United States, the GDP share of financial sector has substantially increased during five decades until 2000’s as the total economy has grown, but this might be spurious due to unit roots. We will conduct some tests for this issue in Section 3.4.3.
We can easily show that \( p \) has the same properties as \( \tilde{p} \), such as \( p' < 0 \) and \( p'' > 0 \). For the purpose of comparison, we call a model such that \( p \) is always unity as the “basic CIA model” in the rest of the paper.

**Optimization** We define the RHS of the budget constraint (3.2) as total wealth \( a_t \). Then using the value function, the above household’s problem can be rewritten as

\[
V(a_t, m_{t-1}, k_{t-1}) = \max_{c_t, k_t, b_t, m_t} \left[ u(c_t) + \beta E_t [V(a_{t+1}, m_t, k_{t+1})] + \lambda_t (a_t - c_t - k_t - b_t - m_t) + \mu_t \left( \frac{m_{t-1}}{1 + \pi_t} + \tau_t - \tilde{p}(\varphi_t) c_t \right) \right]
\]

where \( \lambda_t \) and \( \mu_t \) are Lagrange multipliers for the budget constraint (3.2) and for the CIA constraint (3.3), respectively. The first-order conditions and the envelope conditions are

\[
\begin{align*}
[c_t] & \quad u'(c_t) - \lambda_t - \mu_t \tilde{p}(k_{t-1}) = 0, \\
[k_t] & \quad \beta E_t \left[ V_a(a_{t+1}, m_t, k_t) \left[ f'(k_t) + 1 - \delta \right] - V_k(a_{t+1}, m_t, k_t) \right] - \lambda_t = 0, \\
[b_t] & \quad \beta E_t \left[ \frac{V_a(a_{t+1}, m_t, k_t)}{1 + \pi_{t+1}} \right] - \lambda_t = 0, \\
[m_t] & \quad \beta E_t \left[ \frac{V_a(a_{t+1}, m_t, k_t)}{1 + \pi_{t+1}} + V_m(a_{t+1}, m_t, k_t) \right] - \lambda_t = 0,
\end{align*}
\]

[Envelope] \( V_a(a_t, m_{t-1}, k_{t-1}) = \lambda_t \), \( V_m(a_t, m_{t-1}, k_{t-1}) = \mu_t \frac{1}{1 + \pi_t} \), \( V_k(a_t, m_{t-1}, k_{t-1}) = -\mu_t \tilde{p}'(k_{t-1}) c_t \)

along with the usual transversality conditions and the Kuhn-Tucker conditions. From the equation

\[
\rho' = \frac{\partial \tilde{p}(\varphi_t)}{\partial k_{t-1}} = \frac{\partial \tilde{p}(\varphi_t)}{\partial \varphi_t} \frac{\partial \varphi_t}{\partial k_{t-1}} = \chi \tilde{p}'(\varphi_t) f'(k_{t-1}) < 0,
\]

\[
\rho'' = \frac{\partial [\chi \tilde{p}(\varphi_t) f'(k_{t-1})]}{\partial k_{t-1}} = \chi \left[ \chi \tilde{p}''(\varphi_t)[f'(k_{t-1})] + \tilde{p}'(\varphi_t) f''(k_{t-1}) \right] > 0.
\]
(3.11), $\lambda_t$ is equal to the marginal utility of wealth. From the equations (3.12) and (3.13), the Lagrange multiplier $\mu_t$ is associated with two marginal values; one is the marginal utility of money and another one is that of capital, and thus a household must choose its portfolio between money and capital so as to satisfy these two equations. Note that in the basic CIA model the multiplier $\mu_t$ simply represents the marginal value of money holdings (multiplied with the inflation rate) and is irrelevant to capital.

Eliminating the marginal values from the equations, we have

\[
\begin{align*}
[c_t] & \quad u'(c_t) = \lambda_t + \mu_t \rho(k_{t-1}), \\
[k_t] & \quad \lambda_t = \beta E_t \left[ \lambda_{t+1} \left[ f'(k_t) + 1 - \delta \right] - \mu_{t+1} \rho'(k_t) c_{t+1} \right], \\
[b_t] & \quad \lambda_t = \beta E_t \left[ \lambda_{t+1} (1 + r_t) \right], \\
[m_t] & \quad \lambda_t = \beta E_t \left( \frac{\lambda_{t+1} + \mu_{t+1}}{1 + \pi_{t+1}} \right).
\end{align*}
\]

The real interest rate $r_t$ appeared in the FOC for bonds (3.16) is defined by the following Fisher relationship.

\[
1 + r_t \equiv E_t \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right).
\]

From the FOC for bonds (3.16) and the FOC for money (3.17), a ratio of the Lagrange multipliers equals the net nominal interest rate.

\[
\frac{\mu_{t+1}}{\lambda_{t+1}} = i_t.
\]

Since both $\mu_t$ and $\lambda_t$ are non-negative, the CIA constraint (3.3) will bind if $i_t > 0$. In other words, $\mu_t = 0$ when the nominal interest rate is zero. This property holds in the basic CIA model as well, since the equations (3.16)-(3.18) are the same between both models. Differences between the models are that in the basic CIA model, (i) the second term $\mu_t \rho(k_{t-1})$ in the RHS of the FOC for consumption (3.14) is replaced with just $\mu_t$, and (ii) the last term $-\mu_{t+1} \rho'(k_t) c_{t+1}$ in the RHS of the FOC for capital (3.15) is eliminated.

We will focus only on the binding case, i.e., the case that $i > 0$ and the equation (3.3) holds.
with strict equality, since clearly our model is equivalent to the basic CIA model when \( \mu_t = 0 \) from the above discussion.

### 3.2.2 Government and Equilibrium

In this economy, there is neither government bonds, taxes nor purchases. All of lump-sum transfers are financed by seigniorage. The government budget constraint (GBC) is

\[
\tau_t = \frac{M_t^s - M_{t-1}^s}{P_t} \equiv m_t^s - \frac{m_{t-1}^s}{1 + \pi_t}
\] (3.20)

where \( M_t^s \) is nominal money supply, \( P_t \) is the price level, \( m_t^s \equiv \frac{M_t^s}{P_t} \) and \( 1 + \pi_t \equiv \frac{P_t}{P_{t-1}} \). Suppose that the nominal money supply grows at rates \( \theta_t \), then the law of motion for real money is

\[
m_t^s = \frac{1 + \theta_t}{1 + \pi_t} m_{t-1}^s.
\] (3.21)

We assume that \( \theta_t \) follows some stochastic process disturbed by a noise \( u_t \), and later Section 6.3 analyzes short-run responses of the economy to monetary shocks. Before that, however, in the next Section 3.3 we investigate steady-state properties and set \( \theta_t = \theta \) (constant) for a while.

Now we define a general equilibrium with the endogenous CIA constraint. Note that our definition has an aspect that is a bit similar to the planner’s problem in the sense that a household does not take the transaction technology as given. In our model the household can affect strength of the CIA constraint by changing a level of capital via the function \( \rho(k_{t-1}) \). We will discuss how results change if this assumption is altered.

**Definition:** Given the initial condition on real money supply \( m_0^s \) and the exogenous monetary shocks \( \{u_t\} \), an equilibrium with an endogenous CIA constraint consists of

- quantities \( \{c_t, k_t, b_t, m_t\} \),
- prices \( \{r_t, i_t, \pi_{t+1}\} \) and
- fiscal and monetary policies \( \{m_t^s, \theta_t, \tau_t\} \)

such that
1. given the prices and the government policies, the quantities \( \{c_t, k_t, b_t, m_t\} \) solve the problem of the representative household (3.1) subject to the budget constraint (3.2) and the endogenous CIA constraint (3.3) with the financial technology \( \rho(k_{t-1}) \),

2. the government satisfies its budget constraint (3.20) and the real money growth rule (3.21), and

3. money and bond markets clear, i.e., \( m_t = m_t^e \) and \( b_t = 0 \).

### 3.3 Steady State Analysis

In this section, we study comparative statics between steady states, focusing on the case that the CIA constraint is binding as discussed in Section 3.2. Steady-state equilibrium values are denoted by a subscript \( ss \). As we will see later, our model does not exhibit superneutrality of money. Furthermore, it also shows that the optimal nominal interest rate is not zero.

#### 3.3.1 Equilibrium

**Inflation and Interest Rates** The steady-state inflation rate \( \pi_{ss} \), the real interest rate \( r_{ss} \) and the nominal interest rate \( i_{ss} \) are exactly the same as the basic CIA model. The law of motion for real money supply (3.21) implies that

\[
\pi_{ss} = \theta. \tag{3.22}
\]

The FOC for bonds (3.16) implies that the steady-state gross real interest rate equals the inverse of the discount factor \( \beta \).

\[
1 + r_{ss} = \frac{1}{\beta}. \tag{3.23}
\]

At last the Fisher relationship (3.18) gives us the steady-state nominal interest rate.

\[
1 + i_{ss} = \frac{1 + \theta}{\beta}. \tag{3.24}
\]

The last equation (3.24) implies that the nominal interest rate \( i_{ss} \) is strictly positive if \( 1 + \theta > \beta \),
in other words we can focus on the case of the binding CIA constraint if the gross nominal money growth rate \(1 + \theta\) is greater than the discount factor \(\beta\).

**Velocity of Money** The rest of equilibrium values, however, can be different from ones in the basic CIA model. Combining the binding CIA constraint (3.21) with the GBC (3.20) to eliminate \(\tau_{ss}\) and \(\frac{m_{ss}}{1+\pi_{ss}}\), we have

\[
\frac{c_{ss}}{m_{ss}} = \frac{1}{\rho(k_{ss})} \equiv v(k_{ss}) > 1. \tag{3.25}
\]

So the velocity of money \(\frac{c_{ss}}{m_{ss}}\) is not unity as with the usual cash-and-credit goods model, while the velocity in the basic CIA model always equals one. In our model, the steady-state velocity equals the inverse of the transaction technology \(\rho(k_{ss})\). We define it as \(v(k_{ss})\).

**Consumption** Since this is a representative agent model, bonds \(b_t\) should be zero at optimum to clear the bond market. Substituting the GBC (3.20) into the household budget constraint (3.2) with equality, lump-sum transfers \(\tau_{ss}\) are canceled out by seigniorage \(m_{ss} - \frac{m_{ss}}{1+\pi_{ss}}\). With these facts, the resource constraint which determines steady-state consumption is

\[
c_{ss} = f(k_{ss}) - \delta k_{ss} \equiv c(k_{ss}). \tag{3.26}
\]

This expression itself is the same as the one in the basic CIA model, but in general the equilibrium value of capital stock, \(k_{ss}\), is different from the one in the basic model as we will see immediately below. Therefore \(c_{ss}\) can be different from the value in the basic CIA model as well.

**Capital Stock** A main difference from the basic CIA model appears in an equation that determines equilibrium capital stock. Combining the FOC for capital (3.15) with the equations (3.14), (3.23) and (3.24), we obtain

\[
\underbrace{f'(k_{ss}) - \delta}_{\text{return from production}} \underbrace{-i_{ss}\rho'(k_{ss})c(k_{ss})}_{\text{additional return}} = r_{ss}. \tag{3.27}
\]

Note that in the basic CIA model this condition becomes a very common equation \(f'(k_{ss}) - \delta = r_{ss}\), which means the marginal product of capital (after depreciation) must equal the real interest rate.
In this case, steady-state capital stock $k_{ss}$ is independent of inflation as the steady-state interest rate $r_{ss}$ is just a constant. In our model, however, the inflation rate $\pi_{ss}$, or equivalently, the money growth rate $\theta$ can affect $k_{ss}$ through the new term $-i_{ss}\rho'(k_{ss})c(k_{ss})$ because the nominal interest rate $i_{ss}$ depends on $\theta$ as the equation (3.24) shows. Since $\delta$ and $r_{ss}$ are given, capital stock is the only variable that a household can adjust to satisfy the optimality condition (3.27) with equality.

With the endogenous CIA constraint, monetary superneutrality no longer holds even in the long run.

Before discussing intuition, we try to do comparative statics to see how the equilibrium capital react to inflation. First we define the following two functions.

$$A(k) = f'(k) - \delta,$$  \hspace{1cm} (3.28)

$$B(k) = r_{ss} + i_{ss}\rho'(k)c(k).$$  \hspace{1cm} (3.29)

$A(k)$ is the first term in the LHS of the equation (3.27) and $B(k)$ is the RHS plus the second term in the LHS of the same equation. So if the curves $A(k)$ and $B(k)$ have an intersection for positive level of $k$, it is the steady-state equilibrium value of capital.

$A(k)$ is a curve representing the marginal product of capital, which is monotonically decreasing in $k$ and moves from $+\infty$ to $-\delta$ as $k$ goes from 0 to $+\infty$. As for $B(k)$, theoretically the term $i_{ss}\rho'(k)c(k)$ (this term is negative) can be either increasing or decreasing in $k$, depending on curvatures of $\rho'(k)$ and $c(k)$. However, we can guess that in most cases it is monotonically increasing in $k$ and approaches to 0 from the negative side because of the assumptions \(\lim_{\varphi \to 0} \rho'(\varphi) = -\infty\) and \(\lim_{\varphi \to \infty} \rho'(\varphi) = 0\), in which case the economy has a unique steady-state equilibrium. (See Appendix C.1 for more detailed discussion about the shape of $B(k)$.) For illustrative purpose, now we consider the following simple specifications. We put a restriction that $\kappa < \alpha^{-1}$ to satisfy the assumptions (3.7)-(3.9).

4The specification for the transaction technology (3.31) implies

\[
\begin{align*}
\rho(k_{t-1}) &= (1 + \chi k_{t-1}^{\alpha})^{-\gamma}, \\
\rho'(k_{t-1}) &= -\alpha \kappa \chi \gamma (1 + \chi k_{t-1}^{\alpha})^{-\gamma - 1} k_{t-1}^{\alpha - 1} < 0, \\
\rho''(k_{t-1}) &= \alpha \kappa \chi \gamma \left[ \alpha \chi (\gamma + 1) (1 + \chi k_{t-1}^{\alpha})^{-\gamma - 2} k_{t-1}^{\alpha (\alpha - 1)} + (1 - \alpha \kappa) (1 + \chi k_{t-1}^{\alpha})^{-\gamma - 1} k_{t-1}^{\alpha - 2} \right].
\end{align*}
\]
Under these specifications, we checked various sets of parameter values and confirmed that in each case there exists a unique equilibrium. Figure 3.3 shows how equilibrium capital is determined in this model.

In the basic CIA model (or equivalently, in the case that the CIA constraint is endogenous but not binding), equilibrium capital is determined at the point $E_0$ in Figure 3.3, which satisfies the classical condition $f'(k) - \delta = r_{ss}$.

When the CIA constraint is endogenous and binding ($i > 0$), an equilibrium is determined at the point $E$. Since $A(k)$ has a downward slope and $B(k)$ is below the horizontal line at $r_{ss}$, equilibrium capital is higher than the one in the case that $i = 0$. Furthermore, an increase in $\theta$ raises $i_{ss}$ and thereby shifts the curve of $B(k)$ downward. In our model inflation promotes capital accumulation.

---

If $1 - \alpha \kappa > 0$, the second derivative $\rho''(k_{t-1})$ is always positive and the assumptions (3.8) and (3.9) are also satisfied.

The equilibrium capital in the basic CIA model can be algebraically solved as $k_{ss} = \left[ \frac{\alpha \beta}{1 - \beta(1 - \gamma)} \right]^{\frac{1}{1 - \gamma}}$. 

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The intuition is as follows. As discussed above, in the basic CIA model the net marginal return on capital is given by $f'(k) - \delta = r_{ss}$ and not affected by inflation. In our model, a household can receive an additional return from capital investment, which appears as the second term in the LHS of the equation (3.27). This term, $-i_{ss} \rho' (k_{ss}) c(k_{ss})$, represents the discounted marginal utility received in the next period from improvement of credit transactions caused by additional capital accumulation. If a household gives up one unit of today’s consumption and invests on capital, then in the next period the CIA constraint is relaxed via the development of financial sector and the household can increase consumption by $-\rho' (k_{ss}) c(k_{ss})$ (this term is positive since $\rho'$ is negative).

The steady-state nominal interest rate $i_{ss}$ is a ratio of shadow prices $\mu_{ss}$ and $\lambda_{ss}$ as the equation (6.14) shows, and thus converts the amount of consumption to utility units. In this case, $\mu_{ss}$ can be interpreted as a measure to evaluate the next-period additional consumption in terms of discounted utility via the easing of the endogenous CIA constraint and $\lambda_{ss}$ is a utility measure for the decrease in current-period consumption. (See also the equations (6.11)-(6.13).)

In summary, both in the basic CIA and our models, higher interest rate makes cash goods more expensive because a household needs cash to buy goods although cash can be allowed to use only in the next period. In the basic model, the nominal interest does not affect the long-run equilibrium as the return on capital is determined only from the real side of the economy. In our model, however, a household has an incentive to add capital stock due to its additional return that relaxes the next-period CIA constraint through the development in the financial sector.

3.3.2 Failure of the Friedman Rule

The Friedman rule does not hold in our model, i.e., the zero nominal interest rate is no longer desirable and moreover a nonnegative inflation rate can be optimal depending on parameter values. Since our utility function depends only on $c_t$, we can use the value of steady-state consumption as a measure to find the optimal inflation rate. Given the current specifications, $k_{ss}$ is monotonically increasing in $\theta$ as described in Section 3.3.1. Since $c_{ss}$ is a hump-shaped function of $k_{ss}$ as expressed by the equation (3.26), the model always has a unique optimal inflation rate $\theta^*$ as long as an equilibrium exists. Figure 3.4 shows an example. We will briefly discuss about a numerical result.
about the optimal rate of inflation in Section 3.5.3 after finishing calibration.

Figure 3.4: Optimal Inflation Rate

3.4 Responses to Monetary Shocks

In this section, we explore short-run dynamics of the economy by computing impulse responses to monetary shocks. In previous literature, Cooley and Hansen (1989) is the first exercise to quantitatively analyze effects of inflation using a CIA model. They constructs a tractable stochastic CIA model based on a theoretical framework provided by Lucas and Stokey (1983). Compared with the Cooley and Hansen's model, our model described below is simpler in the sense that there is no labor choice, but is extended in the sense that the CIA constraint is endogenous.

3.4.1 Additional Assumptions and Specifications

First we specify the stochastic process \( \{ \theta_t \} \), assuming that now the nominal money growth rate fluctuates around its long-run average \( \theta_{ss} \).

\[ \theta_t = \theta_{ss} + u_t. \]

\(^6\) Figure 3.4 is drawn under parameters \( \alpha = 0.36, \delta = 0.019, \chi = 0.044 \) and \( \gamma = 1.6 \). These parameter values are the same as our benchmark calibration described in Section 3.4.3. Note that the vertical axes for \( k_{ss} \) and \( c_{ss} \) have different scales respectively for easier viewing.
The disturbance \( u_t \) is assumed to follow an AR(1) process.

\[
    u_t = \Phi_u u_{t-1} + e^u_t
\]

(3.32)

where \( \Phi_u \in (0, 1) \) and the innovation \( e^u_t \) is a serially uncorrelated mean zero process with variance \( \sigma^2_{e^u} \). Note that we define \( u_t \) as a deviation from the steady-state money growth rate \( \theta_{ss} \) and thus changes in \( u_t \) do not affect \( \theta_{ss} \). This means that we are focusing on effects of transitory shocks rather than permanent shocks which change the steady-state equilibrium. Therefore the framework presented in this section cannot analyze the transition dynamics when the steady state itself changes. Such analysis is beyond the scope of this paper.

Next, we assume that the flow utility function has a CRRA specification.

\[
    u(c_t) = \frac{c_t^{1-\phi}}{1-\phi}.
\]

(3.33)

The above two additional specifications are borrowed from Walsh (2010) as with many other assumptions about the basic CIA model.

### 3.4.2 System of Equilibrium Conditions

To find a rational-expectations equilibrium, we use a method by Blanchard and Kahn (1980). Collecting the equilibrium conditions, our model can be expressed by the following 6 equations with 6 variables. We eliminated \( \mu_{t+1} \) using the FOC for consumption (3.14) and the relationship (3.19).

The system is also simplified by substituting for \( c_t, y_t \) and \( 1 + r_t \), i.e., the CIA constraint (3.13), the production function (3.30) and the Fisher relationship (3.18) no longer appear explicitly. (Note that the binding CIA constraint can be expressed as \( c_t = (1 + \chi k_{t-1}^{\alpha})^\gamma m_t \).)
\[ \lambda_t = \beta E_t \left[ \lambda_{t+1} \left( \alpha k_t^{\alpha-1} + 1 - \delta \right) + \lambda_{t+1} i_t \frac{\alpha \chi k_t^{\alpha-1}}{1 + \chi k_t^{\alpha}} m_{t+1} \right], \quad (3.34) \]

[Real return on capital]

\[ \lambda_t = \beta E_t \left( \frac{1 + i_t}{1 + \pi_t} \right), \quad (3.35) \]

[Euler equation for bond]

\[ \lambda_t = \beta E_t \left[ \frac{[1 - (1 + \chi k_t^{\alpha})^\gamma] \lambda_{t+1} + (1 + \chi k_t^{\alpha})^{(1 - \phi)} m_{t+1}}{1 + \pi_t} \right], \quad (3.36) \]

[Money demand]

\[ (1 + \chi k_{t-1}^{\alpha})^\gamma m_t = k_{t-1}^\alpha - \delta k_{t-1}, \quad (3.37) \]

[Resource constraint]

\[ m_t = 1 + \theta_{ss} + u_t \frac{m_{t-1}}{1 + \pi_t}, \quad (3.38) \]

[Real money growth path]

\[ u_t = \Phi_u u_{t-1} + e_t^u. \quad (3.39) \]

Comparing this, in the basic CIA model the equations (3.34), (3.36) and (3.37) turn into the following simpler forms. The other three equations are the same.

\[ \lambda_t = \beta E_t \left[ \lambda_{t+1} \left( \alpha k_t^{\alpha-1} + 1 - \delta \right) \right], \]

[Real return on capital]

\[ \lambda_t = \beta E_t \left( \frac{m_{t+1}^{-\phi}}{1 + \pi_t} \right), \]

[Money demand]

\[ m_t = k_{t-1}^\alpha - \delta k_{t-1}. \]

[Resource constraint]

Differences from the basic CIA model are (i) the second term in the RHS of the equation (3.34), (ii) the numerator in the RHS of the equation (3.36), and (iii) the LHS of the equation (3.37). The first one, \( \frac{i_t \alpha \chi k_t^{\alpha-1}}{1 + \chi k_t^{\alpha}} m_{t+1} \), represents the additional return on capital through relaxing the CIA constraint, which corresponds to the term \( -i_{ss} \rho'(k_{ss}) c(k_{ss}) \) in the equation (3.27). The second one looks a bit messy as a result of algebraic manipulation, but it is just equivalent to \( \lambda_{t+1} + \mu_{t+1} \). (See the equation (3.17).) Generally in CIA models, an increase in the nominal interest rate raises the shadow price of the binding CIA constraint (\( \mu_{t+1} \), which represents a kind of cost for liquidity services), reducing household’s money demand. In my model, however, the new term \( \rho(k_t) = (1 + \chi k_t^{\alpha})^\gamma \) will create another effect to restrain the reduction in money (and thus consumption) as explained later.
Next we log-linearize the above equations and express it as first-order difference equations. In matrix representation, it can be written as

\[ M_1 E_t(X_{t+1}) = M_2 X_t + e_{t+1} \]

where \( M_1 \) and \( M_2 \) are 6 x 6 matrices in which each element is a function of parameters, \( X_{t+1} \equiv \left[ \hat{\pi}_{t+1}, \hat{\lambda}_{t+1}, \hat{k}_{t}, \hat{m}_{t}, \hat{\ell}_{t}, u_{t+1} \right]' \) and \( e_{t+1} \equiv \left[ 0_{1 \times 5}, e^u_{t+1} \right]' \). The hat symbol (\(^\hat{}\)) represents a deviation of a variable from its steady-state value. In this model \( \hat{\lambda}_{t+1} \) and \( \hat{\pi}_{t+1} \) work as forward-looking variables and the others are predetermined variables. If \( M_1 \) and \( M_2 \) satisfy some conditions, the system can be solved uniquely. See Appendix C.2 for computational details.

### 3.4.3 Calibration

The purpose of this Section 3.4 is essentially to provide a basic framework for analyzing short-run effects of money and inflation when a CIA constraint is endogenous. It is not to obtain a well calibrated model that highly matches actual data, because our model is simple and thus lacks some important components such as endogenous labor supply which is needed to replicate business cycle facts. From this perspective, we borrow values for some parameters simply from a textbook by Walsh (2010). Only for parameters which are not available from the textbook, we choose them so that model-implied values match U.S. monetary data.

Table 3.1 shows parameters for our benchmark model. A length of model period is assumed to be one quarter following Cooley and Hansen (1989).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>1.5</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.989</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.36</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.019</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>1</td>
</tr>
<tr>
<td>( \chi )</td>
<td>0.044</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1.6</td>
</tr>
<tr>
<td>( \rho_u )</td>
<td>0.68</td>
</tr>
<tr>
<td>( \sigma_{e^u} )</td>
<td>4.1</td>
</tr>
<tr>
<td>( \theta_{ss} )</td>
<td>0.017</td>
</tr>
</tbody>
</table>

First, the risk aversion \( \phi \) is set to 1.5 as a standard value, although Walsh (2010) set it to 2.0. The reason why we do not use the same value is our model does not have a unique solution when setting \( \phi = 2 \). Probably this stems from a lack of labor choice and we will discuss more in Section 3.5.2.

\( \beta, \alpha, \delta, \rho_u, \sigma_{e^u} \) and \( \theta_{ss} \) are from Chapters 2 and 3 in Walsh (2010). For parameters of money growth, \( \rho_u, \sigma_{e^u} \) and \( \theta_{ss} \), Walsh (2010) offers two types of values: one corresponds to the growth rate of M1 and another one corresponds to the growth rate of MZM (Money Zero Maturity). We
adopt the values corresponding to MZM. MZM is a monetary aggregate constructed by the Federal Reserve Bank of St. Louis, which measures balances available immediately for transactions at zero cost. Some researchers argue that MZM is a more appropriate measure of the transactions demand for money relative to other aggregates such as M1 and M2. In Walsh (2010) $\theta_{ss}$ is an average quarterly growth rate of MZM for the period from 1984 to 2007, and $\rho_u$ and $\sigma_{v_u}$ are yielded by estimating an AR(1) process for MZM growth.

Next, we have to set a value of $\kappa$ as a parameter which can capture a linear/non-linear relationship between the financial sector’s development $\varphi_t$ and total output $y_t$. As we mentioned in Section 3.2, if $\kappa = 1$ then $\varphi_t$ is a linear function of $y_t$, although it is unclear whether the assumption is correct or not just by looking at Figure 3.1. Moreover, seeing Figure 3.5, it is true that the financial sector’s GDP share had risen over time until the early 2000’s as the total economy has grown, but after that it sharply declined and recently looks roughly flat.

---

MZM is broader than M1 but essentially narrower than M2. It is defined as M2 less small-denomination time deposits plus institutional money funds. This concept excludes all securities, which are subject to risk of capital loss, and time deposits, which carry penalties for early withdrawal. This new aggregate was proposed in response to an empirical breakdown of stable relations between interest rates and other monetary aggregates such as M1, M2 and the monetary base (Carlson and Keen, 1996; Lucas and Nicolini, 2015). Carlson and Keen (1996) and Teles and Zhou (2005) argue that MZM is an appropriate monetary aggregate because the response of MZM to changes in its opportunity cost has remained stable even after 1980 unlike those of M1 and M2. Altig et al. (2011) work with MZM for their New Keynesian model due to the following reasons: (i) MZM is a natural empirical counterpart to cash balances in their model, and (ii) their statistical procedure requires that the velocity of money is stationary. The MZM velocity is reasonably characterized as being stationary, while the stationarity assumption is more problematic for the velocity of other aggregates.
To determine the value of $\kappa$, we briefly investigated an empirical relationship between the financial sector’s GDP share and total output of economy. We used annual data from 1963 to 2016 in real terms provided by the Bureau of Economic Analysis (BEA). For the financial sector’s GDP share, we excluded insurance carriers and related activities as they seem not to contribute to offer transaction services such that can relax the CIA constraint. We can estimate the parameters for the equation (3.10) using the following equation.

$$\ln s_t = a + b \ln y_t + \epsilon_t$$  \hfill (3.40)

where $s_t$ is the GDP share of financial sector ($= \varphi_t / y_t$), $\epsilon_t$ is an error term, $a \equiv \ln \chi$ and $b \equiv \kappa - 1$.

If we test a null hypothesis that $b = 0$ using this equation and if it is not rejected, then we may conclude that $\kappa = 1$. Unit root and cointegration tests, however, suggested that both $s_t$ and $y_t$ have unit roots but are not cointegrated.\footnote{We conducted the augmented Dickey-Fuller test as a unit root test and the Phillips-Ouliaris test as a cointegration test. As for the ADF test, we set lag order to 1 based on minimum AIC for both GDP and financial sector’s share. The Dickey-Fuller statistics were -1.877 for GDP and -1.551 for the financial sector’s share, respectively, which implies a null hypothesis that each series has a unit root is not rejected even at 10 percent significance level ($p$-values for stationarity were 0.652 for GDP and 0.799 for the share). As for the Phillips-Ouliaris test, we tested the relationship in the equation (3.10) and the resulting statistic was -35.65, which implies a null hypothesis that $\ln s_t$ and $\ln y_t$ are not cointegrated is not rejected even at 10% significance level ($p$-value for cointegration is 0.15).} Therefore, we took the first difference of the equation (3.40).

Note: Real estate and rental and leasing industries are excluded from the financial sector.

Source: the Bureau of Economic Analysis.
and estimated it by OLS. The result is

\[ \Delta \ln s_t = 0.389 \Delta \ln y_t \]

(0.215)

where \( \Delta x_t \equiv x_t - x_{t-1} \) and the number in parenthesis is the standard deviation of \( b \). The corresponding \( t \)-value is 1.81 and we cannot reject the null hypothesis that \( b = 0 \) (or, equivalently \( \kappa = 1 \)) at 5% significance level. We also confirmed that these differentiated series are stationary by applying the ADF test to \( \Delta s_t \) and \( \Delta y_t \) again. Based on the above considerations, \( \kappa \) is set to 1.0, which means we assume that the financial sector develops linearly in output. This result also has a potential advantage that is consistent with the balanced growth path when we add an exogenous growth component into the model.

Since now we assume that the share \( s_t \) is constant at least in the long run, we set \( \chi = 0.044 \), which equals the average GDP share of the financial industry in the United States between 2007-2016.

At last, we set the curvature parameter of transaction technology \( \gamma \) to 1.6 so that the implied steady-state velocity of money \( (v(k_{ss}) = [\rho(k_{ss})]^{-1}) \) matches data. According to the FRED database by the Federal Reserve Bank of St. Louis, the quarterly-based velocity of MZM is roughly flat at around 1.3 after 2016.

3.4.4 Results

We assume that the economy is in the steady state at time \( t = 0 \) and there is a monetary shock of one percentage point at \( t = 1 \) (i.e., \( e^u_1 = 1 \)). Under the parameters in Table 3.1, Figure 3.3 shows responses of selected variables to the positive money supply shock.9

<table>
<thead>
<tr>
<th></th>
<th>( k_{ss} )</th>
<th>( c_{ss} )</th>
<th>( m_{ss} )</th>
<th>velocity</th>
<th>( r_{ss} )</th>
<th>( t_{ss} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic CIA model</td>
<td>48.2457</td>
<td>3.1202</td>
<td>3.1202</td>
<td>1.0000</td>
<td>0.0111</td>
<td>0.0283</td>
</tr>
<tr>
<td>Endogenous CIA model</td>
<td>48.5526</td>
<td>3.1236</td>
<td>2.4033</td>
<td>1.2997</td>
<td>0.0111</td>
<td>0.0283</td>
</tr>
<tr>
<td>Alternative assumption</td>
<td>48.2457</td>
<td>3.1202</td>
<td>2.4020</td>
<td>1.2990</td>
<td>0.0111</td>
<td>0.0283</td>
</tr>
</tbody>
</table>

9Note that levels of steady-state values are different from each other across models. Especially the steady-state value of money holdings in our model is about 23 percent below that in the basic CIA model, while capital and consumption are at almost the same levels.
The solid blue lines represent our benchmark model (i.e. with the endogenous CIA constraint), and the dashed black lines represent the basic CIA model. The dotted red lines are the alternative experiment which will be explained in Section 3.5.1. First we see the results of our benchmark model and then compare it with the basic CIA model.

Responses in the Benchmark Model In our model, (i) an effect of the initial shock $e_t$ (i.e., the response function of $u_t$) gradually diminishes and almost disappears by $t = 20$ according to the AR(1) process (3.32). (ii) Reflecting the rise in the money growth rate, the inflation rate $\pi_t$ jumps...
up at $t = 1$ and then gradually declines as the effect of monetary shock fades out. (iii) Real money balances $m_t$ jump down at $t = 1$, since inflation raises the nominal interest rate, leading to an increase in the opportunity cost of holding money. (iv) The response of consumption $c_t$ has the same direction as money because the CIA constraint is binding. Inflation is a tax on consumption since a household must hold money until the next period to finance consumption. (v) Since labor supply is fixed, capital stock $k_t$ gradually increases simply reflecting the decreases in money and consumption, but eventually turns to decrease. (vi) At last, the increase in capital supply leads to a fall in the real interest rate $r_t$. It takes a long time until all variables return to the initial steady state. (It is around $t = 200$.)

**Comparison with the Basic CIA Model: Stabilization Effect** Comparing our model with the basic CIA model, we can see that the endogenous CIA constraint has an effect to restrain fluctuations of an economy. In our model, amplitude of each response in absolute terms is smaller than the one in the basic CIA model. Only exception is the nominal interest rate $i_t$ which is abbreviated in the graph. That response is very slightly amplified compared with the basic CIA model, although the difference is extremely small, almost zero.

To have intuition about this stabilization effect, first let’s consider why the extent of decline in consumption is more moderate than in the basic CIA model. In the basic model, the FOC for consumption can be expressed as

$$u'(c_t) = \lambda_t (1 + i_{t-1}).$$

An important implication of this equation is that a positive nominal interest rate acts as a tax on consumption (Walsh, 2010). When $i_{t-1} > 0$, the marginal utility of consumption $u'(c_t)$ exceeds the marginal utility of income $\lambda_t$, which means the relative price of consumption to output is $1 + i_{t-1}$, not 1. Since a household cannot buy consumption goods without cash, the positive opportunity cost makes consumption more expensive above its production cost. On the other hand, in our model, recalling that the FOC for consumption and the equation, we can obtain the following expression.

---

The response of $i_t$ is very similar to the response of $u_t$ except for their scales.
\[ u'(c_t) = \lambda_t[1 + \sigma_{t-1}\rho(k_{t-1})]. \]

The difference from the basic CIA model is \( \rho(k_{t-1}) \in (0, 1] \) multiplied with \( \sigma_{t-1} \). This equation suggests that our model also has the same channel that a rise in the nominal interest rate reduces consumption through an increase in the marginal utility. With the endogenous CIA constraint, however, this inflation tax effect is partially offset because the increase in capital \( k_t \) eases the constraint via development in the financial sector. In other words, improvement of transaction technology accompanied with output growth partially offsets an increase in the opportunity cost caused by inflation.

As a result of the above, in terms of level (not rate of change), both \( c_t \) and \( m_t \) are greater relative to the basic CIA model, which implies capital stock \( k_t \) is smaller. Less capital supply makes the fall in the real interest rate \( r_t \) more moderate, thus slightly lowering the inflation rate \( \pi_t \) according to the Fisher relationship \((3.18)\). Our model suggests that the presence of a financial sector may weaken effects of monetary policy.

### 3.5 Discussions

In this section, we examine some alternative assumptions and parameter values as a sensitivity analysis and thereby discuss some shortcomings of this paper.

#### 3.5.1 Alternative Assumption: A Household Takes the CIA Constraint as Given

The results so far are based on the assumption that raising capital by the representative household relaxes the CIA constraint and the agent knows it. However, it is difficult to justify this assumption as realistic one. In the real world, each household is too small in an economy to affect aggregate capital alone. Therefore, it may be more appropriate to assume that the tiny individual household does not internalize the relationship between raising its own capital and the transaction technology in its decision-making process, in other words, the household makes its decisions taking \( \bar{\rho}(\varphi_t) \) in the CIA constraint \((3.3)\) as given. Then the definition of equilibrium is rewritten as follows.
**Definition:** Given the initial condition on real money supply $m^g_0$ and the exogenous monetary shocks $\{ u_t \}$, an equilibrium with externality consists of quantities $\{ c_t, k_t, b_t, m_t \}$ and fiscal and monetary policies $\{ m^f_t, \tau_t \}$ such that: (i) given the prices, the government policies and the financial technology $\rho(k_{t-1})$, the quantities $\{ c_t, k_t, b_t, m_t \}$ solve the problem of the representative household (3.1) subject to the budget constraint (3.2) and the endogenous CIA constraint (3.3), (ii) the government satisfies its budget constraint (3.20) and the real money growth rule (3.21), and (iii) money and bond markets clear, i.e., $m_t = m^g_t$ and $b_t = 0$.

With this alternative definition, the first order condition for capital (3.15) turns into

$$\lambda_t = \beta \left( \lambda_{t+1} \left[ f'(k_t) + 1 - \delta \right] \right).$$

This condition is exactly the same as the one in the basic CIA model. Therefore the steady-state capital stock is constant at a level determined by $f'(k_t) - \delta = r_{ss}$, i.e., the Tobin effect in the long run disappears.

Even in this case, however, still there is the stabilization effect in the short run. The FOC for consumption (3.14) does not change and the term $\mu_t \rho(k_{t-1})$ remains in the equation, which implies that the mechanism described in the end of Section 3.4.4 is still valid. If each individual does not internalize the transaction technology in his decision making, he does not choose to raise capital at least in the long run, and therefore aggregate capital does not change as well. In the short run, however, a temporary expansion of the financial sector can partially offset damage from an increase in the opportunity cost by relaxing the CIA constraint.

Red dotted lines in Figure 3.6 represent impulse responses under this alternative assumption. Now the second term in the FOC for capital (3.34) is eliminated, but we found that the presence of this term was numerically trivial. The assumption on the household’s optimization does not make a difference to the result in the short run.\(^{11}\)

Another possible argument is that it is also unclear whether the transaction technology is affected by output fluctuation. Even if economic growth is truly correlated with financial sector’s development, the assumption that the payment technology is directly connected with the business

\(^{11}\)To be more precise, there are slight differences between two, but in the graph they are so small that one can not recognize them by eyes.
cycle may need more empirical evidence to justify it. Although it seems to be challenging to prove the correlation between financial sector’s growth and advance in transaction technology (as an instrument easing the CIA constraint), these assumptions are keys to our model and the above issues remain as a future work.

3.5.2 Sensitivity Analysis

Next we show results of sensitivity analysis by changing a few parameter values. We found that in our model the existence of a unique equilibrium is not so robust to changes in some parameters.

The first problem is the risk aversion parameter $\phi$. We set it to 1.5 in the previous sections but actually the Walsh (2010)’s textbook sets $\phi = 2$. If using the value greater than 1.54, and all other things equal, our model does not have a unique rational-expectations equilibrium, since the matrix $M_1^{-1}M_2$ (defined as $M$ in Appendix C.2) does not satisfy the Blanchard and Kahn condition, i.e., the number of eigenvalues which are outside of unit circle does not equal the number of forward-looking variables. We do not know the exact reason why the model has no stationary solution for high values of $\phi$, but it seems to stem from exogeneity of labor supply. We found that the basic CIA model is also not robust to changes in $\phi$. When we set values of $\phi$ greater than 2, the basic CIA model does not satisfy the B-K condition, although if labor supply is endogenous then 2 of $\phi$ works as Walsh (2010) demonstrates.

The existence of a unique steady-state equilibrium also depends on the curvature parameter for financial technology, $\gamma$. When we set values of $\gamma$ greater than 1.78, the model does not satisfy the B-K condition. This implies that our model cannot simulate an economy with the velocity greater than 1.4. As seen in Section 3.4.3, the MZM velocity is around 1.3 in recent years, but before 2014 it was greater than 1.4. Moreover, seeing the velocities of other aggregates such as M1 and M2, in general their levels are higher than that of MZM. For example, the M1 velocity is much higher and more volatile than that of MZM. Its recent value is 5.5 in Q2 of 2017 and the model does not have a rational-expectations equilibrium if we want to calibrate the model based on M1.

\[12\] The restriction on the value of $\phi$ for the unique solution depends on other parameter values such as $\gamma$. If other parameters are the same, the values of $\phi$ greater than 1.54 do not satisfy the B-K conditions. The range of restriction widens if we set lower values of $\gamma$. We also confirmed that the alternative assumption discussed in Section 3.5.1 does not affect this range of restriction for $\phi$. 

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3.5.3 Optimal Inflation Rate

At last we would like to briefly discuss about a problem regarding optimal inflation. In the previous parts, we picked up the exogenous average inflation rate $\theta_{ss} = 0.017$ from data to calibrate the model. Initially, however, we tried to set $\theta_{ss}$ to its optimal level discussed in Section 3.3.2 as well as to set $\gamma$ to match the implied velocity of money with data as described in 3.4.3. In this case, the calculated optimal inflation rate was 494%, which is unrealistically high. The corresponding value of $\gamma$ was 1.27 and still the model satisfies the B-K condition, but this result strongly suggests that our model is not so rich to determine the desirable rate of inflation on a practical basis. One possible reason for this phenomenon may be associated with the absence of endogenous labor. Since inflation is a tax on cash goods (consumption) and causes a shift to credit goods (leisure), people might no longer prefer such distortion from inflation. If so, the optimal inflation rate might come to more realistic range.

3.6 Conclusions

This paper incorporated an endogenous CIA constraint into a very basic CIA model by assuming that there is a financial sector offering services that can facilitate credit transactions and thereby can ease the CIA constraint endogenously. Our steady-state analysis found that with the endogenous CIA constraint the monetary superneutrality no longer holds even in the long run and inflation can have a positive effect on capital accumulation. The Friedman rule also does not hold and the optimal inflation rate can be positive depending on parameter values. Then we investigated effects of transitory monetary shocks in the short run. In our stochastic model, the endogenous CIA constraint has an effect to stabilize an economy against money growth shocks. When the representative household does not internalize the transaction technology in its decision making, the Tobin effect in the long run disappears but the short-run stabilization effect still remains.
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Appendix A

Appendix for Chapter 1

A.1 Solving the Problem

This appendix provides details about computational procedures to solve the problem.

A.1.1 Exogenous Variables and Parameters

First of all, note that we have the following exogenous variables.

- The initial level of assets when households enter the economy is assumed to be zero ($k_1 = m_1 = 0$).

- The total factor productivity is set to one by just normalization ($\zeta = 1$).

- The nominal money growth rate $\theta$ is set to zero in the benchmark case and and is exogenously raised from 0% to 10% in the counter-factual experiments.

- The risk aversion ($\sigma$), the elasticity of money demand with respect to interest rate ($v$), the capital’s share ($\alpha$) and the depreciation rate ($\delta$) will be borrowed from literature. (See Table for sources.)

- The population growth rate ($n$), the conditional survival probability $\{s_j\}$, the age-wage profile $\{\eta_j\}$, the persistency of idiosyncratic productivity shocks ($\rho$) and the standard deviation of age-20 households’ productivity ($\sigma_{z_1}$), the government expenditure per output ($G/Y$) and
the pension contribution rate ($\dot{\phi}$) are given by data or institution. (See Table 1.1 and the corresponding explanations for sources.)

The remaining four parameters — the discount factor ($\beta$), the preference weight on consumption relative to money ($\omega$), the preference weight on leisure ($\gamma$) and the standard deviation of error term in the AR(1) productivity process ($\sigma$) — are also exogenous parameters, but these parameters will be set to match the model with actual data. (See Table 1.2.) We need to guess their initial values.

**A.1.2 Procedure for Finding Equilibrium**

The basic strategy to find the equilibrium is as follows.

1. Guess initial values of $K$, $M$, $L$ and $b$.

2. Calculate the equilibrium values of $r$, $w$, $\pi$, $ss$ and $\tau^l$ according to Section A.1.3.

3. Solve the households’ problem based on the values calculated in Step 2 according to Section A.1.4.

4. Check whether the initial guesses for $K$, $L$, $M$ and $b$ are close enough to the calculated equilibrium values. If the distances between the model values and the initial guesses are not small enough, then set new values of $K$, $L$, $M$ and $b$.

5. Continue Steps 2 to 4 to find the equilibrium.

**A.1.3 Equilibrium Values**

Given values of $K$, $M$, $L$ and $b$, some equilibrium values are calculated as follows. From the firm’s FOC, we can easily obtain the real interest rate and the real wage.

\[
\begin{align*}
    r &= \alpha \zeta \left( \frac{K}{L} \right)^{\alpha - 1} - \delta, \\
    w &= (1 - \alpha) \zeta \left( \frac{K}{L} \right)^\alpha.
\end{align*}
\]

(A.1) (A.2)

In the steady state, per capita real money supply should be unchanged over time.
\[ M_{t+1} = (1 + n)M_t. \]

Substituting the money growth rule \((1.8)\) into \(M_{t+1}\) in the above equation, we have

\[ 1 + \pi_t = \frac{1 + \theta}{1 + n}. \]

In particular, the inflation rate \(\pi\) equals the money growth rate \(\theta\) if \(n = 0\). \((\pi_t = \theta).\)

From the FOC’s for firm (the equations \((A.1)\) and \((A.2)\)) and the production function \((1.6)\), we can obtain the standard Euler’s rule.

\[ Y = (r + \delta)K + wL. \]

Combining this with the market clearing conditions \((1.12)\) and \((1.9)\), the equilibrium aggregate consumption \(C\) is given by

\[ C = \sum_j \mu_j \int X c(x,j) d\psi_j = (r - n)K + wL - G \quad (A.3) \]

where we used the relationship that \(K' = (1 + n)K\) in the steady state.

From the balanced pension system \((1.13)\) and the labor market clearing condition \((1.11)\), the equilibrium wage replacement rate is given by

\[ ss = \frac{\phi L}{\mu R} \]

where \([\mu R \equiv \sum_{j=R} \mu_j]\).

From the government budget constraint for purchases \((1.15)\) and the labor market clearing condition \((1.11)\), the equilibrium labor income tax rate is given by

\[ t^l = \frac{1}{wL} \left( G - \frac{\theta}{1 + \pi} M \right). \]
A.1.4 Solutions for Households’ Problem

Essentially we use backward induction of value functions with the method of discretized asset grid points, like Huggett (1996). Households have to choose \( c, k', m' \) and \( l \) simultaneously. Now we ignore the time subscript \( t \) and the lump-sum transfers \( \tau \) because we are focusing on the stationary equilibrium in the benchmark model. We have to solve the Bellman Equation in (1.3) under the borrowing constraint \( k_{j+1} \geq 0 \) and the non-negativity of labor \( l_j \geq 0 \). The non-negativity for consumption \( c_j \), real balances \( m_{j+1} \) and leisure \( (1 - l_j) \) will never bind because they are involved in utility, but the borrowing constraint for capital and the non-negativity constraint for labor supply can be binding. Reproducing the households’ problem in the recursive form,

\[
V(x, j) = \max_{c, k', m', l} u(c, m', l) + \beta s_{j+1} E \left[ V(x', j+1) \mid z \right] \\
\text{s.t.} \quad c = a + \tilde{w}_j l + b + ss_j w - k' - m', \\
\quad a' = (1 + r)k' + \frac{m'}{1 + \pi}, \\
\quad k' \geq 0 \text{ and } l \geq 0
\]

where \( \tilde{w}_j \equiv (1 - \tau^l - \phi)z_j \eta_j w \).

A.1.4.1 Method 1 (Using Money and Labor Grids as Choice Variables)

In this subsection, we consider the algorithm that solves the problem by using a total asset grid as a state variable and money and labor grids as choice variables (Method 1). In the next subsection, we will provide the alternative algorithm which adopts the total asset grid as a choice variable and uses the first order conditions (Method 2). The main difference between the two methods is that the choices of money holdings and labor are treated as discrete in Method 1 and as continuous in Method 2. Also, choices of next-period total assets are continuous in Method 1 but discrete in Method 2. We believe that Method 1 provided in this subsection is theoretically more appropriate than Method 2 for our purpose of analysis because an individual choice of money (cash) and labor are more likely to be discrete than a choice of total assets. However, a computational burden in Method 1 is much heavier than the one in Method 2. Therefore, we utilize Method 2 as well and
will confirm that the basic results are similar in both algorithms.

In Method 1, we set following three grids to evaluate the value function.

\[
\begin{align*}
\text{a}_{\text{grid}} &= \{a_{\text{min}}, \ldots, a_{\text{max}}\}_{1 \times N_a}, \quad (a_{\text{min}} \geq 0) \\
\text{m}_{\text{grid}} &= \{m_{\text{min}}, \ldots, m_{\text{max}}\}_{1 \times N_m}, \quad (m_{\text{min}} > 0 \text{ and } m_{\text{max}} < a_{\text{min}}) \\
\text{l}_{\text{grid}} &= \{l_{\text{min}}, \ldots, l_{\text{max}}\}_{1 \times N_l}, \quad (l_{\text{min}} \geq 0 \text{ and } l_{\text{max}} < 1)
\end{align*}
\]

where \(N_a, N_m\) and \(N_l\) are the number of grid points for total assets, real money balances and labor supply, respectively. Households choose the level of next-period capital \(k'\) continuously. To do this, we use an interpolation technique.

Given values of \(k' \in \mathbb{R}_+, m' \in \text{m}_{\text{grid}}, l \in \text{l}_{\text{grid}}, a \in \text{a}_{\text{grid}}, z \in \mathbb{Z}\) and \(j \in \{1, 2, \ldots, J\}\), a value of \(c\) is determined by the budget constraint (1.3) and a value of \(a'\) is determined by its definition (1.2). Then we can calculate a value of flow utility \(u(c, m', l)\). With the assumption \(s_{J+1} = 0\) that implies terminal utility equals zero, we can solve the value function by backward induction. In other words, we can start from \(j = J\) and pick up values of \(k'\) so that the chosen \(k'\) maximizes the value function for each state \((x\text{ and } j)\) using \(s_{J+1}V(x', J+1) = 0\). The value of \(a'\) is not necessarily on the nodes in the asset grid because we are choosing \(k'\) continuously, thus we linearly interpolate the expected value \(E[V(a', z', j + 1)|z]\). For example, if we choose \(k'\) from \(\mathbb{R}_+\) and the calculated value of \(a'\) is in \([a_i, a_{i+1}]\) where \(a_i\) and \(a_{i+1}\) are adjacent nodes in the asset grid, then the linearly interpolated value function in the next period is

\[
V(a', z', j + 1) \approx V(a_i, z', j + 1) + \frac{a' - a_i}{a_{i+1} - a_i} \left[ V(a_{i+1}, z', j + 1) - V(a_i, z', j + 1) \right].
\]

Using this expression and the Markov transition matrix, we can find an optimal value of \(k'\) which maximizes the current-period value function \(V(a, z, j)\) by numerical computation.

**Finding Distribution** When calculating the stationary distribution, we should notice that in general the decision rule \(a'(x, j)\) calculated above is not necessarily on a node in \(a_{\text{grid}}\). Therefore, we allocate the population in the next period corresponding each value of optimal \(a'\) linearly into two adjacent nodes \(a_i\) and \(a_{i+1}\) such that \(a' \in [a_i, a_{i+1}]\). Given the exogenous distribution of age-1
households, \( \psi_1 \), the algorithm to calculate the probability \( \psi_j(x') \) for \( j \geq 2 \) is done by recursively as follows. First, we set \( \psi_j(x') = 0 \) \( \forall x' \) and \( j \geq 2 \). Then, for \( i = 1, \ldots, N_a \) and \( s = 1, \ldots, S \),

1. Find the index \( i' \) that satisfies \( a_{i'} \leq a'(x, j) < a_{i'+1} \).

2. Calculate the interpolation weight \( \bar{\omega} \) as

\[
\bar{\omega} = \frac{a'(x, j) - a_{i'}}{a_{i'+1} - a_{i'}}.
\]

3. For each \( s' = 1, \ldots, S \), update \( \psi_j \) as

\[
\psi_j(a_{i'}, z_{s'}) = \psi_{j-1}(a_{i'}, z_{s'}) + (1 - \bar{\omega})Pr(z_{s'}|z_s)\psi_j(a_i, z_s),
\]

\[
\psi_j(a_{i'+1}, z_{s'}) = \psi_{j-1}(a_{i'+1}, z_{s'}) + \bar{\omega}Pr(z_{s'}|z_s)\psi_j(a_i, z_s)
\]

where \( z_{s'} \) is the productivity shock at the \( s' \)th state.

### A.1.4.2 Method 2 (Using Total Asset Grid as State and Choice Variables)

To reduce computational burdens, we can use an alternative method to solve the households’ problem by combining the first order conditions and the total asset grid. In this algorithm, we use the total asset grid as both state and choice variables. First we set an asset grid to evaluate the value function.

\[
a_{grid} = \{ a_{min}, \ldots, a_{max} \}_{1 \times N_a} \quad (a_{min} \geq 0)
\]

where \( N_a \) is the number of grid points. For any set of values of \( a \in a_{grid}, a' \in a_{grid}, z \in \mathbb{Z} \) and \( j \in \{1, 2, \ldots, J\} \), we calculate values of \( c, k', m', l \) in accordance with the following procedure, and then compute a value of flow utility for each state. This means that we will have a utility matrix (more rigorously speaking, a four-dimensional array) that has \( N \times N \times S \times J \) elements, where \( S \) is the number of productivity states. Based on this utility matrix, we will choose the optimal values of \( c, k', m', l \) by backward induction of value functions.

**First Order Conditions** The necessary conditions for the households’ problem are
\[ k' = -u(c, m', l) + \beta s_{j+1}(1 + r) E \left[ V_a(x', j + 1) \right] \geq 0, \]
\[ m' = -u(c, m', l) + u_{m'}(c, m', l) + \frac{\beta s_{j+1}}{1 + \pi} E \left[ V_a(x', j + 1) \right] = 0, \]
\[ l = \tilde{w}_j u(c, m', l) + u_l(c, m', l) \geq 0 \]

and the usual transversality conditions. Note that the FOC’s for \( k' \) and \( l \) are expressed with inequality due to the presence of constraints \( k' \geq 0 \) and \( l \geq 0 \). The envelope condition is

\[ V_a(x, j) = u(c, m', l). \]

Under our specification for the utility function, we can rewrite the above conditions as

\[ [k'] \quad -\Lambda \omega c^{\frac{1}{\nu}} + \beta s_{j+1}(1 + r) E \left[ \Lambda' \omega (c')^{-\frac{1}{\nu}} \right] \geq 0, \quad (A.4) \]
\[ [m'] \quad -\Lambda \omega c^{\frac{1}{\nu}} + \Lambda(1 - \omega)(m')^{-\frac{1}{\nu}} + \frac{\beta s_{j+1}}{1 + \pi} E \left[ \Lambda' \omega (c')^{-\frac{1}{\nu}} \right] = 0, \quad (A.5) \]
\[ [l] \quad \tilde{w}_j \omega c^{\frac{1}{\nu}} - \left( \omega c^{\frac{\nu - 1}{\nu}} + (1 - \omega)(m')^{\frac{\nu - 1}{\nu}} \right) (1 - l)^{1-\sigma} \geq 0 \quad (A.6) \]

where \( \Lambda = \left( \omega c^{\frac{\nu - 1}{\nu}} + (1 - \omega)(m')^{\frac{\nu - 1}{\nu}} \right) (1 - l)^{1-\sigma} \geq 0 \). Particularly, the FOC for \( l \) is also expressed as

\[ \frac{c}{1 - l} \leq \frac{\tilde{w}_j}{\gamma} \left[ 1 + \frac{1 - \omega}{\omega} \left( \frac{m'}{c} \right)^{\frac{\nu - 1}{\nu}} \right]^{-1}. \quad (A.7) \]

**Interior Solutions** When the borrowing constraint \( k' \geq 0 \) is not binding (i.e., when \( k' > 0 \)), the FOC \((A.4)\) holds with equality. Then, the FOC \((A.5)\) with equality and the FOC \((A.6)\) imply that

\[ \frac{m'}{c} = \left( \frac{1 - \omega}{\omega} \frac{1}{\Pi} \right)^{\nu} \equiv d_1 \quad (A.8) \]

where \( \Pi = 1 - \frac{1}{(1 + r)(1 + \pi)} \). Therefore if \( k' > 0 \) and values of \( r \) and \( \pi \) are given, the money-consumption ratio should be constant. Next, when the constraint \( l \geq 0 \) is not binding (i.e., when \( l > 0 \)), the FOC \((A.6)\) holds with equality, which implies that
This means that when \( k' > 0 \) and \( l > 0 \), the leisure-consumption ratio should be constant because \( \frac{m'}{c} \) is constant if \( k' > 0 \). Finally, rewriting the definition of total asset (1.2), next-period capital stock \( k' \) can be rewritten as

\[
k' = \frac{a'}{1 + r} - \frac{m'}{(1 + r)(1 + \pi)}.
\]  

Substituting this into the budget constraint (1.3),

\[
c = \frac{1}{D} \left( a - \frac{a'}{1 + r} + \tilde{w}_j + b + ss_j w \right)
\]

where \( D \equiv 1 + \Pi \frac{m'}{c} + \tilde{w}_j \frac{1 - l}{c} \). Therefore if once we set values of \( a \) and \( a' \) and if \( k' > 0 \) and \( l > 0 \), we can calculate a value of \( c \). Now we denote this value of \( c \) as \( c_b \), i.e.,

\[
c_b = \frac{1}{D_b} \left( a - \frac{a'}{1 + r} + \tilde{w}_j + b + ss_j w \right)
\]

where \( D_b \equiv 1 + \Pi d_1 + \tilde{w}_j d_2 \). Once getting the value of \( c \), we can easily calculate \( m' \), \( l \) and \( k' \) as follows.

\[
m'_b = d_1 c_b,
\]

\[
l_b = 1 - d_2 c_b,
\]

\[
k'_b = \frac{a'}{1 + r} - \frac{m'_b}{(1 + r)(1 + \pi)}.
\]

**Case 1: Non-binding FOC for labor**  
Note that the FOC for labor (A.7) holds with equality only if the solution satisfies \( l \in [0,1] \). Otherwise, the labor supply \( l \) takes on a corner value. In other words, if the value of \( l \) calculated from the equation (A.7) with equality is below zero, then we can say that \( l = 0 \). This condition is rewritten as
\[ c \geq \frac{\bar{w}_j}{\gamma} \left[ 1 + \frac{1 - \omega}{\omega} \left( \frac{m'}{c} \right)^\frac{\nu - 1}{\nu} \right]^{-1}. \]

Therefore if the value of \( c_b \) calculated by the equation (A.11) is greater than or equal to the RHS of the above equation, we have \( l = 0 \) as a solution. In this case, note that \( c_b \) itself cannot be a solution because the FOC for labor (A.7) no longer holds with equality. We can separate this case into further two cases as follows.

**Case 1-1 \((c_b > 0 \text{ and } k'_b > 0)\):** When \( l = 0 \) and \( k'_b > 0 \), the FOC for labor (A.6) does not bind but the FOC for capital (A.4) holds with equality, which means that the equation (A.9) no longer holds but the equation (A.8) still holds. In this case, the budget constraint (1.3) and the equation (A.10) imply that

\[ c = \frac{1}{D_0} \left( a - \frac{a'}{1 + r} + b + ss_jw \right) \]

where \( D_0 \equiv 1 + \frac{m'}{c} \). As with before, \( m' \) and \( k' \) are given by

\[ m' = d_1c, \]
\[ k' = \frac{a'}{1 + r} - \frac{m'}{(1 + r)(1 + \pi)}. \]

**Case 1-2 \((c_b > 0 \text{ but } k'_b < 0)\):** If \( k'_b < 0 \), it violates the borrowing constraint \( k' \geq 0 \) and the FOC for capital (A.4) no longer binds. In this case, given \( a \) and \( a' \), we can calculate \( k' \), \( m' \) and \( c \) as follows.

\[ k' = 0, \]
\[ m' = (1 + \pi)a' \quad (\because \text{ eq. (1.2))}, \]
\[ c = a + b + ss_jw - m' \quad (\because \text{ budget constraint (1.3) and } l = 0). \]
Case 2: Binding FOC for labor  In contrast to Case 1, if \( c_b < \frac{\tilde{w}_j}{\gamma} \left[ 1 + \frac{1-\omega}{\omega} \left( \frac{m'}{c} \right)^{\frac{\nu-1}{\nu}} \right]^{-1} \), then the FOC for labor (A.6) holds with strict equality and the quantity of labor supply is determined by it.

Case 2-1 (\( c_b > 0 \) and \( k'_b > 0 \)): In this case, we can obtain interior solutions by combining the FOC’s from (A.4) to (A.6), which are the same as \( c_b, m'_b, l_b \) and \( k'_b \) derived above.

Case 2-2 (\( c_b > 0 \) and \( k'_b < 0 \)): In this case, the borrowing constraint \( k' \geq 0 \) is binding. Hence, as with Case 1-2, \( k' \) and \( m' \) are given by

\[
\begin{align*}
k' &= 0, \\
m' &= (1 + \pi)a'.
\end{align*}
\]

The difference from Case 1-2 is that now labor supply \( l \) will be determined by the FOC for labor (A.6) with equality. From the budget constraint (1.3) and the binding borrowing constraint \( k' = 0 \), labor supply \( l \) has to satisfy

\[
l = \frac{c + m' - a - b - ss_j w}{\tilde{w}_j}.
\]

Substituting this into the FOC for labor (A.6) and rewriting,

\[
g \frac{1-\omega}{\omega} (m')^{\frac{\nu-1}{\nu}} c^{\frac{1}{\nu}} + (1 + \gamma) c + m' - a - \tilde{w}_j - b - ss_j w = 0.
\]

In general, we need numerical computation to solve this equation for \( c \). As a special case, however, we can solve this analytically when \( \eta = 0.5 \), i.e.,

\[
c = \frac{-(1 + \gamma) + \sqrt{(1 + \gamma)^2 + 4g \frac{1-\omega}{\omega} \frac{1}{m'} (a + \tilde{w}_j + b + ss_j w - m')}}{2g \frac{1-\omega}{\omega} \frac{1}{m'}}.
\]

Using this, we can calculate a value of \( l \). If the value of \( l \) is positive, we can use it to determine the optimum. However, the calculated \( l \) sometimes takes negative values. In that case, we have to set
\[ l = 0, \]
\[ c = a + b + ssjw - m'. \]

**Case 3** \((c \leq 0):\) In each case, if the calculated value of \(c\) is below zero, then it violates the condition that \(c > 0\). We have to rule out such cases as optimum.

**Backward Induction and Maximizing Value Function** As explained above, given values of \(a, a', z\) and \(j\), we can calculate the values of \(c, k', m'\) and \(l\) and thus can obtain the values of utility corresponding to each state (i.e., utility matrix). We can solve the problem by backward induction because the assumption \(s_{J+1} = 0\) implies that terminal utility always equals zero. In other words, we can start from \(j = J\) and pick up values of \(a'\) so that the chosen \(a'\) maximizes the value function for each state \((x\) and \(j)\) using \(s_{J+1}V(x',J+1) = 0\).

**A.2 Simple Two-period Model with Capital and Money**

In this Appendix, we show that the monetary superneutrality does not necessarily hold in OLG models using a simple example.

Consider a two-period OLG model with capital and money. We focus on the steady-state equilibrium and ignore time subscripts. Suppose that population of each generation is fixed and normalized at one, thus total population equals two. A young household exogenously provides one unit of labor supply and an old household does not work, which implies aggregate labor supply equals one.

A household maximizes a following simple lifetime utility without uncertainty subject to the budget constraints over two periods.
\[
\max_{c_1, c_2, m_1, m_2, k_1} \quad u(c_1) + \mu_1 v(m_1) + \beta [u(c_2) + \mu_2 v(m_2)],
\]
\[
s.t. \quad c_1 + m_1 + k_1 \leq w, \quad (A.12)
\]
\[
c_2 + m_2 \leq Rk_1 + \frac{1}{1+\pi} m_1. \quad (A.13)
\]

where \( u(\cdot) \) is utility from consumption \( c \) and \( v(\cdot) \) is utility from liquidity service by holding real money \( m \). The subscript “1” denotes young and “2” denotes old. \( k \) is capital, \( w \) is the real wage, \( R \) is the gross real interest rate \( \left(R \equiv 1 + r\right) \) and \( \pi \) is the inflation rate.

Suppose that \( \mu_2 = 0 \) for simplicity. Then there is no incentive to hold money when old \( (m_2 = 0) \).

Now we specify that the lifetime utility as a logarithmic form.

\[
U = \ln c_1 + \mu_1 \ln m_1 + \beta \ln c_2
\]

where \( \mu_1 \in (0, 1) \). Under this specification, the first order conditions are

\[
\frac{1}{c_1} = \frac{\beta R}{c_2}, \quad (A.14)
\]
\[
\frac{\mu_1}{m_1} = \beta \left( R - \frac{1}{1+\pi} \right) \frac{1}{c_2}. \quad (A.15)
\]

Note that \( c_1 \) and \( c_2 \) in the equation \( (A.14) \) cannot be canceled out even in the steady state and thus \( R \) depends on the pattern of intertemporal substitution of consumption \( (\frac{\partial c_2}{\partial c_1}) \). Combining the above two conditions, the money demand function is

\[
m_1 = \frac{\mu_1}{\Pi} c_1 \quad (A.16)
\]

where \( \Pi \equiv 1 - \frac{1}{\pi(1+\pi)} \), which represents the opportunity cost.

For the firm sector, we assume 100% depreciation of capital. A representative firm maximizes profits under the Cobb-Douglas technology.

---

\(^{1}\)This functional form is a special case of general CES utility when \( \nu = \sigma = 1.0 \) and labor is exogenous. Applying the l'Hopital's rule, the utility function specified by \( (A.16) \) is equivalent to \( u(c, m') = \omega \ln c + (1 - \omega) \ln(m') \).
$$\max_{K} K^\alpha - RK - w$$

where $K$ is aggregate capital stock. The first order conditions are

\begin{align*}
R &= \alpha K^{\alpha-1}, \quad (A.17) \\
w &= (1 - \alpha)K^\alpha. \quad (A.18)
\end{align*}

In each period the government issues new money at a constant rate $\theta$. In the steady state equilibrium, the inflation rate $\pi$ equals the money growth rate $\theta$ and the seigniorage is given by $\frac{\pi}{1+\pi}M$ as with the benchmark model, but for simplicity we assume that inflation taxes are thrown to the ocean.

The market clearing conditions are

\begin{align*}
\text{[Goods]} & \quad K^\alpha = c_1 + c_2 + k_1, \\
\text{[Capital]} & \quad K = k_1, \quad (A.19) \\
\text{[Money]} & \quad M = m_1, \\
\text{[Labor]} & \quad L = 1.
\end{align*}

Combining the budget constraints (A.12) and (A.13), the Euler equation (A.14) and the money demand function (A.16), given $r$ and $w$ the solutions for $c_1$, $m_1$, $k_1$ and $c_2$ are

\begin{align*}
c_1 &= \frac{w}{1 + \beta + \mu_1}, \\
m_1 &= \frac{\mu_1}{\Pi} \frac{w}{1 + \beta + \mu_1}, \\
k_1 &= w \left[ 1 - \frac{1}{1 + \beta + \mu_1} \left( 1 + \frac{\mu_1}{\Pi} \right) \right], \\
c_2 &= R k_1 + \frac{1}{1 + \pi} m_1. \quad (A.20)
\end{align*}

The equation (A.20) implies that savings by capital ($k_1$) are increasing in inflation, while real
balances $m_1$ are decreasing (because $\Pi \equiv 1 - \frac{1}{R(1+\pi)}$). There is substitution from money to capital at least at the individual level. We can show that equilibrium capital is also affected by inflation. Substituting the equilibrium conditions (A.17), (A.18) and (A.19) into the equation (A.20), we have an equation that gives aggregate capital stock in the steady-state equilibrium.

$$K - (1 - \alpha)K^\alpha \left[ 1 - \frac{1}{1 + \beta + \mu_1} \left( 1 + \frac{\mu_1}{1 - [\alpha K^{\alpha-1}(1 + \pi)]^{-1}} \right) \right] = 0.$$  

Defining the LHS in the above equation as $\varphi(K, \pi)$, the implicit function theorem implies that

$$\frac{dK}{d\pi} = -\frac{\varphi_\pi(K, \pi)}{\varphi_K(K, \pi)} = \frac{w\mu_1 [\Pi R(1 + \pi)]^{-2}}{1 + [1 - (1 - \alpha)R] (\beta + \mu_1) + (1 - \alpha)R\mu_1 \Pi^{-1} + w\mu_1 (1 - \alpha)\Pi^{-2} [R(1 + \pi)K]^{-1} \neq 0.}$$

Hence, superneutrality of money does not hold in this life-cycle economy. If $(1 - \alpha)R < 1$, then $\frac{dK}{d\pi}$ is positive and inflation increases capital (Tobin effect).
Appendix B

Appendix for Chapter 2

B.1 Solving the Problem

This appendix provides details about computational procedures to solve the problem.\footnote{We used the software R (ver.3.4.2) to compute most of things mentioned in this paper. Some parts of the code that require high-speed computing are written in the C language and called from the R session.}

B.1.1 Exogenous Variables and Parameters

First of all, note that we have the following exogenous variables.

- The initial level of assets when households enter the economy is assumed to be zero ($a_1 = 0$).
- The total factor productivity is set to one by just normalization ($\zeta = 1$).
- The consumption tax rate $\tau^c$ is set to 5% in the benchmark case and is exogenously raised to 10% in the counter-factual experiments.
- The risk aversion ($\sigma$), the capital’s share ($\alpha$) and the depreciation rate ($\delta$) will be borrowed from literature. (See Table \ref{tab:parameters} for sources.)
- The population growth rate ($n$), the conditional survival probability ($s_j$), the age-wage profile ($\eta_j$), the persistency of idiosyncratic productivity shocks ($\rho$) and the standard deviation of age-20 households’ productivity ($\sigma_z$), the government expenditures per output ($G/Y$), the pension contribution rate ($\phi$), the fraction of consumption taxes which are used for pension

\begin{itemize}
  \item The initial level of assets when households enter the economy is assumed to be zero ($a_1 = 0$).
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  \item The population growth rate ($n$), the conditional survival probability ($s_j$), the age-wage profile ($\eta_j$), the persistency of idiosyncratic productivity shocks ($\rho$) and the standard deviation of age-20 households’ productivity ($\sigma_z$), the government expenditures per output ($G/Y$), the pension contribution rate ($\phi$), the fraction of consumption taxes which are used for pension
\end{itemize}
expenditures \((p_c)\) and the capital income tax rate \((\tau^a)\) are given by data or institution. (See Table 1.1 and the corresponding explanations for sources.)

The remaining three parameters — the discount factor \((\beta)\), the preference weight on leisure \((\gamma)\) and the standard deviation of error term in the AR(1) productivity process \((\sigma_e)\) — are also exogenous parameters, but these parameters will be set to match the model with actual data. (See Table 1.2.) We need to guess their initial values.

**B.1.2 Procedure for Finding Equilibrium**

The basic strategy to find the equilibrium is as follows.

1. Guess initial values of \(K, L\) and \(b\).

2. Calculate the equilibrium values of \(r, w, ss\) and \(\tau^l\) according to Section B.1.3.

3. Solve the households’ problem based on the values calculated in Step 2 according to Section B.1.4.

4. Check whether the initial guesses for \(K, L\) and \(b\) are close enough to the calculated equilibrium values. If the distances between the model values and the initial guesses are not small enough, then set new values of \(K, L\) and \(b\).

5. Continue Steps 2 to 4 to find the equilibrium.

**B.1.3 Equilibrium Values**

Given values of \(K, L\) and \(b\), some equilibrium values are calculated as follows. From the firm’s FOC, we can easily obtain the real interest rate and the real wage.

\[
\begin{align*}
    r &= \alpha \zeta \left( \frac{K}{L} \right)^{\alpha-1} - \delta, \quad \text{(B.1)} \\
    w &= (1 - \alpha) \zeta \left( \frac{K}{L} \right)^{\alpha}. \quad \text{(B.2)}
\end{align*}
\]

From the FOC’s for firm (the equations (B.1) and (B.2)) and the production function (1.1), we can obtain the standard Euler’s rule.
\[ Y = (r + \delta)K + wL. \]

Combining this with the market clearing conditions (1.12) and (1.9), the equilibrium aggregate consumption \( C \) is given by

\[ C = \sum_j \mu_j \int_X c(x, j) d\psi_j = (r - n)K + wL - G \]  

(B.3)

where we used the relationship that \( K' = (1 + n)K \) in the steady state.

From the balanced pension system (1.13), the equilibrium wage replacement rate is given by

\[ ss = \frac{1}{\mu_R} \left( \phi L + \frac{p_c \tau^c}{w} C \right) \]  

(B.4)

where \( \mu_R = \sum_{j=R}^J \mu_j \).

From the government budget constraint for purchases (1.15), the equilibrium labor income tax rate is given by

\[ \tau_l = \frac{G - \tau^a r K - (1 - p_c) \tau^c C}{wL}. \]  

(B.5)

### B.1.4 Solutions for Households’ Problem

Essentially we use backward induction of value functions with the method of discretized asset grid points, like Huggett (1995). Households have to choose \( c, a' \) and \( l \) simultaneously. Now we ignore the time subscript \( t \) because we are focusing on the stationary equilibrium. We have to solve the Bellman Equation in (1.7) under the borrowing constraint \( a_{j+1} \geq 0 \) and the non-negativity of labor \( l_j \geq 0 \). The non-negativity for consumption \( c_j \) and leisure \( (1 - l_j) \) will never bind because they are involved in utility, but the borrowing constraint for capital and the non-negativity constraint for labor supply can be binding. Reproducing the households’ problem in the recursive form,
\[
V(x, j) = \max_{c, a', l} \left[ u(c, l) + \beta s_{j+1} E \left[ V(x', j + 1) \right] \right] \\
\text{s.t.} \quad c = \frac{1}{1 + \tau^c} \left[ (1 + \tilde{r})a + \tilde{w}_j l + b + ss_j w - a' \right], \\
a' \geq 0 \text{ and } l \geq 0.
\]

where \( \tilde{r} \equiv (1 - \tau^a)r \) and \( \tilde{w}_j \equiv (1 - \tau^l - \phi)z_j \eta_j w \).

**B.1.4.1 Method 1 (Using Money and Labor Grids as Choice Variables)**

In this subsection, we consider the algorithm that solves the problem by using an asset grid as a state variable and a labor grid as choice variables (Method 1). In the next subsection, we will provide the alternative algorithm which adopts the asset grid as a choice variable and uses the first order conditions (Method 2). The main difference between the two methods is that the choice of labor is treated as discrete in Method 1 and as continuous in Method 2. Also, choices of next-period assets are continuous in Method 1 but discrete in Method 2. We believe that Method 1 provided in this subsection is theoretically more appropriate than Method 2 for our purpose of analysis because an individual choice of labor supply is more likely to be discrete than a choice of assets. However, a computational burden in Method 1 is much heavier than the one in Method 2. Therefore, we utilize Method 2 as well and will confirm that the basic results are similar in both algorithms.

In Method 1, we set following three grids to evaluate the value function.

\[
a_{\text{grid}} = \{a_{\text{min}}, \ldots, a_{\text{max}}\}_{1 \times N_a}, \quad (a_{\text{min}} \geq 0)
\]
\[
l_{\text{grid}} = \{l_{\text{min}}, \ldots, l_{\text{max}}\}_{1 \times N_l}, \quad (l_{\text{min}} \geq 0 \text{ and } l_{\text{max}} < 1)
\]

where \( N_a \) and \( N_l \) are the number of grid points for assets and labor supply, respectively. Households choose the level of next-period assets \( a' \) continuously. To do this, we use an interpolation technique.

Given values of \( a' \in \mathbb{R}_+, \ l \in l_{\text{grid}}, \ a \in a_{\text{grid}}, \ z \in \mathbb{Z} \) and \( j \in \{1, 2, \ldots, J\} \), a value of \( c \) is determined by the budget constraint (1.3). Then we can calculate a value of flow utility \( u(c, l) \). With the assumption \( s_{j+1} = 0 \) that implies terminal utility equals zero, we can solve the value function.
by backward induction. In other words, we can start from \( j = J \) and pick up values of \( a' \) so that the chosen \( a' \) maximizes the value function for each state \((x \text{ and } j)\) using \( s_{J+1} V(x', J + 1) = 0 \). The value of \( a' \) is not necessarily on the nodes in the asset grid because we are choosing it continuously, thus we linearly interpolate the expected value \( E[V(a', z', j + 1)|z] \). For example, if we choose \( a' \) from \( \mathbb{R}_+ \) and the calculated value of \( a' \) is in \([a_i, a_{i+1}]\) where \( a_i \) and \( a_{i+1} \) are adjacent nodes in the asset grid, then the linearly interpolated value function in the next period is

\[
V(a', z', j + 1) \approx V(a_i, z', j + 1) + \frac{a' - a_i}{a_{i+1} - a_i} \left[ V(a_{i+1}, z', j + 1) - V(a_i, z', j + 1) \right].
\]

Using this expression and the Markov transition matrix, we can find an optimal value of \( a' \) which maximizes the current-period value function \( V(a, z, j) \) by numerical computation.

**Finding Distribution**  When calculating the stationary distribution, we should notice that in general the decision rule \( a'(x, j) \) calculated above is not necessarily on a node in \( a_{grid} \). Therefore, we allocate the population in the next period corresponding each value of optimal \( a' \) linearly into two adjacent nodes \( a_i \) and \( a_{i+1} \) such that \( a' \in [a_i, a_{i+1}] \). Given the exogenous distribution of age-1 households, \( \psi_1 \), the algorithm to calculate the probability \( \psi_j(x') \) for \( j \geq 2 \) is done by recursively as follows. First, we set \( \psi_j(x') = 0 \) \( \forall x' \) and \( j \geq 2 \). Then, for \( i = 1, \ldots, N_a \) and \( s = 1, \ldots, S \),

1. Find the index \( i' \) that satisfies \( a_{i'} \leq a'(x, j) < a_{i'+1} \).

2. Calculate the interpolation weight \( \varpi \) as

\[
\varpi = \frac{a'(x, j) - a_{i'}}{a_{i'+1} - a_{i'}}.
\]

3. For each \( s' = 1, \ldots, S \), update \( \psi_j \) as

\[
\psi_j(a_{i'}, z_{s'}) = \psi_{j-1}(a_{i'}, z_{s'}) + (1 - \varpi) Pr(z_{s'}|z_s) \psi_j(a_i, z_s),
\]

\[
\psi_j(a_{i'+1}, z_{s'}) = \psi_{j-1}(a_{i'+1}, z_{s'}) + \varpi Pr(z_{s'}|z_s) \psi_j(a_i, z_s)
\]

where \( z_{s'} \) is the productivity shock at the \( s' \)th state.
B.1.4.2 Method 2 (Using Total Asset Grid as State and Choice Variables)

To reduce computational burdens, we can use an alternative method to solve the households’ problem by combining the first order conditions and the total asset grid. In this algorithm, we use the total asset grid as both state and choice variables. First we set an asset grid to evaluate the value function.

\[
a_{\text{grid}} = \{a_{\min}, \ldots, a_{\text{max}}\} \times \mathbb{N}_a \quad (a_{\min} \geq 0)
\]

where \(N_a\) is the number of grid points. For any set of values of \(a \in a_{\text{grid}}, a' \in a_{\text{grid}}, z \in \mathbb{Z}\) and \(j \in \{1, 2, \ldots, J\}\), we calculate values of \(c, a'\) and \(l\) in accordance with the following procedure, and then compute a value of flow utility for each state. This means that we will have a utility matrix (more rigorously speaking, a four-dimensional array) that has \(N_a \times N_a \times S \times J\) elements, where \(S\) is the number of productivity states. Based on this utility matrix, we will choose the optimal values of \(c, a'\) and \(l\) by backward induction of value functions.

**First Order Conditions** The necessary conditions for the households’ problem are

\[
\begin{align*}
[a'] & \quad - \frac{1}{1 + r^c} u_c(c, l) + \beta s_{j+1} E \left[ V_{a'}(x', j + 1) \big| z \right] \geq 0, \\
[l] & \quad \frac{\bar{w}_j}{1 + r^c} u_c(c, l) + u_l(c, l) \geq 0
\end{align*}
\]

and the usual transversality conditions. Note that the FOC’s for \(a'\) and \(l\) are expressed with inequality due to the presence of constraints \(a' \geq 0\) and \(l \geq 0\). The envelope condition is

\[
V_a(x, j) = \frac{1 + \tilde{r}}{1 + r^c} u_c(c, l).
\]

Under our specification for the utility function, we can rewrite the above conditions as

\[
\begin{align*}
[a'] & \quad - \Lambda + \beta s_{j+1} (1 + \tilde{r}) E \left( \Lambda' \big| z \right) \geq 0, \\
[l] & \quad \frac{\bar{w}_j}{1 + r^c} (1 - \gamma) - \gamma c(1 - l)^{-1} \geq 0
\end{align*}
\]
where \( \Lambda \equiv (1 - \gamma) \left[ c_j^{\sigma(1-\gamma)-\gamma}(1 - l_j)^{\gamma(1-\sigma)} \right] \geq 0 \). Particularly, the FOC for \( l \) is also expressed as

\[
\frac{c}{1 - l} \leq \frac{1 - \gamma \cdot \tilde{w}_j}{\gamma \cdot 1 + \tau^c}.
\] (B.8)

**Solutions** For a moment we assume that both \( a \) and \( a' \) are given and picked up from the grid \( a_{\text{grid}} \). Then the budget constraint (1.3) implies

\[
c = \frac{1}{1 + \tau^c} \left[ (1 + \tilde{r})a + \tilde{w}_j l + b + ss_j w - a' \right].
\]

The FOC for labor (B.8) holds with strict equality when \( l > 0 \) and holds with strict inequality when \( l = 0 \). Let \( c_0 \) denote a value of \( c \) calculated by the above equation setting \( l = 0 \). If \( c_0 \) with \( l = 0 \) violates the FOC (B.8) with strict inequality, i.e., if \( c_0 \geq (1 - \gamma)\tilde{w}_j (1 + \tau^c)^{-1} \), then we can say that \( l > 0 \) by contradiction.

In summary, if \( c_0 \geq (1 - \gamma)\tilde{w}_j (1 + \tau^c)^{-1} \), then

\[
l = 0,
\]

\[
c = \frac{1}{1 + \tau^c} \left[ (1 + \tilde{r})a + b + ss_j w - a' \right].
\]

If \( c_0 < (1 - \gamma)\tilde{w}_j (1 + \tau^c)^{-1} \), then the FOC for labor (B.8) with strict equality implies that

\[
l = 1 - \frac{\gamma}{1 - \gamma} \frac{1 + \tau^c}{\tilde{w}_j} c,
\]

\[
\Rightarrow \quad c = \frac{1}{1 + \tau^c} \left[ (1 + \tilde{r})a + \tilde{w}_j \left( 1 - \frac{\gamma}{1 - \gamma} \frac{1 + \tau^c}{\tilde{w}_j} c \right) + b + ss_j w - a' \right]
\]

\[
= \frac{1 - \gamma}{1 + \tau^c} \left[ (1 + \tilde{r})a + \tilde{w}_j + b + ss_j w - a' \right].
\]

If the calculated value of \( c \) is below zero, then it violates the condition that \( c > 0 \). We have to rule out such cases as optimum.

**Backward Induction and Maximizing Value Function** As explained above, given values of \( a, a', z \) and \( j \), we can calculate the values of \( c, a' \) and \( l \) and thus can obtain the values of utility
corresponding to each state (i.e., utility matrix). We can solve the problem by backward induction because the assumption $s_{J+1} = 0$ implies that terminal utility always equals zero. In other words, we can start from $j = J$ and pick up values of $a'$ so that the chosen $a'$ maximizes the value function for each state $(x$ and $j)$ using $s_{J+1}V(x', J + 1) = 0.$
Appendix C

Appendix for Chapter 3

C.1 Discussion about a Shape of $B(k)$

In this appendix, we explore how a shape of function $B(k)$ in the equation (3.29) changes depending on values of parameters.

In general, the shape of the curve $B(k)$ is determined by curvatures of $\rho'(k)$ and $c(k)$. By assumption, $\rho'(k)$ is a concave function that goes from $-\infty$ to zero as $k$ increases. On the other hand, $c(k)$ is a hump-shaped function that is initially increasing in $k$ but turns to be decreasing at some point. We can consider the following three cases: (i) If $\rho'(k)$ moves enough quickly compared with $c(k)$, then $\rho'(k)c(k)$ is monotonically increasing and approaches to zero. $B(k)$ is also monotonically increasing and approaches to $r_{ss}$ from the lower side; (ii) If the increase in $\rho'(k)$ is relatively slow and the increase in $c(k)$ is fast, $\rho'(k)c(k)$ can be decreasing especially in an area that $k$ is close to 0. In this case $B(k)$ eventually turns to be increasing due to the assumption (3.31), $\lim_{\varphi \to \infty} \rho'(\varphi) = 0$; (iii) The last case is when $c(k)$ reaches its peak at an very early stage and then dives into the negative area so rapidly. In this case $\rho'(k)c(k)$ may diverge to $+\infty$ and possibly there may not exist an equilibrium when $B(k)$ do not intersect with $A(k)$ within a range that $c(k) > 0$, but it seems to be unlikely as long as considering a usual production technology with reasonable values of capital share and depreciation rate. We can exclude this case if assuming that the economy is dynamically efficient and the necessary condition $f'(k_{ss}) - \delta \geq 0$ is satisfied.

Since it is difficult to discuss further without additional specifications, we assume that the production function $f(k)$ and the transaction technology $\rho(k)$ follows the equations (1.4) and (3.31),
respectively. Still there exists so many cases and it is hard to argue about a general condition, but as far as we have checked a unique equilibrium seems to always exist. The most typical case is described in Figure C.1 which corresponds to the above case (i).

![Figure C.1: Curves of Functions A and B](image)

This graph is essentially the same as Figure 3.3 but an upper bound of $B(k)$ and a lower bound of $A(k)$ are explicitly added. Roughly speaking, unless the value of $\alpha$, $\kappa$ or $\delta$ is substantially high (note that we have the restriction $\alpha\kappa < 1$), $B(k)$ has a monotonic upward slope. In this case, clearly the model has a unique equilibrium.

The case (ii) can occur when $\alpha$ or $\kappa$ is very large. Even in this case, however, we found that $B(k)$ eventually turns to be increasing at some point due to the assumption (3.8) and approaches to $r_{ss}$ from the negative side. Figure C.2 illustrates a curve of $B(k)$ with a high value of $\alpha$.

![Figure C.2: Curve of Function B with High $\alpha$](image)

\[\text{Note that the “upper bound of } B(k)\text{” may be a bit misleading word. } B(k) \text{ itself can be above } r_{ss} \text{ as } k \text{ goes to } +\infty \text{ because } c(k) \text{ diverges to } -\infty, \text{ but clearly the negative value of } c \text{ cannot be an equilibrium. More precisely we should say “} B(k) \text{ is always lower than } r_{ss} \text{ as long as } c(k) > 0\text{.”}\]
The case (iii) seems not to occur under our specifications as far as we have checked. When $\delta$ is very high, the peak of $c(k)$ shifts to left and thus $B(k)$ becomes positive even when $k$ is relatively low. However, even in this case $A(k)$ quickly shifts to left as well, and therefore $A(k)$ and $B(k)$ have a unique intersection satisfying $c(k) > 0$.

We acknowledge that the above approach does not “prove” the existence of equilibrium in this model, but we think we can say that in most cases there exists a unique equilibrium at least under the reasonable specification for the production function and its parameters $\alpha$ and $\delta$ with the restriction $\alpha \kappa < 1$.

C.2 Rational Expectations Equilibrium

We used a software R (ver.3.4.2) to compute most of things mentioned in this paper. When solving the benchmark stochastic model (3.34)–(3.39), we obtained values of coefficients in log-linearized system by utilizing numerical computation. The results reported in Sections 3.4 and 3.5 are based on these numerically calculated coefficients. In this appendix, however, in order to make descriptions more clear, we describe our computational details based on another log-linearized system that is algebraically derived. We confirmed that results based on the following equations have slight differences from the result in the main text but essentially they are the same.

The log-linearized version of the equations (3.34)–(3.39) are as follows.

\footnote{For example, signs of impulse responses are all the same, although their levels are slightly different. We think the computation in the main text is more accurate as a way of log-linearization because the numerical approach can compute the Jacobian matrix with higher precision, while in the algebraic approach we applied some approximation for percentage variables such as $i_t$ and $\pi_t$ to make expressions more simple.}
\[ \hat{\lambda}_t = E_t \left( \hat{\lambda}_{t+1} \right) + \beta \left( \omega_0 \hat{k}_t + \omega_4 \left[ \hat{t}_t + \hat{m}_t + u_{t+1} - E_t (\hat{\pi}_{t+1}) \right] \right), \]  
(C.1)

\[ \hat{\lambda}_t = E_t \left( \hat{\lambda}_{t+1} \right) + \hat{t}_t - E_t (\hat{\pi}_{t+1}), \]  
(C.2)

\[ (1 + i_{ss}) \hat{\lambda}_t = \alpha \omega_8 \hat{k}_t - \omega_7 (\hat{m}_t + u_{t+1}) + (1 - \omega_1) E_t \left( \hat{\lambda}_{t+1} \right) + \left( \omega_7 - \frac{1}{\beta} \right) E_t (\hat{\pi}_{t+1}), \]  
(C.3)

\[ \frac{c_{ss}}{k_{ss}} \hat{m}_t + \hat{k}_t = \omega_9 \hat{k}_{t-1}, \]  
(C.4)

\[ \hat{m}_t = \hat{m}_{t-1} + u_t - \hat{\pi}_t, \]  
(C.5)

\[ u_{t+1} = \Phi_u u_t + e^{u}_{t+1}. \]  
(C.6)

where

\[ \omega_1 \equiv (1 + \chi y_{ss}^\kappa)^\gamma, \omega_2 \equiv \frac{\chi y_{ss}^\kappa}{1 + \chi y_{ss}^\kappa}, \omega_3 \equiv \frac{\alpha y_{ss}}{k_{ss}}, \omega_4 \equiv 1 - \delta, \omega_4 \equiv \frac{1}{\beta} - \omega_3, \]

\[ \omega_5 \equiv \kappa \omega_4 (1 - \omega_2) + \alpha y_{ss} \frac{k_{ss}}{k_{ss}}, \omega_6 \equiv \alpha \left( \omega_5 - \frac{y_{ss}}{k_{ss}} \right) - \omega_4, \omega_7 \equiv \phi (i_{ss} + \omega_1), \]

\[ \omega_8 \equiv \kappa \gamma \omega_2 (i_{ss} - \omega_7) \text{ and } \omega_9 = \omega_5 + \alpha \kappa \gamma \omega_2 \frac{c_{ss}}{k_{ss}}. \]

We replaced \( \hat{m}_{t+1} \) in the equations (C.1) and (C.3) with current-period variables using the equation (C.5) in order to express the system as first-difference equations.\(^3\)

To derive the rational-expectations equilibrium, we followed a method by Blanchard and Kahn (1980). In this model, the endogenous variables can be separated into two groups.

---

\(^3\)In the basic CIA model, the equations (3.34), (3.36) and (3.37) become as follows, respectively.

\[ \hat{\lambda}_t = E_t \left( \hat{\lambda}_{t+1} \right) + \beta \alpha (\alpha - 1) \frac{y_{ss}}{k_{ss}} \hat{k}_t, \]

\[ \hat{\lambda}_t = -\phi (\hat{m}_t + u_{t+1}) + (\phi - 1) E_t (\hat{\pi}_{t+1}), \]

\[ \frac{c_{ss}}{k_{ss}} \hat{m}_t + \hat{k}_t = \omega_3 \hat{k}_{t-1}. \]

In the alternative model discussed in Section 3.5.1, the FOC for capital is the same as the basic CIA model and other equations are the same as the benchmark model.
\[
x_t = \left[ \hat{\lambda}_t, \hat{\pi}_t \right]',
\]

\[
s_t = \left[ \hat{k}_{t-1}, \hat{m}_{t-1}, \hat{i}_{t-1}, u_t \right]',
\]

where \( x_t \) is a vector of jump (forward-looking) variables and \( s_t \) is a vector of state (pre-determined) variables. We can express the system as the form of first differential equations. The matrix representation of the above system is

\[
M_1 \begin{bmatrix} E_t(x_{t+1}) \\ s_{t+1} \end{bmatrix} = M_2 \begin{bmatrix} x_t \\ s_t \end{bmatrix} + \begin{bmatrix} 0_{5 \times 1} \\ e_{t+1}^u \end{bmatrix}
\]

where

\[
M_1 = \begin{bmatrix}
\frac{1}{\beta} & -\omega_4 & \omega_6 & \omega_4 & \omega_4 & \omega_4 \\
1 & -1 & 0 & 0 & 1 & 0 \\
1 - \omega_1 & \omega_7 - \frac{1}{\beta} & \alpha \omega_8 & -\omega_7 & 0 & -\omega_7 \\
0 & 0 & 1 & \frac{\phi_{ss}}{k_{ss}} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
M_2 \equiv \begin{bmatrix}
\frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 + i_{ss} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \omega_9 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \Phi_u
\end{bmatrix},
\]

Given our parameter values, the matrix \( M_1 \) is invertible. Multiplying \( M_1^{-1} \) in both sides,
\[
\begin{pmatrix}
E_t(x_{t+1}) \\
s_{t+1}
\end{pmatrix} = M
\begin{pmatrix}
x_t \\
s_t
\end{pmatrix} + M_1^{-1}
\begin{pmatrix}
0 \\
e_{t+1}^u
\end{pmatrix}
\]

where \( M \equiv M_1^{-1}M_2 \). Given parameter values the model also satisfies the Blanchard and Kahn’s condition that the number of eigenvalues of \( M \) that are outside the unit circle must equal the number of forward-looking variables. By the spectral decomposition, the matrix \( M \) is expressed as

\[
M = V_0 \Lambda V_0^{-1}.
\]

where \( \Lambda \) is a matrix in which diagonal elements are eigenvalues of \( M \) ordered in descending and \( V_0 \) be the matrix of corresponding eigenvectors. Then, we can rewrite the system as

\[
V
\begin{pmatrix}
E_t(x_{t+1}) \\
s_{t+1}
\end{pmatrix} = \Lambda V
\begin{pmatrix}
x_t \\
s_t
\end{pmatrix} + VM_1^{-1}
\begin{pmatrix}
0 \\
e_{t+1}^u
\end{pmatrix}
\]

where \( V \equiv V_0^{-1} \). Now we separate \( \Lambda \) and \( V \) as follows.

\[
\Lambda =
\begin{bmatrix}
\Lambda_x & 0 \\
0 & \Lambda_s
\end{bmatrix},
\]

\[
V =
\begin{bmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{bmatrix}
\]

where \( \Lambda_x \) is a 2 \( \times \) 2 diagonal matrix correcting the eigenvalues which is greater than one in terms of absolute value, and \( \Lambda_s \) is a matrix correcting the rest of eigenvalues. So the diagonal elements of \( \Lambda_s \) are less than one in terms of absolute value and all of non-diagonal elements are zero. The sufficient condition that guarantees a future path of \( E_t(x_{t+1}) \) does not diverge is that coefficients of \( x_t \) are always zero, i.e.,

\[
V_{11}x_t + V_{12}s_t = 0.
\]

Solving this for \( x_t \), we obtain the following policy function \( P \).
\[ x_t = Ps_t \]

where \( P \equiv -V_{11}^{-1}V_{12} \). Now we assume that all of shock terms are zero. Then the system can be rewritten as

\[
\begin{bmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{bmatrix}
\begin{bmatrix}
P_{s_{t+1}} \\
s_{t+1}
\end{bmatrix}
= \begin{bmatrix}
\Lambda_x & 0 \\
0 & \Lambda_s
\end{bmatrix}
\begin{bmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{bmatrix}
\begin{bmatrix}
P_{s_t} \\
s_t
\end{bmatrix}.
\]

Solving the second equation \((V_{21}P + V_{22})s_{t+1} = \Lambda_s(V_{21}P + V_{22})s_t\) for \(s_{t+1}\), we have the transition function \(T\).

\[ s_{t+1} = Ts_t \]

where \( T \equiv (V_{21}P + V_{22})^{-1}\Lambda_s(V_{21}P + V_{22}) \). Given initial values of shocks, we can easily calculate impulse response functions by using the above policy and transition functions.

At last, we would like to briefly mention about steady-state values that are needed to compute the rational-expectations equilibrium. Given a value of the money growth rate \(\theta_{ss}\), the inflation rate and the interest rates are given by the equations (3.22), (A.1) and (3.24). The steady-state equilibrium capital stock \(k_{ss}\) can be obtained as a numerical solution for the equilibrium condition (3.27). Using the value of \(k_{ss}\), we can compute values of \(c_{ss}\) and \(y_{ss}\) by using the equations (3.26) and (1.6), respectively. In addition, if we decide to compute the Jacobian matrix of original equations (3.34)–(3.38) by the numerical approach, we also need values of \(m_{ss}\) and \(\lambda_{ss}\) to evaluate the Jacobian. The equation (3.25) gives the value of \(m_{ss}\) and the equations (3.14), (A.5) and (1.16) imply \(\lambda_{ss} = \frac{c_{ss}^\phi}{1 + i_{ss} \rho(k_{ss})}\).