On the geometric interpretation of certain vertex algebras and their modules

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On the Geometric Interpretation of Certain Vertex Algebras and Their Modules

Jesse Corradino

A Dissertation
Submitted to the University at Albany, State University of New York
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Dedicated to

Christian J. Corradino

and

Lindsay N. Childs
Abstract

We consider the geometric foundations of certain vertex algebras and their modules according to the formalism of Harish-Chandra Geometry that we develop in conjunction with Formal Geometry. Our main result is that we introduce an extension of the torsor of formal coordinates whereby a sheaf of pro-coherent modules is obtained over an affine scheme that corresponds to a certain vertex algebra. As a corollary of this result, we obtain a similar sheaf of pro-coherent modules corresponding to certain representations of the aforementioned vertex algebra.
Acknowledgements

First and foremost I would like to thank Antun Milas for both his support and patience as my thesis advisor. Furthermore, I thank him especially for the trust he gave to me over the course of its research insofar as he allowed me to explore the broad landscape of geometry related to vertex algebras unencumbered by any demand on his part that my exploration be directed toward one region of that world or another.

On that note, below I partially quote a passage from Alexander Grothendieck’s *Reaping and Sowing*. I include this quote because Grothendieck identifies two crucial aspects of creativity in scientific research and overall I consider creativity to be a superior motivation for scientific research to that of narrowly minded commercial applications. Additionally, I feel it pertains to the my time studying with Professor Milas.

First, he identifies the capability one must have to transcend the "despotic circles that delimit a certain milieu" and second, the curiosity of one to venture beyond those "circles" alone as a method to obtain that capability. I am deeply lacking in the first regard, so without the tutelage of Professor Milas to guide me through the study of vertex algebras and the vast geometrical lore entailed by their study, I would have remained a prisoner of those invisible circles, according to Grothendieck’s words. It is not for me to say whether I still remain a prisoner of those circles, but I can say that I have attempted to escape from them. My escape attempt was precious to me and a delight; it was a tour of marvels of the human imagination, so I will remain always grateful to Professor Milas for giving that to me, or sending me on that adventure. More concisely, I could say I am grateful to Antun for setting me free.

My thesis is dedicated to both my brother Christian and my mentor, Professor Childs. In the former case, it is because his life ended too soon and I loved my little brother so much while he was alive that all my efforts in life afterwards are there to betoken my love for him. In the latter case, I owe to Professor Childs more than I can repay for the opportunities and education he has provided to me, most generously, and without any expectation of a reward.
for himself. I hope the work that went into this thesis amounts to a small measure of recompense on that account, but in comparison to his magnanimity, I know that my hope is in vain.

Beside my advisor and the individuals to whom I have dedicated my thesis, I would like to thank Professor Alexandre Tchernov for his conversations about mathematics and algebraic geometry. Those conversations were especially helpful to clarify certain commutative algebra questions I have had in mind. I would be remiss to neglect thanking Steven Plotnick for his career advice and for helping me to learn how to teach. Last, I would like to thank Professor Goldfarb for the concern he shows for students and being a kind ear. Among my peers, I would also like to thank Michael Coleman for years of pleasant conversation, both on mathematical topics and matters of politics and history that we both enjoy. I would like to thank Ian Parnett for the same, but also for his personal friendship beyond our work environment. I would like to thank Brian Bennett for many conversations we have also shared, especially about categorical matters and foundational issues. He, too, has been a dear friend to me over the years. Last, I would like to thank Dan Wood for discussing mathematics and his exuberance—somehow his straightforward manner always made me grin. Finally, I would like to mention my gratefulness for the support of the staff, particularly Joan Mainwaring, for helping me to maintain my composure through the authorship of this thesis.
Introduction

*Without being aware of it, they’ve remained prisoners of those invisible and despotic circles which delimit the universe of a certain milieu in a given era. To have broken these bounds they would have to rediscover in themselves that capability which was their birthright, as it was mine: The capacity to be alone.*

Alexander Grothendieck

As the title suggests, this thesis is primarily concerned with interpreting certain vertex algebras as geometric objects, so one must first adequately express what it means to interpret a mathematical object, such as a vertex algebra, geometrically. Of course interpretations shall vary across specializations within mathematics and according to generality even unto those, yet since we work in the realms of both algebraic and formal algebraic geometry, we shall adopt the conventions of those subjects in order to interpret certain vertex algebras geometrically. Hence, what we mean when we say "interpret a vertex algebra as a geometric object," we mean that, for an affine scheme $X$, to demonstrate that a certain vertex algebra is an object of $\text{QCoh}(X)$, the category of quasi-coherent sheaves on $X$.

The pursuit of vertex algebras geometrically is voluminous in the literature, for there is great motivation to interpret vertex algebras geometrically owing to their role in conformal field theory [HLZ] and the generally geometric flavor of theoretical physics. Indeed, vertex algebras are regarded as the mathematical incarnation of axiomatic two-dimensional conformally invariant quantum field theory, or CFT in short, in the physical literature. The main treatises on the study of vertex algebras from the algebraic perspective are undertaken in the works of I. Frenkel, Lepowsky, Huang, Meurman, and Li [FLM], [LL].

Unto their representation theoretic and algebraic study, this school of thought also produced its own geometric interpretations of vertex algebras, especially [Hua] and [HL], where the former reference adopts the point of view of operads and the latter that of $\mathcal{D}$-modules, which is more closely aligned to the standard works that are written from the perspec-
tive of algebraic geometry. Despite that the geometric point of view adopted in [Hua] was operadic, Huang furnished an important result in his book for those studying the topic algebro-geometrically that declared the coordinate free nature of vertex algebras as geometric objects associated to algebraic curves which was enormously influential.

Insofar as the algebro-geometric interpretations of vertex algebras that exist, the key tomes are [FBZ] and [BD], where, again, the former was heavily influenced by the aforementioned result on coordinate freeness in [Hua]. Alongside of these works, one has [MSV, GMSII] and [KV], where the former reference has matured into an ansatz similar in importance to the subject of [BD, FBZ] nowadays and one that was informative to the conception of the result in this thesis.

The text *Chiral Algebras* [BD] of Beilinson and Drinfeld is the most comprehensive treatment of geometrizing vertex algebras, as each alternative treatment mentioned above is somehow recapitulated in this work. The main point of view adopted therein is to view vertex algebras geometrically as a special case of *chiral algebras*, which are certain $\mathbb{D}$-modules $\mathcal{V}_X$ on the square $X \times X$ of a curve $X$ supported along the diagonal $\Delta : X \to X \times X$. The various structure maps are then be interpreted as the operator product expansion, normally ordered product, translation invariance, etc. of vertex operators familiar to the vertex algebra community. Unto chiral algebras are two categorically related constructions, that of *factorization algebras* and that of *chiral differential operators*, the former of which is shown to be equivalent to chiral algebras [BD, FBZ] on the Ran space of $X$, an algebraic curve.

Factorization algebras and, more generally, factorization monoids, are the perspective from which [KV] develops the topic of interpreting vertex algebras geometrically by treating vertex algebras as sheaves of $\mathbb{D}$-modules on the loop space $\mathcal{L}(X)$ of an algebraic curve $X$, and the latter, although not chronologically, is how the development in [MSV, GMSII] proceeds. Factorization algebras have enjoyed a kind of renaissance since their discovery and application in *loc.cit* owing to recent work by Costello and Gwilliam [CG] employing them.
in perturbative quantum field theory as the factorization algebra of observables.

The work of Frenkel and Ben Zvi takes a somewhat different approach, but one that can ultimately be understood in terms of Beilinson and Drinfeld’s chiral algebras. Their approach to interpreting vertex algebras geometrically proceeded through formal geometry \[\text{[GK]}\], a subject which exists in the interstice between folklore and rigorous mathematics. Perhaps formal geometry remains in this world owing to the comparatively esoteric nature of infinite dimensional constructions in mathematics.

Formal geometry is, at its core, the study of the torsor of formal coordinates \(\mathbb{C} \to X\), a formal \(X\)-scheme \[\text{[Vdb]}\]. As a fine moduli space, it parameterizes trivializations of the jet scheme \(\text{Jet}_X \to X\) \[\text{[Yek]}\], so it is for this reason, broadly speaking, the torsor of formal coordinates is a torsor over the group of automorphisms of the formal disc, as the fibres of \(\mathbb{C} \to X\) are trivializations of the formal disc. Moreover, by construction, it canonically exists for an arbitrary finite dimensional smooth scheme \(X\) \[\text{[BD],[BD2]}\]. What’s peculiar about this space, however, is that the fibres of the tangent scheme are not the fibres of the Lie algebra of the aforementioned group of automorphisms. This discrepancy imposes a non-classical structure on \(\mathbb{C} \to X\) when compared to the finite setting of principal bundles in geometry or torsors in algebraic geometry. This non-classical feature is addressed in this thesis as Harish-Chandra geometry so that formal geometry is viewed as a special case thereof; to the author’s knowledge, formal geometry is the only extant substantial example of Harish-Chandra geometry, although there are more modern constructions in the setting of derived geometry closely related \text{cf.} \[\text{[Gwil]}\].

We consider the foundations of Harish-Chandra geometry in the generality of pro-finite modules, hence the phrase pro-finite Harish-Chandra ansatz, below. Harish-Chandra geometry takes as its input the data of a torsor \(S \to X\) over a base space \(X\) and a Harish-Chandra pair \((\mathfrak{g}, K)\) consisting of both a Lie algebra \(\mathfrak{g}\) and an algebraic group \(K\). The Lie algebra \(\mathfrak{k}\) of the algebraic group is furnished with an embedding \(\mathfrak{k} \to \mathfrak{g}\) into the Lie algebra it is paired with, and certain compatibilities are required to be satisfied by this embedding. The torsor
$S$ is then both a principal $\mathfrak{g}$-space and a $K$-torsor, so that the compatibility of the Harish-Chandra pair adopted to this torsor reflects the compatibility of the two, in general, distinct structures. This perspective generalizes the classical situation encoded by Lie’s theorem, where the Lie algebra of the Lie group is isomorphic to the Lie algebra of symmetries of the underlying principal bundle. In particular, it is well-suited to the the above situation in formal geometry, where a discrepancy between principal Lie algebra structure and algebraic group torsor differ.

Harish-Chandra geometry also supports the associated bundle construction. Classically, one is able to associated a geometric vector bundle to a representation $V$ of an algebraic group $K$ together with a $K$-torsor $S \to X$, say $\mathcal{V}_X$. This construction is obviously valid in the ansatz described above, however, we now require that the representation $V$ is also a representation of $\mathfrak{g}$ and a compatibility of the two module structures. Accordingly, the associated bundle $\mathcal{V}_X$ is equipped with the additional data of a flat connection by the representation of the Lie algebra $\mathfrak{g}$. We call such representations Harish-Chandra modules. We consider these matters in the generality of pro-modules because of the examples we have in mind from the theory of vertex algebras. Formal geometry emerges as a special case of Harish-Chandra geometry by setting $S = \mathcal{C}or_X$, $\mathfrak{g} = \mathfrak{w}$, the so-called Lie algebra of formal vector fields, and $K = Aut(\mathcal{O})$, the group of formal automorphisms.

Indeed, the formal geometry required to justify the procession of Frenkel and Ben Zvi through formal geometry was that, often, vertex algebras are defined to be conformal ab initio. A vertex algebra is conformal if it is a representation of the Virasoro algebra. Alternatively, there exists a weight 2 vector $\omega$ whose corresponding vertex operator $Y(\omega, z)$ is the generating function for this action. Such representations furnish gradings of the underlying vector space and several other features of a vertex algebra’s definition in general which identify them as a particularly nice class of objects from which to begin developing the entire subject. This is to say nothing of their special relevance to the aforementioned physical motivations.
Geometrically, however, the conformal hypothesis can be expressed in terms of formal geometry as the statement that a conformal vertex algebra is a Harish-Chandra module over the pair $\mathfrak{w}_1$ and $\text{Aut}(O_1)$, where $\mathfrak{w}_1$ is the fibre of the tangent sheaf of the torsor of formal coordinates $\text{Aut}_X$ associated to an algebraic curve of genus greater than or equal to 2 in [FBZ]. In their text, $\text{Aut}(O_1)$ is interpreted as the structure group of this torsor, which is to say, the group of automorphisms preserving the origin of the formal disc $D_x$ about a point $x$ on the curve $X$. It is crucial to notice that the Lie algebra of $\text{Aut}(O_1)$ is not $\mathfrak{w}_1$, but rather a pro-nilpotent subalgebra of the same. The reason for this discrepancy is that the exponential under such a correspondence of the $L_{-1}$ mode does not act locally nilpotently on a conformal vertex algebra $V$, and therefore fails to exponentiate to an automorphism of the disc or an element of $\text{Aut}(O_1)$. Harish-Chandra geometry, as mentioned above, is designed to address such discrepancies.

Frenkel and Ben Zvi’s work is important because, among other things, they are able to represent vertex operators $\mathcal{Y}_x$ as sections of the sheaf of modules $\mathcal{V}_X$ they obtain by the associated bundle construction applied to a Harish-Chandra module $V$ over $(\mathfrak{w}_1, \text{Aut}(O_1))$ as flat sections of the same. This is possible because formally a point on a curve is represented by a single parameter, as are the vertex operators $\mathcal{Y}(-, z)$.

The pro-finite vector bundle $\mathcal{V}_X$ that they obtain through this assignment is an example of a chiral algebra in the sense of Beilinson and Drinfeld, above, by demonstrating how the aforementioned flat sections can be interpreted as the structure maps defining a chiral algebra. Moreover, and importantly to the present work, modules $M$ over a conformal vertex algebra were geometrized in the above manner, owing to the transitive action of $\mathfrak{w}_1$ via the representation of $V$ upon $M$. Accordingly, both sheaves of vertex algebra modules and chiral algebra modules were obtained in their work.

The present work is concerned with an extension of their approach to geometrizing vertex algebras and their modules in the not-necessarily conformal setting. The attempt is motivated by the work of Antun Milas [Mil] wherein modules over a vertex algebra $V$ with a
non-semisimple action of the Virasoro algebra are both defined and examined. The modules in _loc.cit_ are referred to as _logarithmic_ for the non-semisimple action of the Virasoro algebra entails the presence of an additional formal variable, denoted \( \log(z) \) by the author, as it must satisfy properties analogous to the derivative of the logarithm. Such an extension would be necessary because logarithmic modules cannot be treated as Harish-Chandra modules over \((\mathfrak{m}_1, \text{Aut}(O_1))\) since by definition they are non-conformal in the above sense.

Initially, the topic of these thesis was to render logarithmic modules as geometric objects over a curve \( X \) by the outlined procedure, but the task proved to be too nettlesome. As an alternative, we have undertaken the labor of rendering a class of vertex algebras related to more straightforward notions in algebraic geometry, called vertex \( O \)-algebras below, whose representation theory is merely indecomposable or _weak_ in the sense of [Miy]. Logarithmic modules are in particular weak modules themselves, so geometrizing weak modules emerged as a worthy alternative to geometrizing logarithmic modules. Vertex \( O \)-algebras have particularly nice Zhu algebras in terms of a corresponding classical object. As the Zhu algebra controls a sizable portion of a vertex algebra’s representation theory in general, both identifying that representations of the Zhu algebra are in general indecomposable and geometrizing them is enough to satisfy the objectives of this thesis.

A vertex \( O \)-algebra is constructed from an affine scheme \( X \) together with a coherent \( O \)-module \( \mathcal{E} \) replete with structure morphisms satisfying certain properties emerging from geometry. The construction is inspired by Beilinson and Bernstein’s famous paper on the Jantzen conjectures [BB]. Therein the authors define the fibred category of Lie schemes whose objects are pairs consisting of a scheme \( X \) together with a sheaf of Lie algebroids \( \mathcal{L} \), referred to as Lie-Rinehart algebroids below. The globalization of the latter object is a Lie-Rinehart algebra [Rin]. It was observed in another famous paper of Liu, Weinstein, and Xu [LWX] that notion of a double of a Lie algebroid \( \mathcal{L} \oplus \mathcal{L}^\vee \) analogous to that of a double of a Lie algebra \( g \oplus g^\vee \) is no longer a Lie algebroid. So unlike the category of Lie algebras, the category of Lie algebroids is no longer closed under the operation of forming doubles.
Instead, it is a Courtant algebroid, or Courant-Dorfman algebroid below, denoted $\mathcal{E}$. These objects are somewhat unusual as their bracket no longer satisfies the Jacobi identity, but only "up to homotopy." Such structures are known in the literature as Leibniz algebras. Beside the weakened Jacobi identity, Courant-Dorfman algebras exhibit extra features which, when taken as operations on the direct sum of $\mathcal{O} \oplus \mathcal{E}$, satisfy the Borcherds identities of an $\mathbb{N}$-graded vertex algebra restricted to its weights 0 and 1 spaces.

Let us consider the global objects $R$ and $E$ corresponding to $\mathcal{O}$ and $\mathcal{E}$. The work of Li and Yamskulna [LiYam1] shows how a certain Lie algebra, called the loop algebra, may be associated to the sum $R \oplus E$ and how that the representation of the loop algebra induced from the trivial representation of a certain subalgebra on $\kappa$, an algebraically closed field, results in a vertex algebra $V$. Quotienting $V$ by a certain invariant vertex ideal $I$ yields the vertex $\mathcal{O}$-algebra, denoted $\mathfrak{V}_E$ discussed below. That this construction may be geometrized as an pro-finite $\mathcal{O}_X$-module $V_X(\mathcal{E})$ over $X$ by an extension of formal geometry is a main result of this thesis.

Our second main result is that representations of a vertex $\mathcal{O}$-algebra may be geometrized, as well. Given that such representations are in general indecomposable, one satisfies an objective consistent in spirit with the problem posed by the geometrization of logarithmic modules. The observation that representations of a vertex $\mathcal{O}$-algebra are indecomposable in general follows from the relationship between its Zhu algebra and the universal enveloping algebroid of $\mathcal{L}$ associated to the underlying Lie scheme $(X, \mathcal{L})$ [ACM]. The process also entails an extension of the techniques of Frenkel and Ben Zvi outlined above. In particular, one must formulate an extension of their construction general enough to treat $\mathfrak{V}_E$ as a Harish-Chandra module over a Harish-Chandra pair adopted to a torsor. Once these objects are revealed, the process of globalizing both the vertex $\mathcal{O}$-algebra and its modules is given by the localization procedure discovered by Beilinson and Bernstein and expounded upon in the generality of Harish-Chandra geometry by Kaledin and Bezrukavnikov [BK].

The resulting object $V_X(\mathcal{E})$ is not, however, a chiral algebra, as its analogue in [FBZ]
originally was; instead, the result can be described euphemistically as a pro-finite module whose global structure is that of a vertex algebra or a module over the same. Representing vertex operators as flat sections is no longer possible, as the dimension of the underlying space $X$ is in general no longer one. Thusly, there is no longer a correspondence between coordinates of the formal disc and formal variables used to describe vertex operators, as there is when $X$ is a curve. Sheaves of vertex algebras in this way are sometimes called chiral differential operators cf. [GMSII]. Perhaps such objects correspond to higher dimensional chiral algebras, although Beilinson and Drinfeld explicitly note that they do not explore higher dimensional chiral algebras. The reader interested in higher dimensional formulations of chiral algebras may wish to confer with [GaiFr]. Indeed, our construction is a sort of hybrid between the chiral differential operators of [GMSII] and the associated bundle construction of [FBZ]. The work [GMSII] emphasizes how one constructs an a sheaf of vertex algebras on the topological space underlying $X$ by examining the transition maps carefully, whereas the latter satisfies descent via the formal mechanism of the associated bundle construction cf. [SGA1]. The former reference allows for base spaces of arbitrary finite dimension, whereas the latter, as already noted, is presented only for curves. A similar construction appeared recently in [GGB].

Techniques to extend formal geometry have been discussed by several authors over the years. The work of Nest and Feigin [FN] is to the author’s knowledge the first to consider this question. Subsequent work of Nest and Tsygan [NT] applies such extensions to Fedosov quantization to produce algebraic index theorems. The work of Kleijn [Kl], continues in this direction, although a note published by Khoroshkin [Kh] was the first to write down their ideas rigorously. Indeed, formal geometry and its extensions are mostly related to deformation quantization problems specifically with respect to the topic of Fedosov quantization. The key work in this regard for this thesis however is the reference [BK], where the authors expound upon the foundations of the subject in the language of algebraic geometry. Their work, in turn, follows more or less directly from the monumental, though unpublished, work.
of Beilinson and Drinfeld [BD2] and ideas latent in another key source of inspiration [BFM] for many routes of inquiry in both formal geometry and the theory of vertex algebras [Ar1].

Let us outline the contents of this thesis. The point of view on formal geometry presented in this emerges from these works. Background and foundational material in this regard is covered in chapters one through three. A novel contribution here is by recognizing the role that transitive Lie algebras in the works of Guillemin and Sternberg [GS] have in the subject. Beside that difference, the material in chapters two and three relies upon the notion of Lie-Rinehart algebras for its exposition, and so we add to the subject by showing how certain concepts in formal geometry may be simplified in their presence. In particular, the construction of the jet scheme \( \mathcal{J}et_X \) and its relation to the torsor of formal coordinates \( \mathcal{C}or_X \).

Chapter four reviews the basics of vertex algebras and the concepts necessary to describe both vertex \( \mathcal{O} \)-algebras and some of their modules.

Chapter five is the main chapter for results. Therein we discuss the definitions of both Lie-Rinehart algebroids and Courant-Dorfman algebroids as specific types of coherent sheaves on \( X = \text{Spec}(R) \), an affine scheme. We introduce an \textit{ad hoc} definition of modules over the latter to avoid the technical difficulties of working in the homotopy setting of \( L_\infty \)-algebras. With this definition, we reprove the result of Li and Yamskulna to produce an \( \mathbb{N} \)-graded vertex algebra \( \mathcal{V} \) by associating to what we call the canonical representation of a Lie-Rinehart double, or equivalently, a Courant-Dorfman algebra, \( E \), the loop algebra. A quotient of this vertex algebra \( \mathcal{V}_E \) is how we define the vertex \( \mathcal{O} \)-algebra. Afterward, we review the result of [ACM] on its Zhu algebra, showing that it is the image of a surjective associative algebra morphism \( \alpha \) with domain the universal enveloping algebroid of the underlying Lie-Rinehart algebra \( L \). By demonstrating this, we obtain a relationship between representations of the Zhu algebra and the vertex \( \mathcal{O} \)-algebra. Finally, we present the extension \( \mathcal{V}or_X \to X \) of the torsor of formal coordinates \( \mathcal{C}or_X \to X \) adopted to an extension \( (\mathfrak{w}_E, \text{Aut}(E, O)) \) of Harish-Chandra pairs of a attached to a transitive Lie subalgebra, in the sense of [GS], of \( \mathfrak{w} \) in Theorem 15. This demonstration, together with a combination of results of Li, Yamskulna,
and Dong, [LiYam2], [Dong], exhibit our vertex $\mathcal{O}$-algebra as a Harish-Chandra module over the pair $(\mathfrak{m}_E, \text{Aut}(E, O))$. The geometrization in theorem 16 then follows from the localization procedure of Harish-Chandra geometry. The corollary is this works for modules, as well, which is theorem 17.
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1 Pro-Nilpotent Lie Algebras, Pro-Unipotent Algebraic Groups, and their Representations

In this chapter we review the definitions of pro-algebraic groups and pro-Lie algebras and their categorical correspondence when restricted to the full subcategories of pro-unipotent algebraic groups and pro-nilpotent Lie algebras. Furthermore, we review a specific topology that we intend to use below in the context of formal geometry to define a transitivity condition on Lie algebras peculiar to Lie algebras imbued with this topology. The transitivity condition is used later geometrically to lift certain structures related to Lie algebras that satisfy the condition, and in turn, to extend structures, as well. Hence clarifying the transitivity condition is quite important for the development later. The material on the linearly compact topology can be found in [Guil] and [VdB].

Our exposition of pro-Lie algebras and pro-algebraic groups closely follows [Kum2] chapter 4, but the insight that the generality of projective objects was natural relevant geometry problems with respect to vertex algebras is in [BFM], section 3. However, we do not make use of the of Tate vector spaces in loc.cit to handle representations of pro-objects. Instead, we consider Harish-Chandra modules we call locally nilpotent, and so we review local nilpotency in this chapter. This entire chapter is included to prepare the hypotheses of Harish-Chandra geometry in the generality of pro-objects, so for this reason, we shall refer to the entire ansatz as the pro-unipotent Harish-Chandra ansatz.

1.1 Pro-Algebraic Groups and Linearly Compact Algebraic Groups

We begin by reviewing algebraic groups and Lie algebras in the generality of projective limits. Let $K$ be a smooth algebraic group over $\kappa$, where $\kappa$ is an algebraically closed field for characteristic 0 for the remainder of this thesis, and $k \in K$ a geometric point thereof. Let $T_k K$ denote the fibre of the Zariski tangent space $T_K$ space at $k \in K$, and $l_k, r_k$ the left, resp. right, translation endomorphisms of $K$. Denote their derivatives [Wat] by $d(l_k)$, resp.
A vector field $\xi \in T_K$ is said to be left-invariant, resp. right-invariant if for all $k, x \in K$, we have

$$d(l_k)_x(\xi_x) = \xi_{l_k(x)}$$

$$d(r_k)_x(\xi_x) = \xi_{r_k(x)}$$

Observe, both left and right invariant vector fields form a Lie algebra loc.cit under the bracket

$$[\xi_1, \xi_2](f) = \xi_1(\xi_2(f)) - \xi_2(\xi_1(f))$$

for $f \in O_K$. It is well known that the $\kappa$-Lie algebras of left and right invariant vector fields are isomorphic. Moreover, the tangent space at the identity $e \in K$ intervenes in this isomorphism, so one may regard these Lie algebras as $T_e K$ cf. [HTT], [Hosch]. Consequently, we shall identify left and right invariant vector fields under this isomorphism.

**Definition 1.** Given a smooth algebraic $\kappa$-group $K$, we say the tangent space at the identity is the Lie algebra of invariant vector fields or is the Lie algebra of $K$ and denote this by $\text{Lie}(K)$.

Given a morphism of algebraic groups $\phi : K \to K'$, one has a corresponding morphism of Lie algebras, $\text{Lie}(\phi) : \text{Lie}(K) \to \text{Lie}(K')$, so Lie is a covariant functor from the category of algebraic groups to the category of Lie algebras. We adopt the notation $d\phi = \text{Lie}(\phi)$.

Having recalled the basic correspondence between algebraic groups and Lie algebras, we turn to the definition of projective limits of these objects. First, we define pro-algebraic groups cf. [Kum].

**Definition 2.** Let $K$ be an algebraic group and $\Lambda$ a non-empty inverse family of normal subgroups of $K$ such that, for every $N \in \Lambda$, the quotient group $K/N$ is an algebraic group. Then we say $K$ is a pro-algebraic group if the following axioms are satisfied.
(Axiom 1) If $N_1, N_2 \in \Lambda$, then $N_1 \times \kappa N_2 \in \Lambda$.

(Axiom 2) If $N_1 \in \Lambda$, then a normal subgroup $N_2 \triangleleft K$ containing $N_1$ belongs to $\Lambda$ whenever $N_2/N_1 \triangleleft K/N_1$.

(Axiom 3) If $N_1, N_2 \in \Lambda$ such that $N_2 \subset N_1$, then the quotient map

$$\pi_{2,1} : K/N_2 \rightarrow K/N_1$$

is a morphism of algebraic groups.

(Axiom 4) The natural homomorphism

$$\pi : K \rightarrow \lim_{N \in \Lambda} K/N$$

is bijective.

A morphism of pro-algebraic groups is defined as a morphism of pro-objects in the category of $\kappa$-schemes. More specifically, $\phi : K \rightarrow K'$ is a morphism of pro-groups if for each $N' \in \Lambda'$, we have $\phi^{-1}(N') \in \Lambda$ and the induced morphism $\phi_N : K/\phi^{-1}(N') \rightarrow K'/N'$ is a morphism of algebraic groups. Composition of morphisms of pro-groups is well-defined, hence these objects form a category.

We endow a pro-group $K$ with the inverse limit topology to render these objects as topological groups. A useful characterization of this topology is as the finest topology such that, for each $N \in \Lambda$ and each projection $\pi_N : K \rightarrow K/N$, the sets $\{\pi_N^{-1}(U)\}$ where $U \subset K/N$ is open, form a base of the inverse limit topology. It follows from this characterization that a morphism of pro-groups is continuous in this topology by the lattice isomorphisms for finite algebraic groups.

Pro-subgroups and quotients are well-defined, and a projective limit version of the first isomorphism theorem holds. We shall not use these, at least explicitly, so the interested reader is referred to the main reference for this section loc.cit.

More specifically, however, we will consider the category of pro-algebraic groups with the linearly compact topology. These are topological abelian groups whose topological base is
generated by a system of neighborhoods of zero induced by \( \Lambda \). In this topology, a subset \( N \subset K \) is open if and only if \( N \) is closed and the quotient \( K/N \) is discrete. Taking successive intersections of \( N \in \Lambda \) we have a descending filtration of \( K \) given by

\[
K = F_0K \supset F_1K \supset \ldots
\]

As the \( N \in \Lambda \) are open, the quotients \( K/F_kK \) are discrete, whence the projective limit

\[
K = \lim_k K/F_kK
\]

is complete. This description accords with the intuition one has that completions are Cauchy sequences by considering increasing \( k \) for \( F_kK \). Replacing bijection in axiom 4 in the first definition by homeomorphism means \( K \) is a \textit{linearly compact algebraic group}. We shall refer to these pro-algebraic groups as algebraic lc-groups.

### 1.2 Pro-Lie Algebras and Linearly Compact Lie Algebras

Let us now review pro-Lie algebras equipped with the linearly compact topology.

**Definition 3.** Let \( \mathfrak{k} \) be a \( \kappa \)-Lie algebra and \( \Lambda \) a non-empty inverse family of ideals of \( \mathfrak{k} \), such that, for every \( \mathfrak{a} \in \Lambda \), the quotient Lie algebra \( \mathfrak{k}/\mathfrak{a} \) is a finite dimensional \( \kappa \)-Lie algebra, that is, so each ideal \( \mathfrak{a} \) is of finite codimension. Then we say \( \mathfrak{k} \) is a pro-Lie algebra if the following axioms are satisfied:

(Axiom 1) If \( \mathfrak{a}_1, \mathfrak{a}_2 \in \Lambda \), then \( \mathfrak{a}_1 \cap \mathfrak{a}_2 \in \Lambda \).

(Axiom 2) If \( \mathfrak{a}_2 \in \Lambda \), then an ideal \( \mathfrak{a}_1 \subset \mathfrak{k} \) containing \( \mathfrak{a}_2 \) belongs to \( \Lambda \) whenever \( \mathfrak{a}_1/\mathfrak{a}_2 \subset \mathfrak{k}/\mathfrak{a}_2 \).

(Axiom 3) If \( \mathfrak{a}_1, \mathfrak{a}_2 \in \Lambda \) such that \( \mathfrak{a}_2 \subset \mathfrak{a}_1 \), then the quotient map

\[
\gamma_{2,1} : \mathfrak{k}/\mathfrak{a}_2 \to \mathfrak{k}/\mathfrak{a}_1
\]

is a morphism of Lie algebras.

(Axiom 4) The natural Lie algebra morphism
\[ \gamma : \mathfrak{k} \to \lim_{a \in \Lambda} \mathfrak{k}/a \]
is an isomorphism.

As with pro-algebraic groups, one calls a Lie algebra morphism \( \gamma : \mathfrak{k} \to \mathfrak{k}' \) a pro-Lie algebra morphism if for all \( a' \in \Lambda' \), one has \( \gamma^{-1}(a') \in \Lambda \). Again, a pro-Lie algebra morphism is continuous in the inverse limit topology. However, unlike the case with pro-algebraic groups, here the converse of this statement is also true: a continuous Lie algebra morphism is a pro-Lie algebra morphism, so one may characterize isomorphisms as bi-continuous maps. Pro-Lie subalgebras and pro-Lie ideals are Lie-subalgebras or ideals such that they are closed subspaces within the inverse limit topology.

One notes that axiom 4 is redundant, but we include it nonetheless to emphasize how this definition is parallel to the definition of pro-algebraic groups. The topological perspective we adopt on pro-Lie algebras is a matter of importance below for its relevance to a specific Lie algebra fundamental to formal geometry, so we shall expound upon this topic in some detail.

Let \( \kappa \) be a field with topological structure that of the discrete topology. Let \( V \) be a \( \kappa \)-vector space. One refers to subsets of the form \( v + W \) as affine sets, for \( v \in V \) and \( W \subset V \) a subvector space. In particular, for \( v = 0 \), we say such affine sets are a system of neighborhoods of 0. Topologize \( V \) as a vector space with the topology generated by a base of affine sets. Open subspaces \( W \) of such a topological vector space \( V \) are equivalent to closed subspaces such that the quotient space \( V/W \) is a discrete topological space. Given our terminology, we have the following definition.

**Definition 4.** Let \( V \) be a topological vector space whose topology is generated by a base of affine sets. Let \( A \) be an affine subset of \( V \). Then we say \( A \) is linearly compact if

(Axiom 1) \( \cap_{j \in J} A_j \neq 0 \) for every family \( \Lambda \) of closed affine subsets \( A_j \) indexed by some set \( J \). Equivalently, every such family \( \Lambda \) satisfies the finite intersection property.
In particular, if $V$ itself is linearly compact, then we say $V$ is a linearly compact vector space.

Familiar properties of compact spaces also hold for linearly compact vector spaces, for example, if \{\{V_j\}\} is a family of linearly compact vector spaces, then $\prod_{j \in J} V_j$ is a linearly compact vector space. We will topologize pro-Lie algebras by imparting the linearly compact topology to the underlying vector space. In order to do this, we first require the following proposition [G3].

**Proposition 1.** Let $V$ be a linearly compact vector space. Then the following are equivalent:

(I) $V$ can be represented as a projective limit of finite dimensional discrete spaces.

(II) $V$ is the topological dual of a discrete space.

(III) $V$ is the product of finite dimensional discrete spaces with the standard product topology.

Denote the category whose objects are linearly compact vector spaces and whose morphisms are vector space morphisms that are continuous in the linearly compact topology by $\mathcal{VL}$. We are interested in Lie objects in this category, so we have the next definition.

**Definition 5.** The category of linearly compact Lie algebras, denoted $\mathcal{LC}$, is the category whose objects are Lie algebras $\mathfrak{k}$ satisfying the following two axioms:

(Axiom 1) the underlying vector space of $\mathfrak{k} \in \mathcal{VL}$

(Axiom 2) the bracket operation is continuous in the linearly compact topology.

The morphisms in this category are continuous Lie algebra homomorphisms.

Pro-Lie algebras are adopted to this definition by taking $\mathfrak{k}/\mathfrak{a}$ for all $\mathfrak{a} \in \Lambda$ with the discrete topology, then by the proposition they are a subcategory of the category of linearly compact Lie algebras. Hereafter, we shall therefore replace the pre-fix "pro" by "lc" to indicate that
we are working with linearly compact Lie algebras predicated upon pro-Lie algebras in the sense of the first definition.

Next the goal is to define a transitive Lie lc-algebra. Our definition will depend upon a filtration of the underlying Lie algebra induced by a particular subalgebra. We shall obtain this filtration by an inductive construction of subspaces. Let $\mathfrak{g}$ be a topological Lie algebra whose topology is generated by affine subspaces and $\mathfrak{a} \subset \mathfrak{g}$ be a subspace thereof. Define

$$F^\theta(\mathfrak{a}) = \{\xi \in \mathfrak{a} | [\mathfrak{g}, \xi] \in \mathfrak{a}\}$$

Given the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, we have $F^\theta(\mathfrak{a})$ is a subalgebra of $\mathfrak{g}$. Moreover, $F^\theta(\mathfrak{a})$ is an ideal of $\mathfrak{a}$. One can then define a sequence of subspaces of $\mathfrak{g}$ inductively by

$$F^\theta_1(\mathfrak{a}) = F^\theta(\mathfrak{a})$$
$$F^\theta_k(\mathfrak{a}) = F^\theta(F^\theta_{k-1}(\mathfrak{a}))$$

for $k = 1, 2, \ldots$

Let us define a filtration for a topological Lie algebra whose topology is generated by affine subspaces from this sequence of subspaces.

**Definition 6.** Let $\mathfrak{g}$ be a topological Lie algebra and $F_{\bullet}\mathfrak{g} = \{F_k\mathfrak{g}\}$ be a collection of closed subspaces of $\mathfrak{g}$ for $k \in \mathbb{Z}$. Then we say $F_{\bullet}\mathfrak{g}$ is a filtration of $\mathfrak{g}$ if

(Axiom 1) $F_{k+1}\mathfrak{g} \subset F_k\mathfrak{g}$ for all $k$.

(Axiom 2) $\cup_{k \in \mathbb{Z}} F_k\mathfrak{g} = \mathfrak{g}$

(Axiom 3) $\cap_{k \in \mathbb{Z}} F_k\mathfrak{g} = 0$

(Axiom 4) $[F_k\mathfrak{g}, F_l\mathfrak{g}] \subset F_{k+l}\mathfrak{g}$ for all $k, l \in \mathbb{Z}$

Moreover, we say a filtration is open if $F_k\mathfrak{g}$ is open for all $k$.

We shall adopt the convention that for all open filtrations $F_k\mathfrak{g} = \mathfrak{g}$ for $k \leq -1$ and that $F_0\mathfrak{g} \neq \mathfrak{g}$. Now observe that our inductive construction of subspaces above does not in general
provide a filtration of a given topological Lie algebra. However, there is a certain class of open filtrations we have in mind predicated upon the inductive construction of subspaces with respect to a special type of open subalgebra.

We shall say that for a topological Lie algebra \( g \) an open subalgebra \( a \subset g \) is **fundamental** if \( a \) contains no ideals of \( g \) except 0. This means that there are no subspaces \( i \) of \( a \) such that \([g, i] \in i \) for all \( g \in g \). A filtration of \( g \) obtained inductively from a fundamental subalgebra is said to be **canonical**.

We shall confirm that a canonical filtration is indeed a filtration. Let \( a \subset g \) be a fundamental subalgebra. Define 
\[
F^{-1}g = g, \quad F_0g = a, \quad \text{and} \quad F_kg = F_k^g(a) \quad \text{for} \quad k > 0,
\]
above. One observes that since \( a \subset g \) is open, \( F_1g \subset a \) is open. Applying this argument inductively, one has that \( F_kg \) is open for all \( k \). Moreover, the intersection \( \cap_{k=1}^{\infty} F_kg \) is an ideal in \( a \), but since \( a \) is fundamental, we have \( \cap_{k=1}^{\infty} F_kg = 0 \). It is clear that \([F_kg, F_lg] \subset F_{k+l}g\). Therefore, the canonical filtration \( F_*g \) is a filtration of \( g \). One imparts a topology to \( g \) by taking the filtrants \( F_kg \) to be its neighborhoods of zero. This is enough to state our main definition in the category of Lie lc-algebras.

**Definition 7.** Let \( g \) be a topological Lie algebra canonically filtered by a fundamental subalgebra \( a \). Then we say \( a \) is transitive if \( a \) is of finite codimension in \( g \) and linearly compact.

Observe that if \( g \) contains a transitive fundamental subalgebra then it is both complete and separated topologically with respect to the topology generated by its canonical filtration. In this manner, \( g \) is a pro-object in the category of ind-Lie algebras \([BD2]\). The most well-known example of a transitive Lie algebra is that of the origin preserving derivations of the Lie algebra of derivations of the ring of formal power series in several variables. Many other non-trivial examples of transitive algebras are obtained from this primary example cf. \([BK]\), \([G2]\), \([G3]\), \([GS]\), \([SS]\).

We conclude with a useful description of the completion of a Lie lc-algebra canonically filtered by a transitive subalgebra and its associated graded object. A Lie algebra \( \mathfrak{g} \) is graded if
where \( \dim_\kappa \mathfrak{g}_p < \infty \) and \([\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}\). There is a functor assigning graded Lie algebras to lc-Lie algebras we call \textit{lc-completion} which assigns such \( \mathfrak{g} \) to \( g \in \mathcal{LC} \) as follows.

Compute the product \( \hat{\mathfrak{g}} = \prod_{p \geq -i} \mathfrak{g}_p \) over the subspaces \( \mathfrak{g}_p \). One topologizes this by imparting the discrete topology to each \( \mathfrak{g}_p \) and the standard product topology to \( \hat{\mathfrak{g}} \) overall. Given these prescriptions, \( \hat{\mathfrak{g}} = g \in \mathcal{LC} \). Observe that \( \mathfrak{g} \to g \) embeds as a dense subset of homogeneous elements in the linearly compact topology.

Here is the quasi-inverse to this functor with respect to an lc-Lie algebra canonically filtered by a fundamental subalgebra. Given the canonical filtration, then \( g/F_k(g) \) is closed and of finite codimension in \( g \). Moreover, by construction, \([F_k g, F_l g] \subset F_{k+l} g\). Therefore, the associated graded object \( \mathfrak{g} \) is given by

\[
gr(g) = \bigoplus_{k \geq -i} F_k / F_{k+1} g
\]

We shall be most interested in the relationship this construction furnishes between the Lie algebra of derivations of the ring of formal power series in \( n \) variables and the fibre of the tangent sheaf of a smooth \( \kappa \)-scheme of dimension \( n \).

### 1.3 Correspondence between Pro-Algebraic Groups and LC-Lie algebras

Now we are able to introduce both the key definitions and concepts in the pro-unipotent Harish Chandra ansatz.

**Definition 8.** Let \( K \) be an algebraic lc-group with inverse family of normal subgroups \( \Lambda \) and for any \( N \in \Lambda \), let

\[
f_N = \text{Lie}(K/N)
\]

Furthermore, let \( d(\pi_{2,1}) = \gamma_{2,1} : f_{N_2} \to f_{N_1} \), where \( \pi_{2,1} : K/N_2 \to K/N_1 \) and \( N_2 \triangleleft N_1 \). Denote the corresponding projective limit of Lie algebras by
Then we say $\mathfrak{k}$ is the lc-Lie algebra of $K$, denoted $\mathfrak{k} = \text{Lie}(K)$.

The definition is natural in algebraic lc-groups, so Lie is a covariant functor from the category of algebraic lc-groups to the category of lc-Lie algebras. Again, given a morphism of algebraic lc-groups $\phi$, we shall denote its image under this functor by $d\phi$.

Conversely, we may consider a functor Exp from the category of lc-Lie algebras to the category of algebraic lc-groups. Abusing language, what we present is actually not a functor, but the theory of finite dimensional Lie algebras and Lie’s theorem will indicate for us below when it is. We shall naively denote by Exp of some Lie algebra a unipotent algebraic group corresponding to it. The reason for this is in general an algebraic group will not correspond to a Lie algebra, however, we reserve this notation for later when it fact one does.

**Definition 9.** Let $\mathfrak{k}$ be a pro-Lie algebra with inverse family of ideals $\Lambda$ and for any $a \in \Lambda$, let

$$K_a = \text{Exp}(\mathfrak{k}/a)$$

Furthermore, for any $a_2 \subset a_1 \in \Lambda$, let $\text{Exp}(\gamma_{2,1}) = \pi_{2,1} : K_{a_2} \to K_{a_1}$, where $\gamma_{2,1} : \mathfrak{k}/a_2 \to \mathfrak{k}/a_1$ and $a_2 \subset a_1$. Denote the corresponding projective limit of algebraic groups by

$$K = \lim_{a \in \Lambda} K_a$$

Then we say $K$ is the pro-algebraic group of $\mathfrak{k}$, and denote this by $K = \text{Exp}(\mathfrak{k})$ for a pro-Lie algebra $\mathfrak{k}$.

Of course this definition does make sense for an arbitrary Lie algebra for there exist examples of finite dimensional Lie algebras such that Lie’s third theorem can fail. However, it is well known that for a finite dimensional nilpotent Lie algebra $\mathfrak{n}$, the Baker-Campbell-Hausdorff formula imparts a unipotent group structure on the underlying set of this Lie algebra, denoted $\text{Exp}(\mathfrak{n})$. Moreover, the Baker-Campbell-Hausdorff formula is natural, so
that morphisms of finite dimensional nilpotent Lie algebras covariantly correspond to morphisms of unipotent groups of finite type. Hence, we have the categorical equivalence of finite dimensional nilpotent $\kappa$-Lie algebras and unipotent groups of finite type over $\kappa$. We need the following definition to state the main theorem.

**Definition 10.** We say an algebraic lc-group $K$, resp. an lc-Lie algebra $\mathfrak{k}$, is pro-unipotent (resp. nilpotent), if $K/N$, (resp. $\mathfrak{k}/a$) is unipotent (resp. nilpotent) for all $N \in \Lambda$ (resp. for all $a \in \Lambda$).

The following theorem is the corollary of the categorical equivalence between finite dimensional nilpotent Lie algebras and unipotent algebraic groups of finite type over $\kappa$ in the generality of pro-objects.

**Theorem 1.** The category of pro-unipotent algebraic groups over $\kappa$ is equivalent to the category of nilpotent Lie lc-algebras over $\kappa$ under $\text{Lie}$, with quasi-inverse $\text{Exp}$.

**Proof.** (Sketch)

Given a pro-nilpotent Lie lc-algebra $\mathfrak{k}$, one has the unipotent group $\text{Exp}(\mathfrak{k}/a) = K_a$ for any $a \in \Lambda$. Further, by definition, $\mathfrak{a}_1 \subset \mathfrak{a}_2$ corresponds to a morphism of unipotent groups $K_{\mathfrak{a}_1} \rightarrow K_{\mathfrak{a}_2}$ by covariance, hence a projective system of unipotent groups. Clearly, one takes $K = \lim_{a \in \Lambda} K_a$ to be the corresponding algebraic lc-group. Moreover, one can show $\text{Lie}(K) = \mathfrak{k}$.

As sets, $\mathfrak{k}$ and $K$ are equal, so that, given a morphism of pro-nilpotent Lie lc-algebras, say $\mathfrak{k} \rightarrow \mathfrak{k}'$, one has a corresponding morphism of algebraic lc-unipotent groups $K \rightarrow K'$ by the Baker-Campbell-Hausdorff formula.

The last item to check is that $\text{Exp}$ and $\text{Lie}$ are quasi-inverses.

The theorem is proved in detail [Kum2], but the proof there ultimately depends on results in [DMG]. It is a main result used for the pro-unipotent Harish-Chandra ansatz of Harish-Chandra geometry later, because it expresses an aspect of the familiar correspondence between Lie algebras and algebraic groups in the generality necessary for dealing with infinite
dimensional Lie algebras. In broad outline, we are interested in transforming representations of Lie algebras into objects of geometry by assigning them to quasi-coherent sheaves over some space. Our approach to this assignment is by descent of the representation along a torsor under an algebraic group corresponding to the Lie algebra via the associated bundle construction. However, the representations of Lie algebras we consider lack corresponding algebraic groups, so to impart geometric quality to their representations, we will rely upon pro-nilpotent subalgebras and the pro-algebraic groups corresponding to the same by the above theorem. In this manner, we are able to recover descent of representations of Lie algebras lacking a corresponding algebraic group in a manner analogous to the previous outline.

1.4 Representations of Pro-Algebraic Groups and Linearly Compact Lie Algebras

Last, before discussing the basics of Harish-Chandra geometry, we discuss the relationship between representations $V$ of a unipotent algebraic lc-group $K$ and its corresponding nilpotent Lie lc-algebra $\mathfrak{k}$. The relationship between these representations is an extension of the relation to the generality of pro-objects.

**Definition 11.** Let $K$ be a pro-algebraic group, then we say $V$ is a pro-module of $K$ (Axiom I) if every $v \in V$ is contained in a finite dimensional $K$-submodule $W$ such that there exists $N \in \Lambda$ acting trivially on $W$ and the representation of $K/N$ on $W$ is algebraic.

Further, let $\mathfrak{k}$ be a Lie lc-algebra, then we say $V$ is a pro-module of $\mathfrak{k}$ (Axiom I)$'$ if every $v \in V$ is contained in a finite dimensional $\mathfrak{k}$-submodule $W$ such that there exists $a \in \Lambda$ acting trivially on $W$.

A morphism of pro-modules of $K$ is a linear map commuting with the $K$-actions. The same remark applies to morphisms of pro-modules of $\mathfrak{k}$. Given a unipotent algebraic lc-group $K$ and a pro-module $V$ thereof, there is a canonical pro-module structure on $V$ over
the nilpotent Lie lc-algebra \( \mathfrak{k} = \text{Lie}(K) \) given by differentiating the action \( \alpha : K \to \text{GL}(V) \) to that of its corresponding Lie lc-algebra i.e. \( d\alpha : \mathfrak{k} \to \mathfrak{gl}(V) \). Accordingly, one has the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{k}_N & \xrightarrow{d\alpha_N} & \text{End}_{\text{loc.fin}}(V) \\
\downarrow\text{Exp} & & \downarrow\exp \\
K/N & \xrightarrow{d\alpha_N} & \text{GL}(V)
\end{array}
\]

where \( \text{End}_{\text{loc.fin}}(V) \) is the set of \textit{locally finite} endomorphisms of \( V \); that is, there exists a finite dimensional \( K \)-invariant submodule of \( V \) containing \( v \) for all \( v \in V \).

**Definition 12.** Let \( K \) be a pro-algebraic group \( \mathfrak{k} \) its corresponding pro-Lie algebra. For every \( N \in \Lambda \), let \( \text{Ad}_N : K/N \to \mathfrak{gl}(\mathfrak{k}_N) \) be the adjoint representation and \( \text{Ad}_\pi = \text{Ad}_N \circ \pi_N \). We define the inverse limit of the maps \( \text{Ad}_{\pi_i} \) to be

\[
\text{Ad} : K \to \mathfrak{gl}(\mathfrak{k})
\]

the adjoint representation of \( K \) on \( \mathfrak{k} \).

Equivalently, for \( k \in K \), one can define \( \text{Ad} \) to be the derivative of the conjugation morphism associated to \( k \). One has \( \text{Exp}(\text{Ad}(k)\xi) = \pi(k)d\pi(\xi)\pi(k^{-1}) \), for any \( k \in K \) and \( \xi \in \mathfrak{k} \). Moreover, for any pro-module \( V \) with action \( \alpha \) one has

\[
d\alpha(\text{Ad}(k)(\xi)) = \alpha(k)d\alpha(\xi)\alpha(k^{-1})
\]

In general, \( \text{Ad} \) is not a pro-module of \( K \), but the following theorem justifies our use of pro-unipotent groups in our ansatz. Recall first a Lie module \( V \) over \( \mathfrak{k} \) is said to be \textit{locally nilpotent} if there exists \( m \) such that \( \xi^m \cdot v = 0 \) for some \( m \in \mathbb{N} \) and any \( \xi \in \mathfrak{k}, v \in V \). The following theorem is in [Kum2]

**Theorem 2.** Let \( K \) be a unipotent lc-group with nilpotent Lie lc-algebra \( \mathfrak{k} \), then the category of pro-modules of \( K \) is equivalent to the category of pro-modules of \( \mathfrak{k} \) such that every \( \xi \in \mathfrak{k} \) acts locally nilpotently. The assignment
\[(V, \alpha) \mapsto (V, d\alpha)\]

\[\phi \mapsto \phi\]

where \(\phi : V \to V'\) is a morphism of pro-modules of \(K\), furnishes this equivalence.

The point we emphasize later is the quasi-inverse of this equivalence. Given a pro-module \(V\) over a nilpotent Lie lc-algebra \(\mathfrak{k}\) with locally nilpotent action \(\rho : \mathfrak{k} \to \mathfrak{gl}(V)\), we say \(\text{Exp}(\rho)\) exponentiates the action of \(\mathfrak{k}\) to locally finite action of the corresponding unipotent lc-group \(K = \text{Exp}(\mathfrak{k})\). In particular, we have \((V, \rho) \mapsto (V, \text{Exp}(\rho))\) is the quasi-inverse of the above equivalence.

In conclusion, hereafter we shall omit the pre-fix "lc" and work exclusively under the ansatz in this chapter. Representations of pro-objects are assumed to be locally nilpotent or locally finite, according to the kind of pro-object supporting the representation.
2 Harish-Chandra Geometry

In this chapter we review the elements of Harish-Chandra geometry relevant to the geometric interpretation of certain vertex algebras we have in mind. The cornerstone of Harish-Chandra geometry is the concept of a \((g,K)\)-structure, \(S\). A \((g,K)\)-structure is a both a principal \(g\)-space for a Lie algebra \(g\) and a torsor for a (pro) algebraic group \(K\), where \(g\) and \(K\) comprise a Harish-Chandra pair. A Harish-Chandra pair, on the other hand, is the fundamental concept from the representation theoretic perspective. A Harish-Chandra pair consists of both a Lie algebra \(g\) and (pro) algebraic group \(K\) together with a compatibility condition that generalizes the relationship between a Lie algebra and its Lie group given by Lie’s theorem. The compatibility condition carries over to their representations. Geometrically, the compatibility condition imparts a connection to the sheaf of \(O_X\)-modules associated to the representation of the algebra group by the associated bundle construction. Thus, by the Riemann-Hilbert correspondence, bundles associated to representations of Harish-Chandra pairs correspond to \(\mathcal{D}\)-modules on the base of a \((g,K)\)-structure. Interesting examples of this situation arise from the theory of vertex algebras that are Harish-Chandra modules.

In this chapter, we shall present Harish-Chandra geometry in the generality of pro-objects; specifically, where the algebraic group of a Harish-Chandra pair and its corresponding Lie algebra are pro-unipotent and pro-nilpotent, respectively. Moreover, we specify a class of pro-modules for these objects that encompass the examples of vertex algebras we have in mind. The ansatz we present is sufficiently general to be specialized to the the framework of formal geometry in chapter three and to include less standard features such as extensions in that framework.

2.1 Harish-Chandra Pairs and Harish-Chandra Modules

As stated in the introduction, Harish-Chandra pairs consist of both a Lie algebra \(g\) and an algebraic group \(K\) that satisfy a compatibility condition. Harish-Chandra modules are,
grosso modo, representations of both the Lie algebra $\mathfrak{g}$ and an algebraic group $K$ together with a compatibility of the two module structures. Such modules often arise in the theory of infinite dimensional Lie algebras for it is well known that Lie’s so-called third theorem can fail for arbitrary Lie algebras; consequently, the compatibility of both representations generalizes the correspondence between representations of a Lie algebra and its Lie group when Lie’s third theorem does not hold.

Non-trivial examples of vertex algebras are constructed from representations of infinite dimensional Lie algebras which do not exponentiate to an algebraic group, so they are an excellent source of both motivation for and examples of Harish-Chandra modules. Therefore, Harish-Chandra geometry provides an inroad whereby vertex algebras may be studied geometrically qua Harish-Chandra modules.

**Definition 13.** Let $(\mathfrak{g}, K)$ be a pair consisting of a Lie algebra $\mathfrak{g}$ and a pro-unipotent group $K$, together with an action $\alpha : K \to \text{GL}(\mathfrak{g})$ of $K$ on $\mathfrak{g}$. Furthermore, let $\text{Ad} : K \to \mathfrak{gl}(\mathfrak{k})$ be the adjoint action of $K$ on $\mathfrak{k} = \text{Lie}(K)$. Then we say $(\mathfrak{g}, K)$ is a Harish-Chandra pair if

(Axiom I) there exists an embedding $\iota$ of Lie algebras,

$$\iota : \mathfrak{k} \to \mathfrak{g}$$

(Axiom II) the derivative of the action $\alpha$ is equal to the induced adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$, that is,

$$d\alpha = \text{Ad} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{g})$$

Harish-Chandra pairs form a category, whose morphisms are pairs themselves $\Phi = (\phi, \phi) : (\mathfrak{g}', K') \to (\mathfrak{g}, K)$ of both a homomorphism of Lie algebras $\phi : \mathfrak{g}' \to \mathfrak{g}$ and a morphism of pro-algebraic groups $\phi : K' \to K$ such that the following diagram commutes:
Next, we recall the definition of a representation over a Harish-Chandra pair \((g, K)\).

**Definition 14.** Let \((g, K)\) be a Harish-Chandra pair. Further, let \(V\) be both a pro-module for \(K\) and a module over \(g\), that is, there exist structure preserving morphisms

\[
\alpha : K \to \text{Aut}(V)
\]

and

\[
\rho : g \to \text{gl}(V)
\]

then we say \(V\) is a \((g, K)\)-module if

(Axiom 1) the following diagram commutes:

\[
\begin{array}{ccc}
\mathfrak{t} & \xrightarrow{d\alpha} & \text{gl}(V) \\
\downarrow{\iota} & & \downarrow{\rho} \\
g & & \\
\end{array}
\]

Explicitly, the axiom requires that the derivative of the action \(\alpha\) is equal to the composition of the Harish-Chandra embedding and the representation of \(g\) on \(V\),

\[d\alpha = \rho \circ \iota.
\]

Morphisms of representations over the same Harish-Chandra pair \((g, K)\) are defined in the obvious way. We denote the category of \((g, K)\)-modules by \(\mathcal{M}((g, K))\). Notice that \(\text{Aut}(V)\) is the group of \(\kappa\)-linear automorphisms of \(V\), as \(\text{GL}(V)\) is not in general well-defined for a pro-module \(V\). We shall address this peculiarity more carefully below.

### 2.2 Harish-Chandra \((g, K)\)-Structures

Notice a \((g, K)\)-module \(V\) is, in part, a representation of the underlying pro-algebraic group \(K\). Hence, one might be inclined to compute an \(\mathcal{O}_X\)-module associated to \(V\) via the associated bundle construction for some torsor \(S\) under \(K\). Such an inclination is only partially complete, as it does not incorporate the feature the Lie algebra representation contributes. Nonetheless, insofar as this impression is correct, it is therefore helpful to recall the definition of \(K\)-torsors.
Definition 15. Let $S$ be a $\kappa$-scheme of arbitrary type, then a fibre space with fibre $F$ over $X$, is a quadruple

$$\langle F, S, \pi, X \rangle$$

where $X$ is a $\kappa$-scheme of finite type and $\pi : S \to X$ is a morphism of $\kappa$-schemes such that

(Axiom I) for every point $x \in X$, there exists an open neighborhood $U$ of $x$, together with an etale covering $\pi^{-1}(U) \to U$

(Axiom II) an isomorphism $\phi$ over $\pi^{-1}(U)$ of the spaces pictured below inducing the commutativity of the attendant triangle:

\[
\begin{array}{ccc}
S \times_X \pi^{-1}(U) & \xrightarrow{\phi} & \pi^{-1}(U) \times F \\
\downarrow \pi \times \text{Id}_{\pi^{-1}(U)} & & \downarrow \pi_{\pi^{-1}(U)} \\
\pi^{-1}(U) & & 
\end{array}
\]

To incorporate an algebraic group into the structure of a fibre space over a scheme $X$, we have the following definition.

Definition 16. Let $K$ be a pro-algebraic group. A fibred space, with fibre $F$ and structure group $K$ over $X$ is the sextuple

$$\langle K, \alpha, F, S, \pi, X \rangle$$

where $\alpha : F \times K \to F$ is an algebraic action of $K$ upon the fibre $F$, $\alpha_S : S \times K \to S$ is a right action of $K$ upon $S$, and $\langle F, S, \pi, X \rangle$ is a fibre space with fibre $F$ such that

(Axiom I) $\phi$ is equivariant with respect to the action of $K$.

Moreover, in the case where

(Axiom II) the action of $K$ is simply transitive along the fibres, that is, for every $f, f' \in F$, there exists a unique $k \in K$ such that
we say \( \langle K, \alpha, F, S, \pi, X \rangle \) is a principal \( K \)-bundle. In the case where \( F \cong K \) and \( \alpha = \mu \) is the group multiplication, we say that \( \langle K, \mu, F, S, \pi, X \rangle \) is a \( K \)-torsor, denoted \( \pi : S \to X \).

Diagrammatically, the conditions the data of a principal \( K \)-bundle must satisfy amount to the commutativity of the following diagram:

\[
\begin{array}{ccc}
S \times_X \pi^{-1}(U) \times K & \xrightarrow{\phi \times Id_K} & \pi^{-1}(U) \times F \times K \\
\downarrow \alpha_S \times \pi_{\pi^{-1}(U)} & & \downarrow \text{Id}_W \times \alpha \\
S \times_X \pi^{-1}(U) & \xrightarrow{\phi} & \pi^{-1}(U) \times F
\end{array}
\]

In the special case of \( K \)-torsors, which are our primary objects of interest, Axiom II is often referred to as the local triviality of the a \( K \)-torsor. Notice that a consequence of our definition is that the map

\[
S \times_X S \to K \times_X S
\]

is a bijection. Whence, when restricted to the fibre of \( \pi \) over \( x \in X \), we have local triviality, for \( \pi^{-1}(U) \times F \cong \pi^{-1}(U) \times K \). Therefore, a good intuition one could have for what a \( K \)-torsor is is that it identifies the projection \( \pi_{\pi^{-1}(U)} \) in the diagram

\[
\begin{array}{ccc}
S \times_X \pi^{-1}(U) & \xrightarrow{\phi} & \pi^{-1}(U) \times K \\
\downarrow & & \downarrow \\
\pi^{-1}(U) & & \pi^{-1}(U)
\end{array}
\]

with the structure morphism \( \pi : S \to X \).

Morphisms of \( K \)-torsors are defined in the obvious way as \( K \)-equivariant morphisms of \( X \)-schemes. Moreover, \( K \)-torsors pull back under morphisms of \( \kappa \)-schemes. Accordingly, \( K \)-torsors form a category; indeed, the category of \( K \)-torsors is a Cartesian category over the
category of $\kappa$-schemes with the obvious projection. Torsors are useful objects in geometric representation theory because given a representation of the structure group, say $V$, one obtains a geometric vector bundle with typical fibre $V$ according to the following well known theorem. Equivalently, one also obtains a quasi-coherent sheaf on the base space $X$ of a torsor $S$ under $K$. We shall now review this equivalence in greater detail.

First, we shall recall what we mean by a vector bundle on a $\kappa$-scheme $X$.

**Definition 17.** A coherent sheaf $\mathcal{V}$ on $X$ consists of the following data. Let $\iota_i : U_i \rightarrow X$ be an open set of $X$.

(Axiom I) For every commutative diagram

\[
\begin{array}{ccc}
U_i & \xrightarrow{j} & U_j \\
\downarrow{\iota_i} & & \downarrow{\iota_j} \\
X & & \\
\end{array}
\]

we have an isomorphism $\iota_i^*\mathcal{V} \rightarrow j^*\iota_j^*\mathcal{V}$.

(Axiom II) For every triple

\[
\begin{array}{ccc}
U_i & \xrightarrow{j} & U_j \\
\downarrow{\iota_i} & \downarrow{\iota_j} & \downarrow{\iota_k} \\
X & & \\
\end{array}
\]

the diagram
(Axiom III) We say $\mathcal{V}$ is a vector bundle if $\iota_i^* \mathcal{V}$ is a locally free $\mathcal{O}_X(U)$-module for each $\iota_i : U \to X$.

An example of a vector bundle in this sense is the tangent sheaf, $T_X$. Moreover, its $\mathcal{O}_X$-module dual $\Omega^1_X := T_X^\vee$ is also a vector bundle with our hypotheses on $X$ below. We say a morphism of vector bundles $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X$ is a connection. Equivalently, by adjunction, a morphism $\nabla : T_X \to \text{Hom}_{\mathcal{O}_X}(T_X, \mathcal{V})$ is a connection 1-form.

We can obtain a vector bundle from a pro-module of $K$ as follows. Let $W$ be a finite dimensional $K$-submodule such that there exists $N \in \Lambda$ acting trivially on $W$ and the representation of $K/N$ on $W$ is algebraic, that is, there is an action $\alpha : K/N \to \text{Aut}(W)$ which is a morphism of algebraic groups. We shall construction from these hypotheses an vector bundle on $X$, called the associated bundle.

Let $\pi : S \to X$ be a $K$-torsor and consider the set $S \times W$. Observe that $K$-acts on this set diagonally by $k(s, w) = (sk, k^{-1}w)$. It is well-known the quotient of $S \times V$ by this action, denoted $S \times_K V$, is geometric [DMG]. We prove the following.

**Theorem 3.** There is an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{L}_W(U) \to \Gamma(U, S \times_K W)$$

from the set of $K$-equivariant maps $f : \pi^{-1}(U) \to W$ to the set of sections of the geometric quotient $S \times_K W$.

**Proof.** Let $U \to X$ be an open set. Given $f \in \mathcal{L}_W(U)$, we have a map to $S \times_K W$ furnished by the assignment of $s \in \pi^{-1}(U)$ to the equivalence class $[f(s), w]$ that clearly factors through $U$. 
by $K$-equivariance. Let $\Phi_U(f) : U \to S \times_K W$ denote this map. The $\Phi_U(f) \in \Gamma(U, S \times_K W)$ is a section.

Conversely, let $\mu : (S \times_K W) \times X \pi^{-1}(U) \to W \times \pi^{-1}(U)$ be the assignment of $([s,w], s') \mapsto (s'w, s)$ by the action of $K$ on $W$. Observe that $K$ acts on such pairs, hence $\mu$ is $K$-equivariant. Therefore, we have a map

$$(S \times_K W) \times_X \pi^{-1}(U) \to W \times \pi^{-1}(U) \rightarrow W$$

where the right arrow is given by projection on the first factor, denoted $p_1$. Thus, $p_1 \circ \mu$ is $K$-equivariant.

Now, suppose $\sigma \in \Gamma(U, S \times_K W)$, then this gives us a map

$$U \to S \times_K W \to W$$

by composing $\sigma$ with $p_1 \circ \mu$. Define $\Psi_U(\sigma) = p_1 \circ (\sigma \circ \pi \circ \alpha)$, where $\alpha$ is the action of $K$ on $S$ in our hypothesis. Then, since $p_1 \circ \mu$ is $K$-equivariant, so is $\Psi_U(\sigma)$. One checks $\Phi$ and $\Psi$ are mutual inverses.

Last, observe that, with the diagonal action of $K$ on $S \times W$, one has $\mathcal{L}_W(U) = (\pi_* \mathcal{O}_S \otimes W)^K$. According to the fact $\pi : S \to X$ is faithfully flat and that $W$ is finite dimensional, one therefore has that $\mathcal{L}_W := W_X$ is a vector bundle, as desired.

We say that $W_X$ in the theorem is the associated bundle. There is a corresponding theorem for geometric vector bundles, $X$-schemes represented by $\text{Spec}(\text{Sym})(W)$.

**Theorem 4.** Let $\pi : S \to X$ be a $K$-torsor under a pro-algebraic group $K$ and $W$ a finite dimensional module for $K$. Moreover, let

$$r : (S \times W^\vee) \times K \to S \times W^\vee$$

be the right action given by $r((s,w),k) = (\alpha_k(s), \alpha(k^{-1})(w))$, then the following statements are true.

I: The quotient $S \times_K W^\vee = (S \times W^\vee)/K$ given by the set of all orbits of $r$ has a unique smooth $\kappa$-scheme structure such that the projection
\[ S \times W^\vee \to S \times_K W^\vee \]

is a surjective submersion. Denote the equivalence class of \((s, w)\) under this projection by \([s, w]\).

II: The morphism \(\nu : S \times_K W^\vee \to X\) defined by \(\nu([s, w]) = \pi(s)\) defines a geometric vector bundle with typical fibre \(W\), called the associated vector bundle.

The theorem is proven in [CPS], although the primary reference for both the proof of the theorem, torsors, and the associated vector bundle is [DMG].

Proof. (Sketch) Let \(W\) be a finite dimensional representation of \(K\), that is, a comodule for the coordinate ring of \(K\). Define a sheaf on \(X\) by the assignment of \(U \subset X\) to the set of \(K\)-invariant morphisms from the pre-image of \(U \subset X\) under \(\pi\) to \(V\), viz.

\[ \pi^{-1}(U) \to W \]

and denote this sheaf by \(W_X\). Equivalently, \(W_X(U) = (\mathcal{O}_S(\pi^{-1}(U)) \otimes V)^K\), exhibiting \(W_X\) as an \(\mathcal{O}_X\)-module. By adjunction one has that \(W_X = (\pi_* \mathcal{O}_S \otimes W)^K \subset \pi_*(\mathcal{O}_S \otimes W)\) so that \(\pi^*W_X\) is isomorphic to \(\Gamma(S, \mathcal{O}_S \otimes W)\), the global sections of the trivial bundle on \(S\).

The inclusion in the pushforward of \(W_X\) corresponds to a morphism of \(S \times_X W^\vee \to \text{Spec}(\text{Sym}(W^\vee))\) [Har] which factors to an isomorphism

\[ S \times_K W^\vee \to \text{Spec}(\text{Sym}(W^\vee)) \]

Since the morphism \(\nu : S \times_X W^\vee \to X\) is affine by construction, the inductive limits of such morphisms satisfy descent. Consequently, one may consider formal schemes obtained from directed systems of such affine \(X\)-schemes. Furthermore, we regard the \(\mathcal{O}_X\)-module \(W_X\) or associated bundle as the object of \(\text{Coh}(X)\) geometrizing the representation \(W\) of \(K\), in the sense discussed in the introduction. Hereafter, when referring to the associated bundle construction, we are referring to the coherent \(\mathcal{O}_X\)-module \(W_X\), or projective limits of the same.
Turning our attention now to the matter of "pro-coherent" modules, intuitively, one has 
the vector bundle associated to a pro-module for a pro-algebraic group \( K \) should correspond 
to a projective limit of such finite dimensional associated bundles. The categorical matters 
underlying this are somewhat delicate, and they are addressed in the literature by several 
authors. Deligne \([\text{HaRD}]\) shows that \( \text{QCoh}(X) \) is equivalent to \( \text{Ind}(\text{Coh})(X) \), that is, any 
quasi-coherent \( \mathcal{O}_X \)-module is an inductive limit of coherent \( \mathcal{O}_X \)-modules. In \([\text{SGA III}]\), the 
category of pro-coherent \( \mathcal{O}_X \)-modules, denoted \( \text{Pro}(\text{Coh})(X) \) is defined to be \( \text{Ind}^\circ(\text{Coh}^\circ)(X) \), 
where \( \mathcal{C}^\circ \) is the opposite category of a category \( \mathcal{C} \). Moreover, in \( \text{loc.cit} \) it is shown that \( \otimes_{\mathcal{O}_X} \) 
extends to \( \text{Pro}(\text{Coh})(X) \), so it is a symmetric monoidal category, and in \([\text{AM}]\), they show 
if \( \mathcal{C} \) is abelian, \( \text{Pro}(\mathcal{C}) \) is abelian. It is well known \( \text{Coh}(X) \) is abelian. Hence, \( \text{Pro}(\text{Coh})(X) \) 
is a symmetric monoidal abelian category. We ultimately shall regard the vector bundle 
associated to a pro-module for \( K \) to be an object of this category, and regard it as an infinite 
dimensional vector bundle over \( X \) under these auspices. To simplify the matter of computing 
projective limits and to bring our examples in line with the intuition, we now include an 
additional hypotheses on representations \( V \) of \( K \) below.

Suppose \( V \) is a \( \kappa \)-module together with an increasing filtration \( 0 \subset V^0 \subset V^1 \subset \ldots \) so 
that \( V = \bigcup_{n \geq 0} V^n \). Consider the Lie subalgebra of \( \text{End}(V) \)
\[ \mathfrak{L} = \{ f \in \text{End}(V) : f(V^n) \subset V^{n-1}, n \geq 0 \} \]
The Lie subalgebra \( \mathfrak{L} \) is a pro-nilpotent Lie algebra which generates the Lie algebra of 
locally nilpotent operators on \( V \). The set of ideals \( a \in \Lambda \) in \( \mathfrak{L} \) are given by 
\[ a_n = \{ f \in \mathfrak{L} : f|_{V^n} = 0, n \in \mathbb{N} \} \]
Observe, the corresponding pro-unipotent group \( \text{Exp}(\mathfrak{L}) = \text{Aut}(\mathfrak{L}) \) is defined by 
\[ \text{Aut}(\mathfrak{L}) = \{ \phi \in \text{Aut}(V) : (\phi - \text{Id})V^n \subset V^{n-1}, n \geq 0 \} \]
whose set of normal subgroups \( A \in \Lambda \) is given by any normal subgroup containing 
\[ A_n = \{ \phi \in \text{Aut}(\mathfrak{L}) : \phi|_{V^n} = \text{Id}, n \in \mathbb{N} \} \]
and such that

\[
\text{Aut}(\mathfrak{L}\mathfrak{N})/A_n \triangleleft \text{Aut}(V^n)
\]

Clearly, \((\text{End}(V), \text{Aut}(\mathfrak{L}\mathfrak{N}))\) is a Harish-Chandra pair. Let \(\{S^n\}\) be a projective system of \(\text{Aut}(\mathfrak{L}\mathfrak{N})/A_n\)-torsors, then the projective limit \(\pi : S \to X\) is by construction an \(\text{Aut}(\mathfrak{L}\mathfrak{N})\)-torsor over \(X\). The following definition is inspired by [FBZ] and [GGW]

**Definition 18.** With notation as above, let \(\{V^n, \phi_n : V^n \to V^{n-1}\}\) be a confinal subset, and form the vector bundle \(V^n_X\) associated to the finite dimensional \(\text{Aut}(\mathfrak{L}\mathfrak{N})/A_n\)-representation \(V^n\) and \(S^n\) for each \(n \geq 0\), then the projective system of coherent \(\mathcal{O}_X\)-modules \(\{V^n_X\}\) is the vector bundle associated to the pro-module for \(\text{Aut}(\mathfrak{L}\mathfrak{N})\), denoted simply by \(V_X\). Moreover, for a Harish-Chandra pair \((\mathfrak{g}, K)\) such that \(\mathfrak{L}\mathfrak{N} \subset \mathfrak{g} \subset \text{End}(V)\) and \(K \cong \text{Aut}(\mathfrak{L}\mathfrak{N})\) we say the vector bundle \(V_X\) associated to the \((\mathfrak{g}, K)\)-module is a bundle of internal symmetries.

Immediately it follows that a bundle of internal symmetries is an object of \(\text{Pro} (\text{Coh}) (X)\). Hereafter, we shall only consider Harish-Chandra pair \((\mathfrak{g}, K)\) such that \(\mathfrak{L}\mathfrak{N} \subset \mathfrak{g} \subset \text{End}(V)\) and \(K \cong \text{Aut}(\mathfrak{L}\mathfrak{N})\) and bundles of internal symmetries unless otherwise noted. In particular, given a pro-module for \(K\) a pro-unipotent algebraic group, the associated bundles \(\{W^n_X\}\) form an inverse system of coherent \(\mathcal{O}_X\)-modules by the definition of a pro-module for \(K\). Consequently, the assignment of this filtering system of coherent \(\mathcal{O}_X\)-modules

\[
\{W^n_X\}_{i \in I} \to \lim_{J \subseteq I} \prod_{j \in J} W^n_X
\]

renders the object \(V_X := \prod_{i \in I} W^n_X\) as an bundle of internal symmetries in the category \(\text{Pro} (\text{Coh}) (X)\). It is therefore in this sense we consider representations of pro-unipotent algebraic groups to be pro-coherent \(\mathcal{O}_X\)-modules throughout this thesis.

Given a Harish-Chandra module, the definition of the associated vector bundle in the theorem is independent of the Lie algebraic half of the Harish-Chandra pair, so it does not effectively express this portion of the data. In point of fact, the associated vector bundle specifies how representations of \(K\) descend to \(X\) along \(\pi\), but this descent property does not
involve the Lie algebra. Yet, the Lie algebra can be represented geometrically and how it is represented expresses the compatibility between the Lie algebra and the algebraic group. Therefore, let us first recall the spaces associated to Lie algebras.

Let $S$ be a $\kappa$-scheme of arbitrary type, $T_S$ its tangent sheaf, that is, the $\mathcal{O}_S$-module and $\kappa$-Lie algebra $\widetilde{\text{Der}}_\kappa(\mathcal{O}_S) = T_S$, and, for a geometric point $s \in S$, denote by $T_sS$ the fibre thereof.

Recall our topological hypotheses for Lie algebras in this thesis, so that, according to our hypotheses, we shall regard the Lie algebra $\mathfrak{g}$ as an lc-Lie algebra. Thus, any statements of continuity are with respect to this topology.

**Definition 19.** We say that a $\kappa$-scheme $S$ of arbitrary type is a $\mathfrak{g}$-space if

**Axiom I** there exists a continuous homomorphism

$$\epsilon : \mathfrak{g} \to T_S$$

of Lie algebras.

Provided the induced morphism of $\mathcal{O}_S$-modules

$$\alpha : \mathcal{O}_S \otimes \mathfrak{g} \to T_S$$

defined by $\alpha(s \otimes g) = s\epsilon(g)$, called the action map, is an isomorphism, then we say $S$ is a principal $\mathfrak{g}$-space. We refer to the action map of a principal $\mathfrak{g}$-space as a simply transitive action, and more generally, to the action map of a $\mathfrak{g}$-space as a transitive action if $\alpha$ is surjective.

The following theorem of Gelfand and Kazhdan [GF] characterizes principal $\mathfrak{g}$-spaces.

**Theorem 5.** Let $S$ be a scheme of arbitrary type, then $S$ is a principal $\mathfrak{g}$-space if and only if the map of Lie algebras from $\mathfrak{g}$ to $T_sS$ via the composition of $\epsilon$ with the evaluation map $ev_s : T_S \to T_sS$ viz.

$$\mathfrak{g} \to T_S \to T_sS$$
is an isomorphism for all \( s \in S \).

A \( \mathfrak{g} \)-valued differential form of degree \( p \) is a continuous \( \mathcal{O}_S \)-linear mapping \( \bigwedge^p T_s S \to \mathcal{O}_S \otimes \mathfrak{g} \); equivalently, \( \bigwedge^p T_s S \to \mathfrak{g} \). We write \( \Omega^p(S, \mathfrak{g}) \) for the \( \mathcal{O}_S \)-module of differential \( p \)-forms and \( \Omega^*(S; \mathfrak{g}) = \bigoplus_p \Omega^p(S; \mathfrak{g}) \) for the direct sum of such modules. As usual, \( \Omega^*(S; \mathfrak{g}) \) is equipped with the degree 1 de Rham differential \( d \), given by the exterior derivative of \( p \)-forms, thus rendering \( \Omega^*(S; \mathfrak{g}) \) as a differentially graded or dg algebra. Specifically, we have, for \( \eta \in \Omega^p(S, \mathfrak{g}) \), \( (\zeta_1, \ldots, \zeta_{p+1}) \in T_s S^{\times(p+1)} \)

\[
(d\eta)(\zeta_1, \ldots, \zeta_{p+1}) = \\
\sum_{i,j} (-1)^{i+j-1} \eta([\zeta_i, \zeta_j], \ldots, \hat{\zeta}_i, \ldots, \hat{\zeta}_j, \ldots, \zeta_{p+1}) \\
+ \sum_i (-1)^i \zeta_i(\eta(\zeta_1, \ldots, \hat{\zeta}_i, \ldots, \zeta_{p+1}))
\]

and we say that \( (\Omega^*(S; \mathfrak{g}), d) \) is the \( \mathfrak{g} \)-valued de Rham complex of \( S \). One notices at \( s \in S \) one has a homomorphism of Lie algebras \( T_s S \to \mathfrak{g} \) in degree one, whence the name \( \mathfrak{g} \)-valued.

Assume now that \( S \) is a principal \( \mathfrak{g} \)-space over \( X \), then the isomorphism included in the definition is invertible, with inverse the map \( \nabla_\mathfrak{g} : T_s S \to \mathfrak{g} \), a \( \mathfrak{g} \)-valued 1-form. This 1-form is characterized by the following fundamental condition according to the proceeding theorem.

**Definition 20.** The Lie algebra homomorphism \( \nabla_\mathfrak{g} \) inverse to the composition \( ev_s \circ \epsilon \) is called a Maurer-Cartan element if

(Axiom I) \( \nabla_\mathfrak{g} \) satisfies the Maurer-Cartan equation

\[
d\nabla_\mathfrak{g} - \frac{1}{2} [\nabla_\mathfrak{g}, \nabla_\mathfrak{g}] = 0
\]

We have the following theorem of Bernstein and Rozenfel’d \([BR]\), although it is stated without proof in \([GF]\).
Theorem 6. Let \( S \) be a principal \( \mathfrak{g} \)-space. Then the inverse of the composition \( \text{ev}_{s} \circ \epsilon = \nabla - \mathfrak{g} \) satisfies the Maurer-Cartan equation.

Before we explain the geometric significance of this theorem, we must recall the construction of the Atiyah sequence associated to a \( K \)-torsor \( \pi : S \to X \). In what follows in the exposition, we prove the following lemma.

Theorem 7. Let \( \pi : S \to X \) be a \( K \)-torsor and \( \mathfrak{k} = \text{Lie}(K) \). Then a \( \mathfrak{k} \)-valued 1-form

\[
\nabla_{\mathfrak{k}} : T_{S} \to \mathcal{O}_{S} \otimes \mathfrak{k}
\]

corresponding to the connection defined by the projection

\[
p : T_{S} \to \mathcal{O}_{S} \otimes \mathfrak{k}
\]

always exists.

Let \( \pi : S \to X \) be a \( K \)-torsor. Define \( \mathcal{A}t(S) \) be the sheaf of \( K \)-equivariant maps from \( \pi^{-1}(U) \) to \( T_{S} \), as above. In other words, consider the \( \mathcal{O}_{X} \)-module \( \mathcal{A}t(S) = (\pi_{*}(T_{S}))^{K} \). This \( \mathcal{O}_{X} \)-module is the canonical example of a Lie-Rinehart algebroid, discussed later, and is referred to as the Atiyah algebroid.

Choose some \( s \in \pi^{-1}(U) \), then we have an evaluation homomorphism from the global sections of \( K \)-invariant vector fields defined on \( U \) to the fibre of the tangent sheaf of \( S \) at each \( s \in S \), viz.

\[
ev_{\pi,s} : \Gamma(\pi^{-1}(U), T_{S}(\pi^{-1}(U)))^{K} \to T_{s}S
\]

These evaluation homomorphisms glue to an isomorphism of sheaves

\[
ev_{\pi} : \pi^{*}\mathcal{A}t(S) \to T_{S}
\]

Now let \( d\pi : T_{S} \to \pi^{*}T_{X} \) be the sheaf morphism induced by the normal sequence corresponding to the structure morphism \( \pi : S \to X \). One has a surjective morphism \( d\pi \circ \text{ev}_{s} : \pi^{*}\mathcal{A}t(S) \to \pi^{*}T_{X} \) and as both sheaves are pulled back from \( X = S/K \), they are \( K \)-modules, as well. Hence, the sheaf morphism \( d\pi \) descends to a morphism of \( \mathcal{O}_{X} \)-modules.
\[ \rho : \mathfrak{at}(S) \to T_X \]

upon dividing by the \( K \)-action. The morphism \( \rho \) is called the anchor map. To reveal how a connection is obtained from this anchor map, consider \( \mathcal{A}d(S) \), the associated vector bundle obtained from the Lie algebra \( \mathfrak{k} \) with respect to the adjoint action of \( K \) on \( \mathfrak{k} \). This associated vector bundle is an \( \mathcal{O}_X \)-module by construction and is known as the adjacent bundle.

Observe, the trivial locally free sheaf \( \mathcal{O}_S \times \mathfrak{k} \) on \( S \) is \( K \)-equivariant. Accordingly, we have a morphism of sheaves

\[ \alpha : \mathcal{O}_S \times \mathfrak{k} \to T_S \]

which induces a short exact sequence of \( \mathcal{O}_S \)-modules,

\[
0 \longrightarrow \mathcal{O}_S \times \mathfrak{k} \xrightarrow{\alpha} T_S \xrightarrow{d\pi} \pi^*(T_X) \longrightarrow 0
\]

on \( S \). In particular, \( \text{Im}(\alpha) = \ker(d\pi) \). Therefore, define \( \text{Im}(\alpha) := \mathcal{V} er(S) \), which is regarded as the vertical sheaf associated to the structure morphism \( \pi : S \to X \). This in turn identifies \( \mathcal{A}d(S) \) as the \( K \)-invariant subsheaf of \( \mathcal{A}t(S) \), giving rise to a short exact sequence of \( \mathcal{O}_X \)-modules

\[
0 \longrightarrow \mathcal{A}d(S) \longrightarrow \mathcal{A}t(S) \xrightarrow{\rho} T_X \longrightarrow 0
\]

known as the Atiyah sequence on \( X \).

Let us show how the splitting of this sequence is related to the existence of a connection. Suppose one has a section \( \sigma \) of \( \rho \). Consider the corresponding sequence of pullbacks

\[
\pi^*T_X \xrightarrow{\pi^*\sigma} \pi^*\mathcal{A}t(S) \xrightarrow{\cong} T_S
\]

The image of \( \pi^*T_X \) under the composition \( \pi^*\sigma \) is then defined to be the horizontal sheaf, denoted \( \mathcal{H} or(S) \). The composition \( \mathcal{V} er(S) \to T_S \to T_S/\mathcal{H} or(S) \) is an isomorphism. Accordingly,

\[
T_S \cong \mathcal{H} or(S) \oplus \mathcal{V} er(S)
\]
As such, one has a decomposition of the tangent sheaf into a direct sum of subsheaves. Classically, this is the definition of a connection. Yet, by construction, for a $K$-torsor $S$, the preceding exposition justifies the following definition.

**Definition 21.** Let $\pi : S \to X$ be a $K$-torsor. Then we say a splitting of the Atiyah sequence of $X$ is a connection on $S$.

Observe, the decomposition of $T_S$ given by the projection $p : T_S \to T_S/Hor(S)$ furnishes a $\mathfrak{t}$-valued one form given by projection to the second factor, denoted

$$\nabla_\mathfrak{t} : T_S \to \mathcal{O}_S \otimes \mathfrak{t}$$

Given a splitting of the Atiyah sequence of $X$, we call this $\mathfrak{t}$-valued one form a connection 1-form.

In general, for a $K$-torsor $\pi : S \to X$, the action map $\alpha : \mathcal{O}_S \otimes \mathfrak{t} \to T_S$ where $\mathfrak{t} = \text{Lie}(K)$ is transitive. Thus, consider the canonical $\mathfrak{t}$-valued Maurer Cartan form $\omega_K \in \Omega^1(K; \mathfrak{t}) \cong \text{Hom}_K(\mathfrak{t}, \mathfrak{t})$ corresponding to the identity operator $\text{Id}_\mathfrak{t} \in \text{Hom}_K(\mathfrak{t}, \mathfrak{t}) = \text{End}(\mathfrak{t}) \cong \mathfrak{t}^\vee \otimes \mathfrak{t} \cong \Omega^1(K)^K \otimes \mathfrak{t}$, where the superscript denotes $K$-invariance. If the 1-form $\nabla_\mathfrak{t} \in \Omega^1(S; \mathfrak{t})$ constructed above satisfies the equations $r_s^*(\nabla_\mathfrak{t}) = \omega_K$ and $\alpha_{S,k}^*(\nabla_\mathfrak{t}) = \text{Ad}(k^{-1})_*(\nabla_\mathfrak{t})$, where $r_s : K \to S$ is the inverse of the isomorphism induced by the fibre of the morphism $S \times_X S \to K \times_X S$, $\alpha_{S,k} : S \to S$ is the action of $K$ on $S$ for fixed $k \in K$, and $\text{Ad}(k^{-1})_*$ is the pushforward of the adjoint action of $K$ on $\mathfrak{t}$, then, for each $s \in S$, the map $\nabla_\mathfrak{t} : T_sS \to \mathfrak{t} = \mathcal{H}or_s(K)$ is a surjection, hence, we can take $\ker(\nabla_\mathfrak{t}) := \mathcal{H}or_s(S)$ and decompose the fibre of the tangent sheaf at $s \in S$ as

$$T_sS = \mathcal{H}or_s(S) \oplus \mathfrak{t}$$

Our decomposition accords with the usual situation when $\mathcal{V}er_s(S) = \mathfrak{t}$. Therefore, $\nabla_\mathfrak{t}$ defines a connection 1-form on the $K$-torsor $\pi : S \to X$. Thus the proof of the above theorem is just a restatement of the torsor condition given the equations on the pullbacks of $r_s, \alpha_{S,k}$, and the pushforward of the adjoint action which must be satisfied. Such projections exist in general, demonstrating the theorem.
The geometric significance of the Maurer-Cartan element provided by the earlier theorem is hopefully now clear: \( \nabla_\mathfrak{g} \) is a connection 1-form on a principal \( \mathfrak{g} \)-space, \( S \). As \( \nabla_\mathfrak{g} \) satisfies the Maurer-Cartan equation, we say this connection is flat.

We now wish to use this interpretation of the Maurer-Cartan element of a principal \( \mathfrak{g} \)-space as a connection 1-form in conjunction with the connection 1-form \( \nabla_\mathfrak{k} \) associated to a \( K \)-torsor \( \pi : S \to X \) by the theorem to express the compatibility we require between both a principal \( \mathfrak{g} \)-space and a \( K \)-torsor \( S \) over \( X \). Given a Harish-Chandra pair \( (\mathfrak{g}, K) \), the obvious and preferable compatibility between a principal \( \mathfrak{g} \)-space \( S \) associated to its Lie half and a \( K \)-torsor \( \pi : S \to X \) associated to its algebraic group half would be for \( \text{Exp}(\mathfrak{g}) = K \); yet, as stressed several times already, Lie’s theorem can fail in general, so the attendant \( \mathfrak{g} \)-space is not necessarily equivalent to the attendant \( K \)-torsor, as it would be if the equality were satisfied. Indeed, this discrepancy was originally the motivation for the development of the Harish-Chandra ansatz, for one would like to retain as much as possible of this correspondence when no algebraic group corresponds to the Lie algebra \( \mathfrak{g} \). The following notion of \( (\mathfrak{g}, K) \)-structure associated to a Harish-Chandra pair \( (\mathfrak{g}, K) \) naturally arose as the solution to the problem posed by extending Lie’s theorem to situations where Lie’s theorem fails.

Specifically, one may still insist the Lie algebra actions coincide, as would be the case if Lie’s theorem were satisfied: geometrically this compatibility would mean that the connection 1-form \( \nabla_\mathfrak{g} \) of \( S \) restricts to the connection 1-form \( \nabla_\mathfrak{k} \) of the fibre \( F \) over an arbitrary point \( x \in X \) under the structure morphism \( \pi \).

Let \( \mathfrak{g} \) be a Lie algebra with Lie subalgebra \( \mathfrak{k} \subset \mathfrak{g} \). Suppose \( S \) is a principal \( \mathfrak{g} \)-space and \( (F, S, \pi, X) \) is a fibre space, with fibre \( F \) over \( X \). The subalgebra \( \mathfrak{k} \) is called stationary whenever there exists a continuous homomorphism from \( \mathfrak{k} \) to the tangent space of the fibre \( F \), that is, \( \epsilon_K : \mathfrak{k} \to T_F \). Recall the normal sequence

\[
\begin{array}{ccc}
T_F & \longrightarrow & T_{S|F} \\
\downarrow & & \downarrow \\
\mathcal{N}_{F/S} & \longrightarrow &
\end{array}
\]

induced by the embedding of \( F \subset S \). Assume the continuous homomorphism \( \epsilon_K : \mathfrak{k} \to T_F \)
of the stationary subalgebra fits into a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\epsilon_F} & T_{S|F} \\
\uparrow & & \uparrow \\
\mathfrak{k} & \xrightarrow{\epsilon_K} & T_F
\end{array}
\]

where \(\epsilon_F\) is, by abuse of notation, the restriction of the isomorphism \(\epsilon : \mathfrak{g} \to T_S\) to the fibre \(F\). Then we say \(S\) is a \textit{fibre} \(\mathfrak{g}\)-\textit{space with stationary algebra} \(\mathfrak{k}\). Moreover, in the above commuting diagram, we say the simply transitive action of \(\mathfrak{g}\) \textit{extends the fibre-wise action of} \(\mathfrak{k}\) cf. [FBZ].

Finally, we may now define the structure which unifies both a principal \(\mathfrak{g}\)-space \(S\) and \(K\)-torsor \(\pi : S \to X\) given a Harish-chandra pair \((\mathfrak{g}, K)\). Suppose \(S\) is a principal \(\mathfrak{g}\)-space, then by hypothesis, \(S\) is a fibre \(\mathfrak{g}\) space with stationary algebra \(\mathfrak{k}\). Observe, when \(S\) is in addition a \(K\)-torsor, the extension of the fibre-wise action \(\mathfrak{k}\) by \(\mathfrak{g}\) induces a square diagram

\[
\begin{array}{ccc}
\mathfrak{g} & \xleftarrow{\nabla_{\mathfrak{g}}|F} & T_{S|F} \\
\uparrow & & \uparrow \\
\mathfrak{k} & \xleftarrow{\nabla_{\mathfrak{k}}} & T_F
\end{array}
\]

Furthermore, when \((\mathfrak{g}, K)\) is a Harish-Chandra pair, this diagram commutes, for \(d\text{Ad}\) is equal to the induced action of \(\mathfrak{k}\) on \(\mathfrak{g}\) given by the embedding \(\mathfrak{k} \to \mathfrak{g}\) included in the definition of a Harish-Chandra pair. Accordingly, the connection 1-form \(\nabla_{\mathfrak{g}}\) on the principal \(\mathfrak{g}\)-space \(S\) provided in the theorem restricts to the connection 1-form \(\nabla_{\mathfrak{k}}\) of the \(K\)-torsor \(\pi : S \to X\) provided by the lemma along the fibres of \(\pi\). Hence, we can state the key definition cf. [FBZ].

**Definition 22.** Let \(S\) be a \(\kappa\)-scheme of arbitrary type over a \(\kappa\)-scheme \(X\), and \((\mathfrak{g}, K)\) be a Harish-Chandra pair. Suppose \(S\) is both a \(K\)-torsor \(\pi : S \to X\) and a principal \(\mathfrak{g}\). Then we say \(\pi : S \to X\) is a \((\mathfrak{g}, K)\)-structure adopted to the Harish-Chandra pair \((\mathfrak{g}, K)\) if
(AXIOM I) \( g \) extends the fibre-wise action of \( \mathfrak{k} \)

So a \((g, K)\)-structure is a \(K\)-torsor \( \pi : S \to X \) together with a flat \(g\)-valued connection 1-form \( \nabla_g \) on \( S \) which agrees with the canonical connection 1-form \( \nabla_\mathfrak{k} \) along the fibres of the \(K\)-torsor \( \pi : S \to X \). A morpism of \((g, K)\)-structures is simply a morpism of \(K\)-torsors which preserves the connection 1-form \( \nabla_g \). In this way, one may consider spaces for Lie algebras which cannot be treated classically under the auspices of Lie’s theorem as principal \(G\)-bundles.

Ultimately, a \((g, K)\)-structure is a very useful object, as it extends the associated vector bundle construction to representations of Lie algebras for which Lie’s theorem may fail by adding a connection to the \( \mathcal{O}_X \)-module obtained from the associated bundle construction applied to a representation of \( K \). A good set of examples to have in mind are representations of infinite dimensional Lie algebras that fit into a Harish-Chandra pair and in particular, those representations which have an additional structure, such as that of a vertex algebra described below. Just as including the action of the Lie algebra \( g \) provided additional structure to the underlying geometric object, namely the torsor, it too imparts additional structure to the vector bundle \( \nabla_X \) associated to a Harish-Chandra module \( V \), namely, a connection \( \nabla \). Clarification of this last point is the topic of localization below.

### 2.3 Extensions and Localization of \((g, K)\)-Structures

This section concludes our review of the structures we require in the generality of the pro-unipotent Harish Chandra ansatz, or Harish-Chandra geometry, to apply to the topics of formal geometry and vertex algebras. We isolate a functorial property as it plays a crucial role in geometrizing both vertex algebras and their modules below. Specifically, we need to describe how \((g, K)\)-structures can be extended given a morphism of Harish-Chandra pairs.

Let \((g, K)\) and \((g', K')\) be Harish-Chandra pairs together with a morphism \( \Phi = (f, \phi) : (g', K') \to (g, K) \). Further, let \( \pi' : S' \to X \) be a \((g', K')\)-structure over \( X \). We want to describe how \( S' \) extends \( S \).
As a torsor, $\Phi_*(S')$ is the quotient

$$K \times_{K'} S'$$

given by the diagonal action of $K'$ on $K$, that is, $k' \in K'$ acts on $K$ via $k' \cdot k = k\phi(k')^{-1}$.

The Atiyah bundle $\mathcal{A}t(\Phi_*(S'))$ is given by the quotient of the direct sum

$$\mathcal{A}d(S) \oplus \mathcal{A}t(S')$$

by the image of the map $u : \mathcal{A}d(S') \to \mathcal{A}d(S)$ induced by $f : g' \to g$ and the inclusion $i : \mathcal{A}d(S') \to \mathcal{A}t(S')$.

The hypothesis $S'$ is a $(g',K')$-structure furnishes a splitting of the Atiyah sequence $\sigma' : T_X \to \mathcal{A}t(S')$. Composing this splitting and pulling back by the same furnishes a morphism

$$\sigma'^* : \pi'^* T_X \to \pi'^* \mathcal{A}t(\Phi_*(S'))$$

which in turn defines the horizontal sheaf $\mathcal{H}or(\mathcal{A}t(\Phi_*(S'))) and therefore, a splitting of the tangent sheaf $T_{\Phi_*(S')}$, in other words, a connection. This categorical construction motivates the following definition.

**Definition 23.** Let $S'$ be a $(g',K')$-structure over $X$ and $\Phi : (g',K') \to (g,K)$ a morphism of Harish-Chandra pairs. Then we say, for a $(g,K)$-structure $S$ over $X$, that $S'$ is an extension if

(Axiom I)

$$\Phi_*(S') \cong S$$

(Axiom II) The induced morphism

$$\nabla^* : \mathcal{A}t(\Phi_*(S')) \to T_{S'}$$

defines a connection on $\Phi_*(S')$. 

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In the special case when $S' \subset S$ and $\Phi$ is an inclusion, we say an extension is a lift.

In this manner, one regards lifts as the analogue of the reduction of the structure group construction for a $K$-torsor to that of a $(\mathfrak{g}, K)$-structure.

Recall that, given a $(\mathfrak{g}, K)$-module, $V$, the associated vector bundle $\mathcal{V}_X$ only reflects the contribution of the group half $K$ of the Harish-Chandra pair $(\mathfrak{g}, K)$. A special case of the lift construction confers the contribution of the Lie algebra half $\mathfrak{g}$ to $\mathcal{V}_X$, as well.

Let $\Phi = (\rho, \alpha) : (\mathfrak{g}, K) \to (\mathfrak{gl}(V), GL(V))$ be the induced morphism of Harish-Chandra pairs. The torsor $\Phi^*(S)$ is the lift of the bundle of frames [Huy] of the associated vector bundle $\mathcal{V}_X$. Namely,

$$\Phi^*(S') \to \mathcal{F}_X(\mathcal{V}_X)$$

is an isomorphism [BK].

Now, by definition of a $(\mathfrak{g}, K)$-structure, the vector bundle associated to the Lie algebra $\mathfrak{g}$ acts upon $\mathcal{V}_X$, for $\mathfrak{g}$ extends the fibre-wise action of $\mathfrak{k}$. Moreover, the Atiyah bundle $At(S)$ acts on $\mathcal{V}_X$ by construction. So, let $U \subset X$, then for $\xi \in At(U)$ and $v \in \mathcal{V}_U$ we have

$$\nabla_\xi(v) = \xi(v) - \nabla_\mathfrak{g}(\xi(v))$$

is a connection on $\mathcal{V}_X$. The associated vector bundle $\mathcal{V}_X$ together with the connection $\nabla_\xi$ is called the localization of a $(\mathfrak{g}, K)$-module $V$ with respect to the $(\mathfrak{g}, K)$-structure $S \to X$.

Another more direct way to understand localization is as follows. Given that $V$ is a $\mathfrak{g}$-module, one has an action $\mathcal{O}_S \otimes \mathfrak{g} \to \mathcal{O}_S \otimes V$ as $\mathcal{O}_S$-modules. By hypothesis, $\pi : S \to X$ is a $(\mathfrak{g}, K)$-structure, so that $\mathcal{O}_S \otimes \mathfrak{g} \cong T_S$. Moreover, $\nabla_\mathfrak{g}$ extends the fibrewise action of $K$, therefore, $\nabla_\mathfrak{g}$ is $K$-equivariant. As such, upon descent, $\pi^*(T_S)^K = At(S) \to (\pi^*\mathcal{O}_S \otimes V)^K = \mathcal{V}_X$. Since the connection $\nabla_\mathfrak{g}$ on $S$ is equivalent to a splitting of the Atiyah sequence, that is, there exists a splitting $\sigma$ of the normal sequence $At(S) \to T_X$, one has by composition

$$\nabla : T_X \to \mathcal{V}_X$$

which is a connection on $\mathcal{V}_X$, by definition.
Localization is the synonym for the associated bundle construction in the generality of \((g,K)\)-structures, that is to say, \(K\)-equivariant descent together with an infinitesimal action by a Lie algebra. Often, below, we shall conflate the two for style without remark. Localization is referred to as such because in the literature it can be shown [BB] that it is a functor right adjoint to the global sections functor, as the localization functor is classically. Localization therefore how one geometrically interprets Harish-Chandra modules, as the result is a quasi-coherent \(\mathcal{O}_X\)-module with a connection.
3 Formal Geometry

The above presentation of Harish-Chandra geometry is sufficiently general to be specialized to the setting of formal geometry discovered by Gel’fand and Kazhdan. Formal geometry is the study of the torsor of formal coordinates, \( \mathcal{C}_{\mathfrak{X}} \) as a \((\mathfrak{g}, K)\)-structure over the Harish-Chandra pair \((\mathfrak{w}, \text{Aut}(O))\) of formal vector fields and formal automorphisms and structures related to the same by lifts and extensions.

Formal Geometry was initiated by Gel’fand and Kazhdan to study the cohomology of infinite dimensional Lie algebras in the seventies [GF]. At the same time, closely related work was studied by Guillemin, Sternberg, and others, on the infinite dimensional Lie groups of Cartan under the guise of pseudo-groups [GS],[SS]. In particular, much effort was expended by these authors investigating the structure of the Lie algebra \( \mathfrak{w} \) of derivations of the ring of power series in several variables underlying formal geometry. Bernstein and Rozenfeld showed how these works were related through the notion of abstract linearly compact Lie algebras that "formalized" a sheaf of Lie algebras associated to a pseudogroup [BR], thus connecting initially disparate paths of inquiry.

The subject was taken up later by Nest, Tsygan, and several others, to study deformation quantization [NTs], [KI], [CF] and by Beilinson and Drinfeld to study Geometric Langlands [BD2]. In the latter work, the efficacy of formal geometry as a tool for studying \( \mathcal{D} \)-modules of arbitrary type and representations of the affine Kac-Moody Lie algebra, specifically vertex algebras, was revealed. Thereafter, Frenkel and Ben-Zvi [FBZ] used the framework of formal geometry to realize vertex algebras as geometric objects, specifically as pro-coherent \( \mathcal{O}_X \)-modules with a flat connection, over an algebraic curve \( X \) of genus greater than or equal to two. The presentation in loc.cit is the primary source of influence in this thesis. Formal geometry vis a vis vertex algebras then resurfaced more than a decade later in the work of Gorbounov and Gwilliam. [GGW] in order to relate chiral differential operators-a species of geometric vertex algebras-to factorization algebras predicated upon the \( \beta\gamma \)-system in the works of Costello and Gwilliam.
Formal geometry is ubiquitous across these various applications because the construction automates the globalization of locally coordinate dependent constructions, thereby fulfilling one objective common to all geometry, that is, to globalize local constructions. One adopts the perspective that in a formal neighborhood of a point, smooth spaces are the same up to a non-canonical identification. One then expresses various coordinate dependent constructions in a formal neighborhood or disc about a point in some ambient space $X$ and then identifies these expressions within the infinite dimensional space $\mathcal{C}or_X$. One is then able to "spread out" this identification over the base $X$ via $\mathcal{C}or_X$, as one regards the torsor of formal coordinates as a "cover by formal neighborhoods" of $X$ in the sense of Cech cohomology $[\text{Kl}]$.

Coordinate dependence exists in different branches of geometry; whenever a notion of coordinates is available, this framework exists. Consequently, formal geometry has been studied in the analytic, $C^\infty$, and algebro-geometric settings. As we focus on the latter, this will entail an increase in topological sophistication, as regular $\kappa$-schemes with the Zariski topology are too coarse to possess a practically useful notion of coordinates by Grothendieck’s theorem $[\text{EGAI}]$. Ergo, to work with coordinate dependent constructions algebro-geometrically, we shall utilize the etale topology on a $\kappa$-scheme $X$ to obtain coordinates. The sense in which a construction is coordinate dependent also varies according to the problem at hand. For example, $[\text{FBZ}]$ view vertex operators as dependent upon coordinates by regarding the formal parameter of the vertex operator as the coordinate of a point of an algebraic curve. In this thesis, however, our construction depends on coordinates because the vertex algebra itself is generated by both coordinates of the underlying regular $\kappa$-scheme in tandem with a pair of frames they induce on a pair of coherent $\mathcal{O}_X$-modules.

Last, Guillemin and Sternberg’s realization theorem $[\text{GS, Bl, Drai}]$ augments formal geometry when viewed as a special case of Harish-Chandra geometry by furnishing this specialization with a useful method whereby the torsor of formal coordinates may be lifted and extended. Indeed, according to their realization theorem, the Harish-Chandra pair
(w, Aut(O)) adopted to the torsor of formal coordinates Cor_X, possesses a universal mapping property in the category of Harish-Chandra pairs by asserting the existence of a morphism of Harish-Chandra pairs (g, K) → (w, Aut(O)) whenever ℱ ⊂ g is transitive. One is then able to construct both lifts and extensions of Cor_X by the realization theorem. One therefore able to extend the localization process to Harish-Chandra modules and (g, K)-structures which are not Harish-Chandra modules over (w, Aut(O)).

This final point is especially relevant to the certain vertex algebras we have in mind, as it allows one to consider the localization of vertex algebras and their modules that are not conformal in the sense of [FBZ]. Essentially, the localization of vertex algebras in loc.cit depended upon this hypothesis. More generally, however, one may regard the contribution of the realization theorem to the framework of formal geometry as a result that distinguishes formal geometry as the correct setting in which to undertake the geometric representation theory of Harish-Chandra modules.

3.1 Coordinates and Formal Coordinates

We begin by discussing how finite dimensional regular κ-schemes may be coordinatized locally in a manner familiar from differential geometry, which is to say that locally our schemes look like affine n-space. It is well known such coordinatization is achieved by means of the anaology between the local diffeomorphisms of differential geometry and the etale morphisms of algebraic geometry. Further, we shall describe what it means to formally coordinatize a regular κ-scheme. To accomplish this second objective, we shall use the language of O_X-Lie-Rinehart algebroids discussed in general in chapter 5 with respect to the tangent sheaf qua the canonical Lie-Rinehart algebroid T_X, the O_X-module of sections of the tangent bundle TX.

Let X be regular scheme over κ of Krull dimension n. Recall that according to this hypothesis X is locally noetherian, thus every point x ∈ X specializes to a closed point, that is, each x ∈ X is contained in the closure of a closed point. Therefore, we can and shall
work with closed points of $X$ in what follows without further remark. Recall that $x \in X$ is closed if and only if $\kappa(x)$ is a finite extension of $\kappa$ by Hilbert’s Nullstellansatz. Moreover, by regularity, this extension is separable of transcendence degree $n$. This entails the following local description of $X$.

Let $X$ be a $\kappa$-scheme as above. Recall that $X$ is locally etale if for every $x \in X$, there exists an open neighborhood $U \subset X$ of $x$ and sections $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ such that the morphism

$$f = \prod_i f_i : U \to \mathbb{A}_\kappa^n$$

is etale. This means that, from the non-intrinsic perspective, one has a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\exists} & \text{Spec}(\kappa[X_1, \ldots, X_n]/(f_1, \ldots, f_n)) \\
\downarrow f & & \downarrow \iota \\
\mathbb{A}_\kappa^n & \end{array}
$$

where $\exists$ is an open immersion and the $f_i \in \mathcal{O}_X(U)$ are regular functions such that the determinant of their Jacobian

$$\det(\text{Jac}_{f_1, \ldots, f_n}) \in \kappa(x)$$

is a unit in the residue field of $x \in X$. One notes that the Jacobian of the $f_i$ is a unit.

We can choose constants so that the affine coordinates $X_i$ are contained in the ideal sheaf of the closed point $x \in X$, so that $f(x) = \emptyset$, the origin in $\mathbb{A}_\kappa^n$. We then have the following definition of local coordinates.

**Definition 24.** Let $U$ be an open neighborhood of $x \in X$, then we say $\{x_i = f_i - f(x)\} \in \mathcal{O}_X(U)$ is a coordinate system at $x$ if

(Axiom 1) the induced morphism
\[ f : U \to \mathbb{A}^n \]

is etale.

**Axiom 2** the induced morphism is centered at \( x \), that is, the induced morphism maps

\[ x \mapsto \mathbb{O} \]

where \( \mathbb{O} \) is the origin in \( \mathbb{A}^n \).

It is in this sense that our schemes \( X \) are alike differentiable manifolds. In particular, the invertibility of the Jacobian matrix guarantees a form of the inverse function theorem as one is disposed of in differential geometry. This feature is one of the major motivations for the introduction of the etale topology in algebraic geometry and what brings the subject closer to classical geometry. This analogy avails one of the use of coordinates, although coordinates in this sense do not separate points as they do in the category of differentiable manifolds.

Next we shall use our definition of coordinates at \( x \in X \) to define formal coordinates. Let \( \Delta : X \to X \times X \) be the diagonal embedding. A choice of coordinates at \( x \) are correspond to a free basis of the conormal sheaf of the diagonal embedding. Indeed, define

\[ v_i = f_i \otimes I_\Delta + I_\Delta \otimes f_i \mod (I_\Delta)^2 \]

where \( I_\Delta \) is the ideal sheaf of the diagonal embedding. Then as is well-known the \( v_i \) are a free basis of \( I_\Delta/(I_\Delta)^2 \). Hence, for open \( U \subset X \) as above, the cotangent sheaf is locally free and we have the expression

\[ \Omega^1_U \cong \bigoplus_{i=1}^n \mathcal{O}_X(U) v_i \]

of the cotangent sheaf.

Let \( x \in U \) be an open neighborhood of \( x \in X \) and \( \{x_i\} \) be a system of coordinates at \( x \). Consider the the tangent sheaf \( T_X \) as the \( \mathcal{O}_X \)-module dual of the cotagent sheaf. As the cotangent sheaf is locally free, so too is \( T_X \) since the canonical pairing is non-degenerate by the regularity hypothesis. Denote by \( \{\partial_{x_i}\} \) the dual local basis. We shall treat the tangent sheaf as the canonical Lie-Rinehart algebroid. This prescription in conjunction with the
definitions of both the universal enveloping algebroid and its dual, the jet construction, are discussed below in chapter five.

Let $\mathcal{U}_X$ be the universal enveloping algebroid of $T_X$. Recall that $\mathcal{U}_X$ is replete with an increasing filtration $[\text{Kap}]$

$$\mathcal{U}_X^{\leq d} = \text{Span}\{\partial_{x_{i_1}} \cdots \partial_{x_{i_p}} \mid \partial_{x_{i_t}} \in T_X, \sum_i i_t \leq d\}$$

as $T_X$ is projective of finite rank over $X$. The universal Lie algebroid $\mathcal{U}_X$ is the inductive limit over $d$ of this filtration. Moreover, $\mathcal{U}_X$ is furnished with a counit $e^\vee$ and an $O_X$-bialgebra structure with respect the left and right counit properties of $e^\vee$. These constructions are all local $[\text{Moerd}]$.

One dualizes this construction to obtain the jet algebra

$$J_X = \text{Hom}_{O_X}(\mathcal{U}_X, O_X)$$

on $X$. By the above, the jet algebra is a sheaf of $O_X$-algebras complete with respect to the ideal $J^c$ given by the kernel of the morphism

$$e : J_X \to O_X$$

dual to the counit morphism. Moreover, one obtains embeddings

$$(s^*, t^*) : O_X \to J_X$$

dual to the projections onto $O_X$ of $\mathcal{U}_X$ by projecting onto the subspace $\mathcal{U}_X^{\geq 1}$ according to whether $\mathcal{U}_X$ is treated as either a right or a left $O_X$-module; in the former case, we write $s^*$, and $t^*$ in the latter.

One can sheafify this construction according to the standard Spec construction for $O_X$-algebras.

**Proposition 2.** The $O_X$-algebra spectrum of $J_X$ is a formal $X$-scheme, denoted $\text{Jet}_X$. Moreover, $\text{Jet}_X$ is a groupoid object in the category of $\kappa$-schemes.
Proof. One has a projective system of $O_X$-algebras $J^d_X = \text{Hom}_{O_X}(U_X^{\leq d}, O_X)$ with respect to the structure morphism $s$. The projective limit of algebras can be sheafified by the standard fact that the projective limit of sheaves is a sheaf. It follows that $\text{Spec}(J^d_X) \to \text{Spec}(J^{d+1}_X)$ forms an inductive system of affine schemes whose projections to $X$ are affine. Accordingly, by [SGA4], [Vis], affine morphisms satisfy descent, so that the inductive limit

$$\mathcal{J}et_X = \text{colim}_d \text{Spec}(J^d_X)$$

glues to a formal scheme.

As for the second statement, this formal scheme is a groupoid object in the category of regular $\kappa$-schemes, where $\mathcal{J}et_X$ is the category of arrows and $X$ is the category of objects with respect to the morphisms

$$(s, t) : \mathcal{J}et_X \to X$$

and the zero section

$$e : X \to \mathcal{J}et_X$$

By definition, $e \circ s = \text{Id}_X$. \hfill \square

We remark that the algebra $J_X$ is isomorphic to the algebra of functions obtained by the completion of the square $X \times X$ along the diagonal $\Delta(X)$, rendering this construction of $\mathcal{J}et_X$ equivalent to the standard one cf. [CVdb]. To compare constructions, observe that the ideal $\mathcal{J}^c$ corresponds to the completion of $I_\Delta$ in the $I_\Delta$-adic topology on $O_{X \times X}$.

We wish to use this description of the Jet scheme in terms of the Lie-Rinehart algebra $(O_X, TX)$ to define formal coordinates. Switching to the affine perspective for the moment, consider a Lie-Rinehart algebra $(R, L)$. There is a convenient interpretation of its jet algebra, due famously to Rinehart [Rin] supplied by a PBW theorem for Lie-Rinehart algebras he proved in loc.cit. Let $\mathcal{J}L$ denote the jet algebra of an arbitrary Lie-Rinehart algebra, $(R, L)$.

**Proposition 3.** Given a Lie-Rinehart algebra $(R, L)$, one has $\mathcal{J}L \cong \hat{\text{Sym}}(L^\vee)$, where the right hand side indicates the completion of the symmetric algebra as an $R$-algebra with respect to
the augmentation ideal, that is, the kernel of the projection of \( \text{Sym}(L) \) onto \( R \). Furthermore, there is an isomorphism of graded \( \mathcal{O}_X \)-algebras

\[
\text{gr}(\mathcal{J}_L) \cong \text{Sym}(L^\vee)
\]

In particular, we shall apply the PBW theorem of Rinehart to the special case of the universal enveloping algebroid of the tangent sheaf \( T_X \) and the corresponding jet algebra \( J_X \). Under the auspices of the proposition, one has the jet algebra is isomorphic to the completion of the cotangent sheaf \( \Omega^1_X \), for

\[
\prod_{k=1}^\infty (J^c)^k/(J^c)^{k+1} \cong \prod_{k=1}^\infty (I_\Delta)^k/(I_\Delta)^{k+1}
\]

and

\[
\prod_{k=1}^\infty (I_\Delta)^k/(I_\Delta)^{k+1} \cong \prod_{k=0}^\infty \text{Sym}^{\leq k}(I_\Delta/(I_\Delta)^2)
\]

So, given a choice of coordinate system at \( x \), one has \( J^c \) is generated by the \( v_i \), because \( J^c/(J^c)^2 \cong \Omega^1_X \) and one lifts the free basis. These facts motivate the first of two definitions of formal coordinates.

**Definition 25.** We say a trivialization of the Jet scheme \( \text{Jet}_X \) is a system of formal coordinates. In other words, given an open neighborhood \( U \) of \( x \) together with an system of coordinate \( \{x_i\} \) centered at the origin, an isomorphism of \( X \)-schemes

\[ \text{Jet}_X(U) \cong \text{Spec}(\mathcal{O}_X(U)[[v_1, \ldots, v_n]]) \]

This first definition is meant to draw one’s attention to the geometric interpretation of the following algebraic definition. Recall, \([\text{VdB}]\), since \( s^* : \mathcal{O}_X \to \mathcal{J}_X \), we have \( \mathcal{J}_X/J^c \cong \mathcal{O}_X \), and that the unit \( e \) is the left inverse of the source, that is, \( e \circ s^* = \text{Id}_X \), and therefore

\[ \mathcal{J}_X \cong \mathcal{O}_X(U)[[v_1, \ldots, v_n]]. \]

for some choice of system of coordinates \( \{x_i\} \) at \( x \). Hence, a choice of coordinate system at \( x \) induces an isomorphism \( \phi_X \) of the jet algebra and the ring of formal power series in the variables \( v_i \) with coefficients in the ring \( \mathcal{O}_X(U) \). We have therefore the following second equivalent definition.
Definition 26. Let $U$ be an open neighborhood of $x$ and $\{x_i\}$ a system of coordinates at $x$ centered at the origin. Then we say $\phi_X$ is a system of formal coordinates on $X$ if

(Axiom 1) the map $\phi_X$ induced by the system of coordinates

$$\phi_X : \Gamma(J_{et X}, U) \to O_X(U)[[v_1, \ldots, v_n]]$$

is an isomorphism.

(Axiom 2) the morphism $\phi_X$ is centered at $x$, that is, the induced morphism maps

$$\mathcal{I}^c \mapsto m$$

where $m$ is the unique maximal ideal of $O_X(U)[[v_1, \ldots, v_n]]$.

Last, we have one final definition of coordinates and an observation that will help to characterize the canonical $(\mathfrak{g}, K)$-structure attached to a regular $\kappa$-scheme $X$ by formal geometry. Observe that, as $J_{et X} \to X$ is an affine morphism, then given a system of coordinates at $x$, the fibre of the structure morphism at $x \in X$ is trivialized by a system of formal coordinates on $X$. Indeed, such a trivialization is realized by an isomorphism

$$J_{X,x} \cong \kappa(x)[[v_1, \ldots, v_n]]$$

via pullback $s^* : O_X \to J_X$. Given a regular section of $O_X(U)$, one should regard its image under this isomorphism as its Taylor series expansion. We single out the isomorphism a system of formal coordinates induces on the fibre in the following definition.

Definition 27. Let $U$ be an open neighborhood of $x$ and $\phi_X$ the induced system of formal coordinates on $X$. Then we say $\phi_x$ is a system of formal coordinates at $x$ if

(Axiom 1) the map $\phi_x$ induced by a system of formal coordinates on the fibre

$$\phi_x : J_{X,x} \to \kappa(x)[[v_1, \ldots, v_n]]$$

is an isomorphism.

(Axiom 2) the morphism $\phi_x$ is centered at $x$, that is, the induced morphism maps
\[ \kappa(x) \otimes \mathcal{O}^c \rightarrow m_x \]

where \( m_x \) is the maximal ideal of \( \kappa(x)[[v_1, \ldots, v_n]] \).

### 3.2 Discs and the Gel’fand-Kazhdan Pair

In this section we define the formal disc centered at a point \( x \in X \) and extract from it both a Lie algebra \( \mathfrak{g} \) and a pro-unipotent algebraic group \( K \) corresponding to a pro-nilpotent Lie subalgebra of the former which is transitive. This pair is the fundamental Harish-Chandra pair of formal geometry, which we call the Gel’Fand-Kazhdan pair.

Let \( x \in X \) and \( U \) an open neighborhood of \( x \) together with a system of coordinates at \( x \). This choice of a system of coordinates at \( x \) determines a basis of the fibre of the cotangent space at \( x \), which we shall denote \( \{v_i\} \). These \( v_i \) generate the \( \kappa(x) \)-vector space \( (m_x/m_x^2)^\vee \).

We have the following definition.

**Definition 28.** Let \( U \) be an open neighborhood of \( x \) and \( \{x_i\} \) a system of coordinates at \( x \). Let \( \{v_i\} \) be the corresponding basis of the fibre of the cotangent sheaf at \( x \). Then we say

\[ \text{Spec}(\mathcal{O}_x) =: D_x \]

is the formal disc centered at \( x \), where \( \mathcal{O}_x \)

\[ \mathcal{O}_x \cong \kappa(x)[[v_1, \ldots, v_n]] \]

and

\[ \mathcal{O}_x = \tilde{\text{Sym}}((m_x/m_x^2)^\vee) \]

is the completion of the symmetric algebra over \( \kappa(x) \) of \( (m_x/m_x^2)^\vee \).

Our description of the disc is somewhat non-standard, as many authors prefer to simply use the fact that the completed fibers of the structure sheaf at \( x \) are rings of formal power series under our hypotheses cf. [FBZ], [BD2], [KL]. However, we prefer this non-standard
description, as it synchronizes it with the constructions presented in the literature on transitive linearly compact Lie algebras cf. [GS], [Bl]. However, we still emphasize that as a scheme $D_x$ is non-canonically isomorphic to

$$\text{Spec}(\kappa(x)[[x_1, \ldots, x_n]])$$

and that such an isomorphism depends on the choice of a system of coordinates at $x$.

We are interested in the Lie algebra of derivations of the formal disc centered at $x$, that is, in the global sections of its tangent sheaf. First, however, a few observations about $\mathcal{O}_x$ are in order to make sense of its vector fields.

Write elements of $\mathcal{O}_x$ as

$$a(v_1, \ldots, v_n) = \sum_{(i_1, \ldots, i_n) \in \mathbb{Z}^n} a_{i_1 \ldots i_n} v_1^{i_1} \cdots v_n^{i_n} \in \mathcal{O}_x$$

and re-write them as

$$a(v_1, \ldots, v_n) = \sum_{d=0}^\infty a_d(v_1, \ldots, v_n) \in \mathcal{O}_x$$

where $a_d(v_1, \ldots, v_n)$ are homogeneous polynomials of degree $d$ in the $v_i$. Let us abbreviate these elements as $a_d$ whenever convenient. This second description lays bare how to define the order of an element of $\mathcal{O}_x$. Define the order of $a \in \mathcal{O}_x$ to be the minimum $d$ such that $a_d \neq 0$, denoted $\text{ord}(a)$. Observe then that the order of elements induces an order filtration on $\mathcal{O}_x$

$$\mathcal{O}_x = F_0 \mathcal{O}_x \supset F_1 \mathcal{O}_x \supset \ldots \supset F_k \mathcal{O}_x \supset F_{k+1} \mathcal{O}_x \supset \ldots$$

where $F_k \mathcal{O}_x = \{a \in \mathcal{O}_x \mid \text{ord}(a) \geq k\}$. Notice that the unique maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$ is given by $F_1 \mathcal{O}_x$ with respect to this filtration and that $\{F_k \mathcal{O}_x\}$ can be regarded as a system of neighborhoods of the origin, thus rendering $\mathcal{O}_x$ as a topological vector space. The Lie algebra of derivations of the disc, denoted $\mathfrak{w}$, then has the following description with respect to the order filtration of $\mathcal{O}_x$. It is interpreted geometrically as the tangent sheaf of $D_x$.

Given a system of coordinates at $x$, define
\[ w(k) = \text{Sym}^k((m_x/m_x^2)') \otimes m_x/m_x^2 \]

where \( m_x/m_x^2 \) is the dual vector space with dual basis \( \{\partial_{x_i}\} \). The algebra \( \text{Sym}^k((m_x/m_x^2)') \) should be regarded as the space of \( k \)-th degree homogeneous polynomials in the \( v_i \) coordinates. Now consider the sum

\[ \text{gr}(w) = \bigoplus_{k=0}^{\infty} w(k) \]

The direct sum \( \text{gr}(w) \) is a graded Lie algebra via the bracket

\[ w(k) \times w(l) \rightarrow w(k + l) \]

so by imparting the discrete topology to \( w(k) \) for each \( k \geq 0 \), we can form its completion to a linearly compact Lie algebra and define its completion

\[ w = \prod_{k=0}^{\infty} w(k) = \text{Sym}((m_x/m_x^2)') \otimes m_x/m_x^2 \]

to be the Lie algebra of formal vector fields. We refer to elements of the Lie lc-algebra \( w \) of vector fields on the disc formal disc centered at \( x \) as formal vector fields.

A formal vector field \( \xi \in w \) is completely determined by where it maps the \( v_i \), so suppose \( \xi(v_i) = a_i(v_1, \ldots, v_n) \); then, the formal vector fields have the following coordinate dependent rendering as

\[ \xi = \sum_{i=1}^{n} a_i \partial_{x_i} \]

where \( a_i \in \mathcal{O}_x \). Moreover, let \( \xi' \in w \) with a similar description, then the formula

\[ \zeta = [\xi; \xi'] = \sum_{i=1}^{n} \{\partial_{x_j}(f_i)g_j - \partial_{x_j}(g_i)f_j\} \]

defines its Lie bracket, where \( f_i, g_i \in \mathcal{O}_x \).

The subspace of derivations of the formal disc \( D_x \) which vanish at the origin, that is, formal vector fields of order greater than 0, form a fundamental subalgebra, which we shall denote by \( w_0 \). It induces a canonical filtration as follows.
The filtration $F_\bullet \mathcal{O}_x$ induces as filtration of $\mathfrak{w}$. Define the order of an element $\xi \in \mathfrak{w}$ to be $\text{ord}(\xi) = -1 + \min\{a_i\}$, where $\xi$ has the description $\xi = \sum_{i=1}^n a_i \partial_{x_i}$. Then we obtain an order filtration of $\mathfrak{w}$ also,

$$\mathfrak{w} = F_{-1} \mathfrak{w} \supset F_0 \mathfrak{w} \supset \ldots \supset F_k \mathfrak{w} \supset F_{k+1} \mathfrak{w} \supset \ldots$$

where $F_{-1} \mathfrak{w} = \mathfrak{w}$ and $F_k^\mathfrak{w} = \{\xi \in \mathfrak{w} \mid \text{ord}(\xi) \geq k\}$ for $k \geq 0$. This definition of order gives $\text{ord}(\xi(a)) \geq \text{ord}(\xi) + \text{ord}(a)$, so $\xi \in F_k \mathfrak{w}$ if and only if $\xi(a) \in F_k \mathcal{O}_x$, so one sees how the order filtration of $\mathcal{O}_x$ determines a filtration of $\mathfrak{w}$. Notice that the least possible order of an element is $-1$, that is, derivations of the form

$$\sum_i \kappa_i \partial_{x_i} + \text{higher order terms}$$

where $\kappa_i \in \kappa(x)$. Continuing, the Lie bracket defined above gives

$$[F_k \mathfrak{w}, F_l \mathfrak{w}] \subset F_{k+l} \mathfrak{w}$$

showing the bracket in $F_0 \mathfrak{w}$ is indeed continuous and that $F_0 \mathfrak{w}$ is a Lie subalgebra of $\mathfrak{w}$, now denoted $\mathfrak{w}_0$. Observe that, importantly, $\mathfrak{w}_0$ is a fundamental subalgebra of $\mathfrak{w}$ by order considerations, hence the order filtration of $\mathfrak{w}$ is a canonical filtration of $\mathfrak{w}$ induced by the fundamental subalgebra $\mathfrak{w}_0$. Therefore $\mathfrak{w}$ is a topological Lie algebra with respect to the topology generated by the canonical filtration induced by $\mathfrak{w}_0$.

The quotient $\mathfrak{w}_0/F_1 \mathfrak{w}_0 \cong \mathfrak{gl}_n$, for $(A_{i,j}) \in \mathfrak{gl}_n$ maps to $\sum_{i,j=1}^n A_{i,j} v_i \partial_{x_j}$ which are precisely the elements of order 0 in $\mathfrak{w}_0$. This map is clearly invertible. Accordingly, we have the semi-direct product decomposition

$$\mathfrak{w}_0 = F_1 \mathfrak{w} \ltimes \mathfrak{gl}_n$$

with respect to the induced action of $\mathfrak{gl}_n$ on $F_1 \mathfrak{w}$. We have the following lemma.

**Lemma 1.** The Lie subalgebra $F_1 \mathfrak{w}$ is a pro-nilpotent Lie algebra with family of ideals $\Lambda$ given by $\{F_k \mathfrak{w}\}$, for $k \geq 2$.
Proof. Recall, the lower central series of a Lie algebra \( \mathfrak{g} \) is a descending series of ideals \( \{C^n \mathfrak{g}\}_{n \geq 1} \) defined recursively by

\[
C^1 \mathfrak{g} = \mathfrak{g} \\
C^n \mathfrak{g} = [\mathfrak{g}, C^{n-1} \mathfrak{g}]
\]

The Lie algebra \( \mathfrak{g} \) is nilpotent if there exists \( n \) such that \( C^n \mathfrak{g} = 0 \) for some \( n \geq 1 \). Given the restriction of the bracket \([F_k \mathfrak{w}, F_l \mathfrak{w}] \subset F_{k+l} \mathfrak{w} \) to \( \mathfrak{w}_0 / F_k \mathfrak{w} \) it follows immediately \( F_1 \mathfrak{w} / F_k \mathfrak{w}_0 \) is a nilpotent Lie algebra, by definition. Moreover, the third isomorphism theorem furnishes a projective system of surjections

\[
F_1 \mathfrak{w} / F_{k+1} \mathfrak{w} \rightarrow F_1 \mathfrak{w} / F_k \mathfrak{w}
\]

Therefore, \( F_1 \mathfrak{w} = \lim_{k \to \infty} F_1 \mathfrak{w} / F_k \mathfrak{w} \), as desired. \( \square \)

A consequence of the lemma is that \( \mathfrak{w}_0 \cong \lim_{k \to \infty} F_k \mathfrak{w} \cong \mathfrak{gl}_n \times F_1 \mathfrak{w} \). Therefore \( \mathfrak{w}_0 \) is a Lie lc-algebra. Moreover, \( \mathfrak{w}_0 \) is of finite codimension in \( \mathfrak{w} \) by order considerations, where we assign order one to elements \( \partial x_i \), thus, by definition, \( \mathfrak{w}_0 \) is transitive.

Since \( \mathfrak{w}_0 \) is an Lie lc-algebra, there exists a corresponding pro-unipotent algebraic group. Indeed, let \( \{Aut^k(O) = \text{Exp}(F_1 \mathfrak{w}_0 / F_k \mathfrak{w})\} \) denote the projective system of unipotent algebraic groups furnished by the lemma. Then there exists a pro-unipotent algebraic group corresponding to \( \mathfrak{w}_0 \), as we may treat \( \mathfrak{gl}_n \) trivially as a projective limit over \( k \) and projective limits commute with products. Hence, we have a \( \text{Exp}(\mathfrak{w}_0) = GL_n \times \text{Exp}(F_1 \mathfrak{w}_0) \), denoted hereafter by \( Aut(O) \). We say \( Aut(O) \) is the group of formal automorphisms.

Our work so far foreshadows the role \( Aut(O) \) will play below in formal geometry. Notice that the set underlying \( Aut(O) \) is equal to \( \mathfrak{w}_0 \) by the Baker Campbell Hausdorff formula. Further, under the order isomorphism exhibited above, formal vector fields of order at least 0 correspond bijectively to elements of \( \kappa[[v_1, \ldots, v_n]] \) of order 1 whose leading coefficients are units in \( \kappa(x) \).
Let us consider automorphisms of the formal disc. Given a vector space of rank \( n \), say \( V \), with dual basis \( v_1, \ldots, v_n \), we say a power series automorphism is a \( n \)-tuple of power series \( (\phi_1, \ldots, \phi_n) \) such that \( \phi_i(v_1, \ldots, v_n) = v_j \), the \( \phi_i \) have zero constant term, and \( \det(Jac_{\phi_1,\ldots,\phi_n}) \) is a unit. A power series automorphism is induced, or defined by, a \( \kappa \)-algebra automorphism

\[
\phi : \kappa[[v_1, \ldots, v_n]] \to \kappa[[v_1, \ldots, v_n]]
\]

such that \( \phi \) preserves the maximal ideal. In particular, elements of \( \text{Aut}(O) \) correspond to power series automorphisms of \( \kappa(x)[[v_1, \ldots, v_n]] \) by construction. In terms of formal geometry, one must recognize that this description imparts an action of \( \text{Aut}(O) \) on the set of formal coordinates centered at \( x \). We shall use this observation below.

Finally, pursuant to the the pro-unipotent Harish-Chandra ansatz, \( \mathfrak{w} \) and \( \text{Aut}(O) \) play the role of the Harish-Chandra pair in formal geometry, as indicated in the following proposition.

**Proposition 4.** The pair \( (\mathfrak{w}, \text{Aut}(O)) \) of the Lie algebra of formal vector fields \( \mathfrak{w} \) and the group of formal automorphisms \( \text{Aut}(O) \) is a Harish-Chandra pair.

**Proof.** All we have to verify is that the adjoint action of \( \mathfrak{w}_0 \) on \( \mathfrak{w} \) is equivalent to the adjoint action of \( \text{Aut}(O) \) on the same. This is clear by construction, cf. chapter 1 construction of the Ad morphism.

We shall refer to this special Harish-Chandra pair as the Gel’fand-Kazhdan pair hereafter.

The Gel’fand-Kazhdan pair is now associated to an arbitrary regular \( n \)-dimensional \( \kappa \)-scheme \( X \), by interpreting \( \mathfrak{w} \) as the completion of the fibre of the tangent sheaf \( T_X \) at \( x \in X \). In terms of Harish-Chandra geometry, the question of whether there exists a \( (\mathfrak{g}, K) \)-structure adapted to the Harish-Chandra pair \( (\mathfrak{w}, \text{Aut}(O)) \) remains unanswered. The \( (\mathfrak{g}, K) \)-structure adopted to this Harish-Chandra pair is the central object of study in formal geometry. The description of it is the topic of the final section in this chapter.
3.3 The Torsor of Formal Coordinates and Transitive Structures

Our first goal is to both define the torsor of formal coordinates and to sketch a proof with references that it is a formal scheme by showing it represents a functor to the category of sets \( \mathcal{E}ns \); that is, to make the assertion the torsor of formal coordinates is a is a formal scheme by demonstrating it is a fine moduli space for the aforementioned functor. The torsor of formal coordinates then emerges in the exposition as the \((g, K)\)-structure adopted to the Gel’fand-Kazhdan pair \((\mathfrak{w}, Aut(O))\).

Our second goal is to introduce transitive structures, which are lifts of the torsor of formal coordinates adopted to transitive Harish-Chandra pairs, in the sense of Harish-Chandra geometry. These lifts may in turn be extended by extensions of Harish-Chandra pairs \((g, K)\) by the Gel’fand-Kazhdan. The extension construction is worthwhile, for our extensions of Harish-Chandra pairs \((g, K)\) by \((\mathfrak{w}, Aut(O))\) are adopted to \((g, K)\)-structures that support vertex algebras as modules \(qua\ (g, K)\)-modules. Later we shall use this technique to construct associated vector bundles with connection on the base space from such vertex algebras. Thus, the torsor of formal coordinates is fundamental to our perspective on the geometric interpretation of certain vertex algebras.

As a set, the torsor of formal coordinates, denoted \(Cor_X\), is defined for \(U \subset X\) to be the set of formal coordinates \(\phi_X\) centered at \(x\). In particular, its fibre over a point \(x \in X\) is the set of systems of formal coordinates at \(x\). Immediately one observes these are nothing more than automorphisms of fibres of the jet scheme, that is, automorphisms of the formal disc about \(x\). Accordingly, viewed as the union of its fibres, this putative \(X\)-scheme consists of pairs \((\phi_x, x)\) and the structure morphism \(\pi : Cor_X \to X\) is given by \((\phi_x, x) \mapsto x\).

The important intuition to have is that \(\phi\) is a formal coordinate when \(U \to X\) is an open neighborhood of \(x \in X\). Accordingly, it is a power series automorphism and this is equivalent to an \(n\)-tuple of power series \(\phi_i\) with zero constant term such that the determinant of their Jacobian is a unit. One has such an \(n\)-tuple because the power series automorphism is completely determined by where it maps the \(v_i\), for \(1 \leq i \leq n\). Such an \(n\)-tuple is therefore
a sequence in \((J^c)^{\times n}\) such that under the composed map
\[
(J^c)^{\times n} \rightarrow (J^c/(J^c)^2)^{\times n} = (\Omega^1_X)^{\times n} \rightarrow \text{det} \Omega^n_X
\]
we have \(\phi_1 \wedge \ldots \wedge \phi_n \neq 0\), where \(\phi_i\) abuses notation to denote the projections of the \(\phi_i\) to \(\Omega^1_X\). Hence, this defines an open subscheme \(\text{Cor}_X \subset (\text{Jet}_X)^{\times n}\) of the \(n\)-fold product of \(\text{Jet}_X\) with itself.

The set description does not exhibit \(\text{Cor}_X\) as an \(X\)-scheme, so for that, we have the following theorem. We adopt the proofs of [Yek], [VdB] to Lie-Rinehart algebroid theoretic perspective by emphasizing the role of the jet algebra together with the pull back of Lie-Rinehart algebroids along schematic morphisms.

**Theorem 8.** The functor \(F : (\text{Sch}/X)^o \rightarrow \mathcal{E}ns\) that assigns an \(X\)-scheme \(Y\) to the set of pairs
\[
\{(x : Y \rightarrow X, \phi_{X/Y}) \mid \phi_{X/Y} : J_{X/Y} \cong \mathcal{O}_Y[[v_1, \ldots, v_n]], \phi_{X/Y}(J^c_{X/Y}) = \mathfrak{m}\}
\]
is representable by a formal scheme \(\text{Cor}_X\), the torsor of formal coordinates.

**Proof.** (Sketch) This theorem is proven in both [Yek] and [VdB]. We denote the \(x^*\text{T}_X\)-jets of a Lie-Rinehart algebra \(x^*\text{T}_X\) obtained by the pull-back of the canonical Lie-Rinehart algebroid \(T_X\) by \(J_{X/Y}\). Pull-backs of Lie-Rinehart algebroids are explained in detail in chapter 5, as well as the construction of corresponding \(L\)-jet, for \(L\) a Lie-Rinehart algebra. We use similar notation for the ideal sheaf \(J^c_{X/Y}\).

So, by hypothesis, we have a morphism of \(\mathcal{O}_Y\)-Lie-Rinehart algebras
\[
x^*\text{T}_X \rightarrow T_Y
\]
and therefore a morphism of universal enveloping algebroids
\[
U_{\mathcal{O}_Y} x^*\text{T}_X \rightarrow U_{\mathcal{O}_Y} T_Y
\]
Dualizing this morphism by applying \(\text{Hom}_{\mathcal{O}_Y}(-, \mathcal{O}_Y)\), we obtain a morphism from \(x^*\text{T}_X\)-jets to \(T_Y\)-jets, viz.
Again, by hypothesis, we have an isomorphism \( \phi_{X/Y} : \mathcal{O}_Y[[v_1, \ldots, v_n]] \to \mathcal{J}_{X/Y} \). This induces a morphism

\[
\mathcal{O}_Y \to \mathcal{J}^c_{X/Y}
\]

\( e^* : \mathcal{O}_Y \to \mathcal{J}_Y \) and by considering the images of the \( v_i \). The images of the \( v_i \) are elements of \( \mathcal{J}^c_{X/Y} \) because \( \phi_{X/Y} \) is a power series automorphism by hypothesis. Then, by duality, this morphism corresponds to an \( \mathcal{O}_Y \)-linear module morphism

\[
x^* U^n_{O_X} \to \mathcal{O}_Y
\]

where \( U^n_{O_X} T_X = U_{O_X} T_X / \mathcal{O}_X \) and the superscript denotes the inductive limit of \( \mathcal{O}_X \)-modules \( \nu T_X (T_X)^n \), cf. chapter 5 for the notation. Thus, by the adjunction \( x^* \dashv x_* \), one has a morphism of \( \mathcal{O}_X \)-modules

\[
U^n_{O_X} T_X \to x_* \mathcal{O}_Y
\]

Now, the former is a coherent \( \mathcal{O}_X \)-module since \( T_X \) is projective. Therefore, we may consider its \( n \)-fold symmetric algebra and the attendant morphism onto the pushforward of \( \mathcal{O}_Y \), viz.

\[
\text{Sym}(U^n_{O_X} T_X)^{\otimes n} \to x_* \mathcal{O}_Y
\]

Geometrically dualizing, this furnishes a schematic morphism

\[
Y \to \text{Spec}(\text{Sym}(U^n_{O_X} T_X)^{\times n})
\]

but the images of the \( v_i \) have non-zero determinant as they are power series isomorphisms by hypothesis, so one actually has a morphism

\[
Y \to \mathcal{C}or_X
\]

that is, an element of the functor of points \( h_{Cor_X} \) as desired.

Reversing this argument word for word provides the converse, therefore, \( \mathcal{C}or_X \) is a fine moduli space for the functor \( \mathcal{F} \) defined above.
There is a nice way to see this result algebraically, also. The coordinate algebra $\Gamma(\mathcal{C}or_X, \mathcal{O}_{\mathcal{C}or_X})$ may be exhibited as a subalgebra of $(\mathcal{J}_X)^{\otimes n}$, cf. [Dolg] localized along a determinant.

The following theorem exhibits $\mathcal{C}or_X$ as a principal $\mathfrak{w}$-space. In the literature, this theorem goes by the name the fundamental theorem of formal geometry.

**Theorem 9.** There exists an isomorphism

$$\mathcal{O}_{\mathcal{C}or_X} \otimes \mathfrak{w} \to T_{\mathcal{C}or_X}$$

restricting on fibres to an isomorphism

$$\mathfrak{w} \cong T_{(x,\phi_x)}\mathcal{C}or_X$$

for $(x, \phi_x) \in \mathcal{C}or_X$.

Furthermore, the inverse of the this isomorphism furnishes a $\mathfrak{w}$-valued 1-form $\nabla_\mathfrak{w} \in \Omega^1(\mathcal{C}or_X; \mathfrak{w})$ satisfying the Maurer-Cartan equation that imparts a flat connection to $\mathcal{C}or_X$.

**Proof. (sketch)** Following [Yek], [GGW], one can write $\mathcal{C}or_X$ as a projective limit of $X$ schemes, the torsors of $k$-th order coordinates $\mathcal{C}or^k_X$, which are given by open subschemes of the $n$-fold product of $k$-th order jet scheme $\mathcal{J}et^k_X$. Specifically, these parameterize isomorphisms of $\phi^k_X : \mathcal{J}_X/\mathcal{J}^k_c \to \text{Sym}^{\leq k}(I_\Delta/I^2_\Delta)$ over an open cover $U \to X$. These are torsors over the unipotent groups $Aut^k_0(O)$ whose nilpotent Lie algebras are $\mathfrak{w}/F_k \mathfrak{w}$. Write $\mathfrak{w}^k_0 = \mathfrak{w}_0/F_k \mathfrak{w}$ with respect to the neighborhoods of zero generating the topology on $\mathfrak{w}$. The $Aut^k_0(O)$-torsor structure $\pi : \mathcal{C}or^k_X \to X$ induces an exact sequence

$$\mathfrak{w}^k_0 \to T_{(x,\phi^k_x)}\mathcal{C}or^k_X \to T_xX$$

which gives $T_{(x,\phi^k_x)}\mathcal{C}or^k_X \cong \mathfrak{w}/F_k \mathfrak{w}$. So, taking the projective limit over $k$ we have $\mathfrak{w} \cong T_{(x,\phi_x)}\mathcal{C}or_X$. Again, from this perspective, $\mathfrak{w}$ appears as a pro-object in the category of ind-Lie algebras. The $\mathfrak{w}$-valued 1-form satisfies the Maurer-Cartan equation by computation [GK], [BR].

\[\square\]
Composition by elements of $Aut(O)$ furnishes an obvious action along the fibres of $Cor_X$, so with the fundamental theorem, we have that the torsor of formal coordinates is a $(\mathfrak{g}, K)$-structure adopted to the Gel’fand-Kazhdan pair $(\mathfrak{w}, Aut(O))$.

**Proposition 5.** The torsor of formal coordinates $Cor_X$ is an $Aut(O)$-torsor. Furthermore, the simply transitive action of the Lie algebra $\mathfrak{w}$ extends the fibre-wise action of $\mathfrak{w}_0 = \text{Lie}(Aut(O))$ on $Cor_X$.

*Proof.* The only item to prove is that the adjoint action is the restriction of the embedding of $\mathfrak{w}_0 \rightarrow \mathfrak{w}$. But since $Aut(O)$ is defined as the pro-algebraic group of the pro-nilpotent Lie algebra $\mathfrak{w}_0$, this follows from the definition of the Ad map given by the correspondence correspondence between pro-nilpotent Lie algebras and their corresponding pro-algebraic groups. 

Finally we define transitive structure of $Cor_X$ from a famous result for transitive Lie algebras discovered by Guillemin and Sternberg [GS]. There is a specialization of the definition of transitivity given in chapter one to the Lie algebra of formal vector fields $\mathfrak{w}$. We say a subalgebra $\mathfrak{w}' \subset \mathfrak{w}$ is transitive if $\mathfrak{w}' \cap \mathfrak{w}_0$ has finite codimension in $\mathfrak{w}$. The result of Guillemin and Sternberg establishes the existence of morphisms of Lie algebras $\mathfrak{g}$ and $\mathfrak{w}$ given the existence of transitive Lie algebras.

**Definition 29.** Let $(\mathfrak{g}, \mathfrak{k})$ be a pair of Lie algebras over $\kappa$ such that $\mathfrak{g}$ is a topological Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ transitive subalgebra. A realization of the pair $(\mathfrak{g}, \mathfrak{k})$ is a Lie algebra homomorphism

$$\mathcal{R} : \mathfrak{g} \rightarrow \mathfrak{w}$$

such that

$$(\text{Axiom I}) \quad \mathcal{R}^{-1}(\mathfrak{w}_0) = \mathfrak{k}.$$ 

A realization is transitive if
The surprising result obtained by Guillemin and Sternberg for Lie algebra pair \((g, \mathfrak{k})\), where \(g\) is a topological Lie algebra and \(\mathfrak{k}\) is a transitive subalgebra is that there exists a realization. A generalization of the following theorem was obtained by Blattner to arbitrary codimension and a computational formula by Draisma in the cases of both finite and arbitrary codimension cf. [Bl],[Drai].

**Theorem 10.** Let \(g\) be a topological Lie algebra and \(\mathfrak{k}\) a transitive subalgebra. Then there exists a transitive realization \(\mathcal{R}\) of the pair \((g, \mathfrak{k})\). Moreover, if \(\mathcal{R}'\) is another such realization, then there exists a system of formal coordinates centered at \(x\) such that

\[
\phi_x \circ \mathcal{R}(\mathfrak{k}) = \mathcal{R}'(\mathfrak{k}) \circ \phi_x
\]

We extend this theorem to the category of Harish-Chandra pairs as follows. Define a Harish-Chandra pair \((g, K)\) to be transitive if \(\mathfrak{k}\) is a transitive subalgebra of \(g\). Then, by the realization theorem, there exists a morphism of transitive Harish-Chandra pairs

\[
\Phi_{\mathcal{R}} = (\mathcal{R}_e, \phi) : (g, K) \rightarrow (w, Aut(O))
\]

unique up to a change of formal coordinate systems centered at \(x\). Given a morphism of transitive Harish-Chandra pairs, the canonical filtration of \(g\) induces maps \(\mathcal{R}_e : g \rightarrow w\) and \(\phi : K \rightarrow Aut(O)\) such that \(F_k g / F_{k+1} g \subset F_k w / F_{k+1} w\) and \(\text{Exp}(F_k g / F_{k+1} g) \subset Aut_k^k(O)\).

Accordingly, we can view \(K\) as a closed subgroup of \(Aut(O)\) and \(g\) as a filtered subalgebra of \(w\). Thus, the morphism of transitive Harish-Chandra pairs is continuous in both the linearly compact topology of the Lie algebras and the linearly compact topology of the pro-unipotent groups.

To define a transitive structure of \(\text{Cor}_X\), we recall that, for a \(K\)-torsor \(S \rightarrow X\) and a subgroup \(H \subset K\), a reduction of the structure group \(K\) to \(H\) is an \(H\)-torsor \(P \rightarrow X\) in tandem with an isomorphism \(P \times_H K \cong S\) of \(K\)-torsors. All of this together motivates the following definition cf. [BR].
Definition 30. Let \((\mathfrak{g}, K)\) be a transitive Harish-Chandra pair and \(\Phi_\mathfrak{g}\) the attendant Harish-Chandra morphism to the Gel’fand-Kazhdan pair. Then we say a \((\mathfrak{g}, K)\)-structure \(\pi : S \to X\) is a transitive structure of \(\mathfrak{C}or_X \to X\) if the following axioms are satisfied.

(Axiom I) \(\pi : S \to X\) is a reduction of the structure group of \(\mathfrak{C}or_X\) to \(K\).

(Axiom II) \(S\) is a principal \(\mathfrak{g}\) space via the restriction of the isomorphism of the fundamental theorem of formal geometry viz.

\[
\mathfrak{g} \cong T_x S
\]

to the filtered subalgebra \(\mathfrak{g} \subset \mathfrak{w}\).

Ultimately our goal is to realize a particular construction in vertex algebras geometrically as sheaves of pro-coherent \(\mathcal{O}_X\)-modules with connections over \(X\). This will entail the construction of an extension of a transitive structure of \(\mathfrak{C}or_X\). This extension is a \((\mathfrak{g}, K)\)-structures such that the Harish-Chandra pair \((\mathfrak{g}, K)\) is an extension of the transitive Harish-Chandra pair generating the transitive structure of \(\mathfrak{C}or_X\). Therefore, transitive structures provide an intermediary space whereby new spaces may be obtained from the torsor of formal coordinates that are general enough to interpret certain vertex algebras geometrically. In particular, this means to exhibit a certain vertex algebra and its module as localizations of \((\mathfrak{g}, K)\)-modules in the sense of chapter 2. Thus, we now turn to describing the exclusive source \((\mathfrak{g}, K)\)-modules in this thesis.
4 Vertex Algebras and Their Modules

Broadly speaking, a vertex algebra is a vector space $V$ and a distinguished vector $|0\rangle$ equipped with an assignment $Y$ of for every vector $v \in V$ to a $\text{End}(V)$-valued Laurent series. The assignment $Y$ is subject to several axioms and satisfies consequences of these that are famously related to the mathematical formulation of physical predictions of 2d-CFT. Therefore, a good motivation to study vertex algebras is as the mathematical objects corresponding to axiomatic 2d-conformal field theory [Huang]. Consequently, from one perspective, their study provides a path whereby mathematicians are able to study rigorously ideas emerging from physics that are otherwise mathematically non-rigorous.

Vertex algebras were discovered by Richard Borcherds [Bo] following the work of Frenkel, Muermann, and Lepowsky in the 1980s who were investigating the Monstrous Moonshine conjecture in representation theory [FML]. Subsequently, they showed the Moonshine module had the structure of a vertex algebra and afterwards, Borcherds settled the conjecture by employing his innovations. Borcherds was awarded a Fields medal for its settlement.

Later, vertex algebras appeared in algebraic geometry vis a vis the geometric Langlands conjecture [BD2] vis a vis as representations of the affine Kac-Moody Lie algebra are well known to possess the structure of a vertex algebra. This aspect of their scope was pursued by several authors investigating the conjecture, e.g. [FBZ], [F], [FG]. The geometric Langlands path of inquiry has substantially expanded their role in algebraic geometry. In particular, through introduction of both chiral algebras and factorization algebras [BD]. The latter are famous nowadays in the works of Costello and Gwilliam [CG] in their investigations of Ads/CFT duality and Costello’s approach to derived geometry via $L_\infty$-spaces. The former are especially well suited to articulating the geometric Langlands conjecture, which relates $\mathcal{D}$-modules on the moduli space of $G$-bundles, $\text{Bun}_G$, on a smooth projective curve, and flat $G^\vee$-bundles on the same, where $G^\vee$ denotes the Langlands dual of the reductive group, $G$. The two categories are equivalent, and conformal vertex algebras are shown to be chiral algebras under these auspices. The author is unaware of any substantial examples of chiral
algebras which do not arise from vertex algebras.

Our treatment of vertex algebras in this chapter is purely expository. We only cover their definition and the definition of their representations, which is seemingly a tradition in practically every paper or book written upon them, and those features necessary to define the Zhu algebra. The Zhu algebra is an associative algebra figured into a categorical correspondence between its representations and those of the vertex algebra from which it is wrought. Although this correspondence is not an equivalence, it is for simple modules. We are interested in its non-simple representations as they are related to representations of certain objects in geometry which in turn generate certain vertex algebras. Accordingly, we will be able to obtain representations for said vertex algebra from geometry.

4.1 Vertex Algebras

A good, albeit vague, intuition for what a vertex algebra is, is that of a meromorphic commutative algebra. As stated above, one can think of a vertex algebra as a vector space \( V \) together a distinguished vector, \(|0\rangle\), thought of intuitively as the unit in this algebra, and the operator \( Y \) is regarded as the multiplication in our putative meromorphic algebra.

Consider that an associative, commutative, unital algebra \( V \) may be defined as a vector space \( V \) together with a degree zero operation \( Y : V \rightarrow \text{End}V \) together with an element \(|0\rangle \in V \) such that, for any \( a \in V \), \( Y(a) \cdot |0\rangle = a \). Furthermore, for any pair \( a, b \in V \), we have \( Y(a)Y(b) = Y(b)Y(a) \). Consequently, the operator \( Y \) is the multiplication of elements in \( V \) and the element \(|0\rangle\) is the unit thereof. Moreover, for any triple, \( a, b, c \in V \) we must also have the associativity law under these auspices, \( id \text{ est, } Y(a) \cdot (Y(b) \cdot Y(c)) = Y(a) \cdot (Y(c) \cdot Y(b)) = Y(c) \cdot (Y(a) \cdot Y(b)) = (Y(a) \cdot Y(b)) \cdot Y(c) \). Hence definition a vertex algebra we provide is meant to proceed according to this analogy. Indeed, our definition is parallel to this definition insofar as one ignores the complexities that arise from allowing singularities into the multiplication, that is to say, the meromorphic aspect of the multiplication.

According to this analogy, let us first examine the conditions whereby a product exists for
Laurent series $Y(a), Y(b), a, b \in V$, above and how we may think of it commuting. Eventually, this is axiomatized as the locality of such operators.

Let $T$ be a $\kappa$-vector space and consider the vector space $T[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$ over $\kappa$ of formal power series in $n$-variables. Elements

$$a(z_1, \ldots, z_n) = \sum_{(i_1, \ldots, i_n) \in \mathbb{Z}^n} a_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n} \in T[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$$

for $a_{i_1, \ldots, i_n} \in T$, are called formal power series. Formal power series can be formally added and formally differentiated. The vector space of formal power series allows arbitrary powers of the $z_i$. This condition distinguishes $T[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$ from the ring of formal Taylor series, where powers are non-negative, and formal Laurent series, where powers are truncated in negative degree.

Notice that the product of $a(z_1, \ldots, z_n)$ and $b(z_1, \ldots, z_n)$ would have coefficients

$$\sum_{i,j \in \mathbb{Z}} z^{i,j} \left( \sum_{k+l=i,j} a_k b_l \right)$$

which in general are not finite sums, so $T[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$ is not closed under multiplication of elements. Indeed, for this reason, the product of formal power series would not be closed as a set under such a naive definition of product. However, the product of formal power series in distinct variables is closed, for the product is an element of $T[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, w_1^{\pm 1}, \ldots, w_m^{\pm 1}]]$.

A key first example is that of the delta function,

$$\delta(z - w) = \sum_{m \in \mathbb{Z}} z^m w^{-m-1} \in k[[z^\pm, w^\pm]]$$

The property that justifies its name is

$$a(z) \delta(z - w) = b(w) \delta(z - w)$$

for $a, b \in T$. In the special case where $a(z) = z$ and $b(w) = w$, this property implies, by induction, that

$$(z - w)^{n+1} \delta_w^n(z - w) = 0$$
that is, \( \delta(z - w) \) is supported on the diagonal of \( \kappa[[z^\pm, w^\pm]] \). We recall a useful lemma which depends upon this example so that we may describe the operator product expansion of vertex operators below.

**Lemma 2.** Suppose \( a(z, w) \in T[[z^{\pm 1}, w^{\pm 1}]] \) satisfies \( (z - w)^N a(z, w) = 0 \) for some \( N \). Then

\[
a(z, w) = \sum_{i=1}^{N} b_i(w) \partial_w^i \delta(z - w)
\]

is the unique decomposition of \( a(z, w) \) in the subspace spanned by the \( b_i(w) \in T[[w^{\pm 1}]] \subset T[[z^{\pm}, w^{\pm}]] \)

Next, we review the Laurent series expansions for formal Taylor series over \( \kappa \). Consider \( a(z, w) \in \kappa[[z, w]] \) and its image in the quotient field \( \kappa((z, w)) \). The quotient field can be topologized in two ways of interest: by taking neighborhoods of 0 to be either powers of \( z \) or powers of \( w \). Completing the quotient field in either topology corresponds to expanding \( a(z, w) \) as a Laurent series in either \( w \) or \( z \), respectively. Analytically, in the former case, one should think of \( z \) as the small variable, and in the latter, \( w \). Continuing the analogy, this corresponds to the expansion in the domains \( |w| > |z| \) and \( |z| > |w| \), respectively.

Now we want to investigate the commutativity of Laurent series in distinct variables. Ultimately, we want to devise a relaxation of a standard commutativity law. So consider the Laurent expansions of \( a(z, w) \) in either variable under both of the inclusions of the completions into the vector space of formal power series \( \kappa[[z^\pm, w^\pm]] \). We shall denote these elements both by \( a(z, w) \) by an abuse of notation. One could ask whether there is a product decomposition of the two expansions

\[
a(z, w) = b(z)c(w) = c(w)b(z)
\]

in \( \kappa[[z^\pm, w^\pm]] \), which is tantamount to inquiring whether the Laurent series expansions of \( a(z, w) \) coincide. In other words, the question of the commutativity of the product of two formal power series in distinct variables is equivalent to whether their Laurent series expansions coincide.
This question is answered positively only by power series contained in the compositum of the topological fields obtained by completions and, as such, emerges as a rather extreme restriction on a potential algebra structure for the set of Laurent series in distinct variables. Yet, instead of insisting that the Laurent series expansions match, we could insist instead that they are equal when their variables are exchanged. According to the above, this is tantamount to the delta function annihilating their commutator, for the condition that $b(z)\delta(z - w) = c(w)\delta(z - w)$ is equivalent to the Laurent series expansions of $a(z, w)$ being equal when the variables $z$ and $w$ are exchanged. We will therefore adopt this perspective for the commutativity of formal power series in distinct variables. We say two formal powers $a(z)$ and $b(w)$ are local whenever

$$\delta(z - w)[a(z), b(w)] = 0$$

The locality condition is pictured in the following diagram.

Let us specialize these considerations in preparation for the definition of a vertex algebra, so let $V$ be a $\kappa$-vector space and consider the space of formal power series in a single variable with $T = \text{End}(V)$. An element of this vector space acts in the obvious way upon $V$. We say such a formal power series $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-j-1}$ is locally nilpotent if, for any $a, v \in V$, then $a_n \cdot v = 0$ for $n$ sufficiently large. The subspace of locally nilpotent formal power series in $\text{End}(V)[[z^\pm]]$ are called fields and the coefficients of the formal variable $z$ are referred to as Fourier modes. Notice that, for $b \in V$, $a(z)b \in \kappa[[z^\pm]]$, so our above discussion of locality
is pertinent to fields. Further, if $V$ is $\mathbb{Z}$-graded, then we say a (homogeneous) field is of conformal dimension $k \in \mathbb{Z}$ if each $a_n$ is a homogeneous linear operator of degree $-n + k$.

We have enough terminology now to define vertex algebras.

**Definition 31.** A vertex algebra is a quadruple $(V, |0\rangle, Y, \mathfrak{D})$, consisting of

(I) a vector space $V$ over $\kappa$

(II) a distinguished vector $|0\rangle$, referred to as the vacuum state

(III) a linear operator, referred to as the state-field correspondence,

$$Y : V \to \text{End}V[[z^{\pm 1}]]$$

assigning to each $a \in V$ a field

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

(IV) an endomorphism $\mathfrak{D}$ of $V$, referred to as the translation operator.

**satisfying the following axioms:**

(Axiom I) $Y(|0\rangle, z) = \text{Id}_V$

(Axiom II) $Y(a, z) \cdot |0\rangle \in V[[z]]$ for every $a \in V$, so that $Y(a, z)|0\rangle|_{z=0} = a$.

(Axiom III) $[\mathfrak{D}, Y(a, z)] = \partial_z Y(a, z)$ for every $a \in V$

(Axiom IV) $\mathfrak{D}(|0\rangle) = 0$

(Axiom V) All fields are local, so for any pair $a, b \in V$, there exists $N$ such that

$$(z - w)^N[Y(a, z), Y(b, w)] = 0$$

Thus, we see the product in our vertex algebra is supported along the diagonal of the affine plane $\mathbb{A}^2$. It is for this reason we think of the product in our algebra as meromorphic. A remark is that one can ignore $\mathfrak{D}$ in the definition of a vertex algebra, as $\mathfrak{D}a = a_{(-2)}|0\rangle$. 64
A vertex algebra is \( \mathbb{N} \)-graded if there exists a decomposition of \( V = \bigoplus_{n \in \mathbb{N}} V^n \) as a graded \( k \)-vector space into weight spaces \( V^n \) of weight \( n \). Furthermore, \( a_n b \in V^{|a| + |b| - (n+1)} \). Often a grading is included in the definition. By an abuse of standard notation, we shall denote the gradation operator of a graded vertex algebra by \( L_0 \), which is the operator that assigns \( a \in V^n \) to \( na \in V \). This abuse of notation reflects the central importance of the Virasoro algebra and conformal vertex algebras [FBZ] whose grading is obtained by the action of the zero Fourier mode of the vertex operator \( Y(\omega, z) \) corresponding to \( \omega \in V \) when \( V \) is a representation of the Virasoro algebra or conformal. Last, we denote the weight for homogeneous \( a \in V^n \) by the symbol \(|a| = n\).

We make vertex algebras into a category by giving the definition of morphisms of these objects.

**Definition 32.** A morphism of vertex algebras \( (V, |0\rangle, Y, D) \) and \( (V', |0\rangle', Y', D') \) is a \( \kappa \)-linear map \( \Psi : V \to V' \) such that

1. \((\text{Axiom 1})\) \( \Psi(|0\rangle) = |0\rangle' \)
2. \((\text{Axiom 2})\) \( \Psi(Y(a, z) \cdot b) = Y'(\Psi(a), z) \cdot \Psi(b) \) for every pair \( a, b \in V \)
3. \((\text{Axiom 3})\) The \( \kappa \)-linear map \( \Psi \) intertwines the translation operators, that is,

\[
[D' \circ \Psi, \Psi \circ D](a) = 0
\]

for every \( a \in V \).

Moreover, if \( V = V' \) we say an endomorphism \( \Psi \) is an automorphism if it is a bijection.

Hereafter, whenever it is clear from the context, we shall denote a vertex algebra simply by \( V \) rather than as a quadruple of data. There is an infinitesimal analogue of an endomorphism, that of a derivation. A derivation of a vertex algebra \( V \) is defined to be an endomorphism \( \Delta \) such that \( \Delta(|0\rangle) = 0 \) and \([\Delta, Y(a, z)] = Y(\Delta(a), z)\), which mimics the standard Leibniz rule. It is immediate that the exponential \( e^\Delta \) of a derivation is an automorphism of \( V \).
One should recognize that the axioms of a vertex algebra include the derivation $\mathfrak{D}$ with this definition. In this way, one may start to think of vertex algebras as algebras with a derivation, or more geometrically, as vector bundles with a connection, by continuing the analogy with commutative algebras.

Morphisms of vertex algebras induce the standard structures associated to objects of an abelian category. Given a subspace $V' \subset V$, we say $V'$ is a vertex subalgebra if $V'$ is an $\mathfrak{D}$ invariant subspace containing the vacuum vector, such that, for every $a, b \in V'$, $Y(a, z) \cdot b \in V'(z))$. A subspace $V'$ is said to strongly generate $V$ if $V$ is spanned by vectors $a^{j_1}_{(-n_1)} \cdots a^{j_r}_{(-n_r)}|0\rangle$ with $a^{j_s} \in V', r \geq 0, j_s \in J, n_s \geq 1$. We say $V$ is finitely strongly generated if the cardinality of $J$ can be taken to be finite. We shall consider certain vertex algebras finitely strongly generated by the weight 0 and 1 spaces extensively below.

Less generally, a $\mathfrak{D}$-invariant subspace $I$ of $V$ is a vertex ideal if $Y(a, z) \cdot b \in I((z))$ for all $a \in I$ and $b \in V$. In particular, the image and kernel of a morphism of vertex algebras are vertex ideals. The skew-symmetry of vertex operators [FBZ] viz.

$$Y(a, z) \cdot b = e^{z\mathfrak{D}}Y(b, -z) \cdot a$$

implies that after interchanging $a$ and $b$, we have isomorphic ideals. Therefore ideals $I$ are two sided so that quotients $V/I$ are well-defined as vertex algebras.

Tensor products of vertex algebras are well-defined. Given vertex algebras $(V, |0\rangle, Y, \mathfrak{D})$ and $(V', |0\rangle', Y', \mathfrak{D}')$, the quadruple

$$(V \otimes \kappa V', |0\rangle \otimes |0\rangle', Y_{V \otimes V'}, \mathfrak{D} \otimes 1 + 1 \otimes \mathfrak{D}')$$

is also a vertex algebra, where, for all $a \in V, a' \in V'$, the state-field correspondence is defined by

$$Y_{V \otimes V'}(a \otimes a', z) = Y(a, z) \otimes Y'(a', z)$$
4.2 Properties of Vertex Algebras

Finally, to complete our analogy with commutative algebras, we need to discuss the associativity property of vertex algebras. This property has several important consequences, and one of its formulations is why vertex algebras are intimately related to theoretical physics.

Already one has that \( Y(a, z)Y(b, w) \cdot c \) and \( Y(b, w)Y(a, z) \cdot c \) are expansions of the same element \( f_{a,b,c} \in V[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}] \) by locality. As mentioned above, this corresponds to the commutativity of the operation \( Y \), but to continue the analogy between vertex algebras and commutative \( \kappa \)-algebras, one should address how \( Y(Y(a)b) \) is related to the former two expansions in order to determine whether the operation \( Y \) is associative in any sense.

It turns out the operator \( Y(Y(a)b) \) is related to the prior pair of expansions by the skew-symmetry property and the identity

\[
e^{w\Xi} Y(a, z)e^{-w\Xi} = Y(a, z + w)
\]

which holds in any vertex algebra. This identity is a consequence of Taylor’s theorem \( e^{z\partial_w} f(w) = f(z + w) \) and the identity

\[
e^{wT}ge^{-wT} = \sum_{n \geq 0} \frac{w^n}{n!} (\text{ad} T)^n g
\]

for \( g \in \mathfrak{g} \), a Lie algebra \( \mathfrak{g} \), and for some \( \Xi \in \text{End}(\mathfrak{g}) \). In particular, for the Lie algebra \( \text{End}(V) \) with bracket the commutator. Accordingly, we have the following proposition. \textit{loc.cit}

**Proposition 6.** In any vertex algebra \( V \), the expressions

\[
Y(a, z)Y(b, w) \cdot c \in V((z))((w))
\]

\[
Y(b, w)Y(a, z) \cdot c \in V((w))((z))
\]

\[
Y(Y(a, z - w) \cdot b, w) \cdot c \in V((w))((z - w))
\]

are expansions of the same element \( f_{a,b,c} \in V[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}] \) for every \( a, b, c \in V \).
By the locality property and the lemma, it is instructive to think of this result as asserting
that
\[ Y(a, z)Y(b, w) = \]
\[ Y(Y(a, z - w)b, w) = \]
\[ = \sum_{n \in \mathbb{Z}} Y(ab, w)(z - w)^{-1} \]
despite the fact the equalities are incorrect. Nonetheless, in this form we obtain the well-known operator product expansion of two fields. The operator product expansion is the special quantity in 2d-conformal field theory vertex algebras express so efficiently.

Operator product expansion allows us to compute the Lie bracket of Fourier modes, which, among other applications, gives Borcherd’s identity. This identity is quite useful for studying \( V \)-modules \( M \) via its Zhu algebra \( A(V) \).

Consider the commutator \( [Y(a, z), Y(b, w)] \). We have by Kac’s lemma that there exist fields \( \gamma_i \) such that
\[ [Y(a, z), Y(b, w)] = \sum_{i=0}^{N-1} \gamma_i(w) \partial_w^i \delta(z - w) \]
therefore, we have \([FBZ]\)
\[ Y(a, z)Y(b, w) = \sum_{i=0}^{N-1} \frac{\gamma_i(w)}{(z - w)^{i+1}} + :Y(a, z)Y(b, w): \]
where \( :Y(a, z)Y(b, w): \) is the normally ordered product of fields emphloc.cit. A computation reveals \( \gamma_i(w) = Y(a_i b, w) \). With this in mind, returning to the commutator, one has
\[ [a_{(m)}, b_{(k)}] = \sum_{n \geq 0} \binom{m}{n} (a_{(n)} b_{(m+k-n)} \]
where by definition, for \( m \in \mathbb{Z} \), one has \( \binom{m}{n} = \frac{m(m-1)\ldots(m-n+1)}{n!} \) for \( n \geq 0 \). More generally, one has the following identity owed to Borcherds which the Lie bracket formula of Fourier modes is a special case.
Proposition 7. *(Borcherds’ Identity)* In any vertex algebra \( V \), and \( a, b \in V \), one has the identity

\[
\sum_{j \geq 0} \binom{m}{j} (a_{(n+j)} b)_{(m+q-|a|-|b|+n+2)} = \\
\sum_{j \geq 0} (-1)^j \binom{n}{j} (a_{(m+n-j-|a|+1)} b_{(q+j-|b|+1)}) - (-1)^n b_{(n+q-j-|b|+1)} a_{(m+j-|a|+1)}
\]

As mentioned, we shall use this important identity to attach an associative algebra \( A(V) \) to \( V \) germane to its representation theory. The second general result we require is a useful proposition for constructing vertex algebras by generators and relations in a manner akin to the Poincare-Birkoff-Witt theorem. Its proof is contained in \([FBZ]\), but is essentially a consequence of induction and Dong’s lemma.

**Proposition 8.** Let \( V \) be a vector space, \( |0\rangle \) a non-zero vector, and \( \mathcal{D} \) an endomorphism of \( V \). Let \( J \) be a countable, ordered set and \( \{a^\alpha\}_{\alpha \in J} \) a collection of vectors in \( V \). Suppose we are also given fields

\[
a^\alpha(z) = \sum_{n \in \mathbb{Z}} a^\alpha_{(n)} z^{-n-1}
\]

such that the following hold:

(I) For all \( \alpha \), \( a^\alpha(z)|0\rangle = a^\alpha + z(\ldots) \).

(II) \( \mathcal{D}(|0\rangle) = 0 \) and \( \mathcal{D}, a^\alpha(z) = \partial_z a^\alpha(z) \) for all \( \alpha \).

(III) All fields \( a^\alpha(z) \) are mutually local.

(IV) \( V \) has a basis of vectors

\[
a_{(n_1)}^{\alpha_1} \cdots a_{(n_r)}^{\alpha_m} |0\rangle
\]

where \( n_1 \geq n_2 \cdots \geq n_r \geq 1 \), and if \( n_i = n_{i+1} \), then \( \alpha_i \leq \alpha_{i+1} \) with respect to the given order on the set \( J \).

Under these assumptions, the above data with the assignment
defines a state-field correspondence and so a vertex algebra structure on $V$. Moreover, if $V$ is $\mathbb{Z}$-graded vector space, $|0\rangle$ has degree 0, the vectors $a^\alpha$ are homogeneous, $\mathcal{D}$ has degree 1, and the fields $a^\alpha(z)$ have conformal dimension $|a^\alpha|$, then $V$ is a $\mathbb{Z}$-graded vertex algebra.

4.3 Representations of Vertex Algebras $V$ and Zhu’s Algebra $A(V)$

In addition to constructing the vertex algebras in the category of pro-coherent $\mathcal{O}_X$-modules via transitive structures in formal geometry, we are equally as interested in constructing its modules in the same category. Hence, we shall review the definition of a representation of a vertex algebra and the Zhu algebra, an associative algebra which is a partial invariant of representations of vertex algebras. In particular, [ACM] relates the Zhu algebra to another associative algebra familiar in geometry, that of the algebra of differential operators, and we shall utilize this relationship with certain vertex algebras below.

**Definition 33.** Let $V$ be a vertex algebra and $M$ a $\kappa$-vector space. We say $M$ is a representation of $V$ or a weak $V$-module if $M$ is furnished with a state-field correspondence $Y_M : V \rightarrow \text{End}(M)[[z^\pm]]$, where

$$Y_M(a, z) = \sum_{n \in \mathbb{Z}} a^M_n z^{-n-1}$$

and $a^M_n \in \text{End}M$ satisfying the following axioms:

(Axiom I) $Y_M(|0\rangle, z) = \text{Id}_M$

(Axiom II) $Y_M(\mathcal{D}a, z) = \partial z Y_M(a, z)$

(Axiom III) All fields are mutually local so $(z - w)^N[Y_M(a, z), Y_M(b, w)] = 0$ for $N$ sufficiently large.
(Axiom IV)

\[ Y_M(a, z)Y_M(b, w) \cdot c \in M((z))(w) \]
\[ Y_M(b, w)Y_M(a, z) \cdot c \in M((w))(z) \]
\[ Y_M(Y(a, z - w) \cdot b, w) \cdot c \in M((w))(z - w) \]

are expansions of the same element \( f_{a,b,c} \in M[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}] \) for every \( a, b \in V \) and \( c \in M \).

We have the Borcherd’s identity for modules

\[
\sum_{j \geq 0} \binom{m}{j} (a_{(n+j)b}^M)_{(m+q-|a|-|b|+n+2)} = \\
\sum_{j \geq 0} (-1)^j \binom{n}{j} (a_{(m+n-j-|a|+1)}^M b_{(q+j-|b|+1)}^M - (-1)^n b_{(n+q-j-|b|+1)}^M a_{(m+j-|a|+1)}^M)
\]

We shall consider representations of \( V \) from two perspectives, one of which is the point of view of the Zhu algebra. The second is perspective is peculiar to that of certain vertex algebras below, as it depends upon a specific Lie algebra involved in its construction. The Zhu algebra, however, is the more general point of view.

Given a vertex algebra \( V \), the construction of the Zhu algebra is motivated by trying to construct \( V \)-modules \( M \) in terms of representations of a more classical construction, such as an associative algebra. In order to find such an associative algebra, Zhu [Zhu] defined an operation \( \ast \) such that the assignment of states \( a \in V \) to Fourier modes \( a_{(j)}^M \in \text{End}(M) \) of \( M \) was an associative algebra map. Moreover, by restriction, one has an assignment of states of \( V \) to endomorphisms of the weight 0 space when \( M \) is graded, viz. \( a \mapsto a_{(0)}^M \).

So, assume \( M \) is \( \mathbb{N} \)-graded \( V \)-module, then this algebra morphism is obtained by the Borcherd’s identity for modules by setting \( m = |a| \) and \( n = -1 \). In fact, when both sides are degree 0 morphisms, we have

\[
\sum_{j \geq 0} \binom{|a|}{j} (a_{(-1+j)b})^M_{(j)} = \sum_{j \geq 0} (a_{j}^M b_{j}^M + b_{j}^M a_{j+1})^M_{(j)}
\]
Restrict this relation to $M^0$ and one sees that the sum $\sum_{j=0}^{\lfloor a \rfloor} \binom{\lfloor a \rfloor}{j} (a_{(j-1)} b)_0^M$ is equal to $a_{(0)}^M b_{(0)}^M$ by the Borcherd’s relations. Therefore, to obtain an algebra morphism from $V$ to $\text{End}(M^0)$, loc.cit defined the operation

$$a \star b := \sum_{j=0}^{\lfloor a \rfloor} \binom{\lfloor a \rfloor}{j} (a_{(j-1)} b)$$

called the Zhu product.

The operation $\star$ is in general not associative on $V$, but Zhu has shown the existence of a subspace $V \star \mathfrak{d} V \subset V$ which is two-sided with respect to $\star$ and such that

$$a \star (b \star c) - (a \star b) \star c \in V \star \mathfrak{d} V$$

Accordingly, so that $\star$ becomes associative, the Zhu Algebra of $V$ is then quotient

$$A(V) := V/V \star \mathfrak{d} V$$

It is well known [Zhu], [Ros] that, for an $\mathbb{N}$-graded $V$-module $M$, its weight 0 space $M^0$ is a module over the associative algebra $A(V)$. Furthermore, there is a functorial correspondence between representations of $A(V)$-modules and modules over $V$ whose weight 0 space is an $A(V)$ that restricts to a bijective correspondence between the same for simple modules loc.cit. In order to prepare the use of the Zhu algebra below in this regard, we recall the left adjoint of the functor which assigns a $V$-module $M$ to an $A(V)$-module $M^0$. We stress that the correspondence between $V$-modules and $A(V)$-modules is far from an equivalence of categories.

Observe that the Borcherds identity canonically attaches a Lie algebra $\mathfrak{g}(V) \subset \text{End}(V)$ to a vertex algebra $V$ whose bracket is given by the Borcherds identity. If $V$ is a graded vertex algebra, this Lie algebra $\mathfrak{g}(V)$ is graded as well. In particular, $\mathfrak{g}(V)_0 \to A(V)$ is a Lie algebra homomorphism. Therefore, if $M^0$ is a module over $A(V)$, we have a $\mathfrak{g}(V)$ module by induction or pullback, viz.

$$M = U(\mathfrak{g}(V)) \otimes_{U(\mathfrak{g}(V)_0)} M^0$$
Dividing by the submodule of Fourier modes of \( Y(a_{(-1)}b, z) - Y(a, z)Y(b, z) \) : one obtains an \( \mathbb{N} \)-graded \( V \)-module \( M \) whose weight 0 space is \( M^0 \). We shall return to this correspondence in chapter five for the special case of what we define in that chapter to be a vertex \( \mathcal{O} \)-algebra.
5 The Vertex $\mathcal{O}$-Algebra $\mathfrak{Y}_E$ and the Vertex Bundle $\mathcal{V}_X(\mathcal{E})$

In this final chapter, we present the algebro-geometric interpretation of anchored sheaves that we mean to employ to generate a vertex algebra suitable for the geometric construction of bundles of weak modules over a vertex algebra. One may think of an anchored module as an $\mathcal{O}_X$-module together with a sheaf morphism whose codomain is the tangent sheaf. This construction is related to the sheaf of Chiral Differential Operators [GMS] in the case when the underlying Lie-Rinehart algebra is the global sections of the tangent sheaf of the underlying space, however, we do not globalize our construction in the same manner as loc.cit. We prefer to undertake our constructions locally in coordinates and globalize via the implements of the formal geometry we have developed in this thesis. Moreover, we do not make any exactness assumptions on the weight 1 space as an anchored module in our construction, but instead are disposed of the generality of transitivity. Geometrically, one may regard this as the distinction between the smooth setting and that of a foliation.

There is a less standard, perhaps, construction of such a suitable vertex algebra given by Li and Yamaskulna, [LiYam]. Indeed, loc.cit remarks that the construction is left to the reader in [GMS] and in the view of those authors, their construction is necessary, so we adhere to their rigor throughout. Their construction works especially well with the construction of vertex algebras over Courant-Dorfman algebras obtained from Lie-Rinehart algebras than the construction of a vertex algebra from the aforementioned global sections of the tangent sheaf, as it is more general insofar as no "chiralizability" hypothesis is necessary. In particular, we consider Courant-Dorfman algebras that are doubles of Lie-Rinehart algebras in the sense of Xu, Liu, and Weinstein [LWX], referred to as Lie bialgebroids in loc.cit. We refer to the vertex algebra generated in weight spaces 1 and 0 by the aforementioned Courant-Dorfman and commutative structure ring respectively as the vertex $\mathcal{O}$-algebra.

Once we obtain the vertex $\mathcal{O}$-algebra from this construction, we then exhibit it as a Harish-Chandra module over the Harish-Chandra pair $(\mathfrak{w}_E, \text{Aut}(E, O))$. This pair is an extension of a transitive Harish-Chandra pair by the tautological Harish-Chandra pair.
(\text{Der}(\mathcal{E}), \text{Aut}(\mathcal{E}))$ of structure ring-module derivations and automorphisms, respectively. This is a consequence of a combination of results pertaining to the grading preserving automorphism group $[\text{Dong}], [\text{LiYam2}]$ and its associated Lie algebra of derivations. The purpose of this demonstration is to form the associated bundle of the vertex $\mathcal{O}$-algebra with respect to a $(\mathfrak{g}, K)$-structure adopted to the Harish-Chandra pair $(\mathfrak{w}_E, \text{Aut}(E, O))$ extending a transitive structure of the torsor of formal coordinates $\mathcal{C}or_X$. At this stage, the geometric interpretation of the vertex $\mathcal{O}$-algebra follows from the localization construction of Harish-Chandra geometry.

Last, there is a relationship between modules over the vertex $\mathcal{O}$-algebra and locally nilpotent representations of the loop algebra $\mathfrak{c}$ underlying its construction $[\text{DLM}]$. In particular, such locally nilpotent representations of the loop algebra are related to representations of the underlying Lie-Rinehart algebra via the Zhu algebra. These modules obtain the structure of vertex $\mathcal{O}$-algebra representations by grade preserving automorphisms of the same, hence they are Harish-Chandra modules $(\mathfrak{w}_E, \text{Aut}(E, O))$ in their own right. Consequently, one may form the associated bundle of these modules via localization, as well. In general, such modules are weak with respect to the vertex $\mathcal{O}$-algebra, therefore we provide a geometric interpretation of non-necessarily conformal modules of a vertex algebra, as desired.

### 5.1 Lie-Rinehart Algebroids and their Representations

In this section, we expound upon the algebraic objects and their corresponding sheafifications underlying our construction of a vertex bundle and its modules over an affine $\kappa$-scheme $X$. This digression entails an explanation of both Lie-Rinehart algebroids and Courant-Dorfman algebroids, together with their representations according to a general heuristic. Our terminology is intended to reflect some of the different conventions in algebraic geometry versus those of differential geometry. We treat both Lie-Rinehart algebroids and Courant-Dorfman algebroids and their corresponding algebraic counterparts under the rubric of anchored sheaves, respectively, anchored modules; that is, coherent $\mathcal{O}_X$-modules equipped with a sheaf mor-
phism to the tangent sheaf $T_X$, respectively $R$-modules with an $R$-module morphism to the module of derivations $\text{Der}_\kappa(R)$.

Let $R$ be a finitely generated commutative $\kappa$-algebra.

**Definition 34.** Let $L$ be an $R$-module. We say $L$ is a Lie-Rinehart algebra over $R$ if $L$ is equipped with a $\kappa$-linear Lie algebra bracket

$$[\ , 
\ ] : L \otimes_\kappa L \to L$$

such that

(Axiom I) there exists a morphism of $R$-modules

$$\rho : L \to \text{Der}_\kappa(R)$$

which is called the anchor map.

(Axiom II) the anchor map is a morphism of $\kappa$-Lie algebras

$$\rho([\lambda, \lambda']) = [\rho(\lambda), \rho(\lambda')]$$

for any $\lambda, \lambda' \in L$ and where the bracket on the right is the commutator in $\text{Der}_\kappa(R)$

(Axiom III) the anchor map satisfies the following compatibility between the $\kappa$-Lie algebra and $R$-module structures that exist upon $L$

$$[\lambda, r\lambda'] = r[\lambda, \lambda'] + \rho(\lambda')(r)\lambda$$

for any $\lambda, \lambda' \in L$ and $r \in R$.

We denote a Lie-Rinehart algebra over $R$ by the pair $(R, L)$

Lie-Rinehart algebras form categories over both the same algebra and distinct algebras. We give the general definition of morphisms and take the case where the underlying algebras are equal as a special case thereof. We formulate the definition so that it fits into a broader pattern of morphisms of anchored modules.
Definition 35. Let \((R, L)\) and \((R', L')\) be Lie-Rinehart algebras over \(R\) and \(R'\), respectively. An Lie-Rinehart morphism \((\varphi, \ell) : (R, L) \Rightarrow (R', L')\) is a pair consisting of

(Axiom I) a \(\kappa\)-algebra morphism \(\varphi : R \to R'\)

(Axiom II) a \(\kappa\)-Lie algebra morphism \(\ell : L \to L'\)

(Axiom III) an additive map \(\ell : L \to L'\) such that \(\ell(re) = \varphi(r)\ell(e)\)

(Axiom IV) \(\varphi(\rho(\lambda)(r)) = \rho'(\ell(\lambda))(\varphi(r))\)

The definition of Lie-Rinehart algebra we have given is somewhat general, so hereafter we shall impose the additional hypothesis that \(L\) is a finitely generated projective \(R\)-module. Of course this hypothesis is not the most general, but it is our intention to imitate the Lie algebroids of differential geometry, whose global sections are finitely generated projective modules, by Swan’s theorem, in the setting of algebraic geometry. The correspondence between finitely generated projective modules and locally free sheaves or geometric vector bundles is well known [Har]. Therefore, the object in algebraic geometry analogous to that of differential geometry that we shall consider is a sheaf of \(\mathcal{O}_X\)-modules \(\mathcal{L}\) over a ringed space \((X, \mathcal{O}_X)\) such that \((\mathcal{O}_X(U), \mathcal{L}(U))\) is a Lie-Rinehart algebra for any open subset \(U \subset X\). We call such a coherent \(\mathcal{O}_X\)-module a Lie-Rinehart algebroid. In particular, for an affine scheme \(X = \text{Spec}(R)\), the category of Lie-Rinehart algebras and the category Lie-Rinehart algebroids are equivalent.

In order for Lie-Rinehart algebroids to really be geometric, they must pull-back along morphisms of schemes \(\phi : Y \to X\), or, alternatively, satisfy descent for \(Y\)-points of \(X\). First, let us consider the algebraic perspective given a morphism of \(\kappa\)-algebras \(\varphi : R \to R'\). Let \((R, L)\) be a Lie-Rinehart algebra, then we define the pull-back of \((R, L)\) to \(R'\) to be the Lie-Rinehart algebra \((R', \varphi^\dagger L)\) as the kernel of the map

\[\Xi : (R' \otimes_R L) \oplus \text{Der}_\kappa(R') \to \text{Hom}_\kappa(R, R')\]

defined by \(\Xi(r' \otimes \lambda, \xi)(r) = r'\varphi^*(\lambda(r) - \xi(\varphi^*(r)))\) on \(r \in R\).
The bracket is given by the formula
\[
[(r'_1 \otimes \lambda_1, \xi_1), (r'_2 \otimes \lambda_2, \xi_2)] = (r'_1 r'_2 \otimes [\lambda_1, \lambda_2] + \xi_1 (r'_2) \otimes \lambda_2 - \xi_2 (r'_1) \otimes \lambda_1, [\xi_1, \xi_2])
\]

In particular, one has the pull-back of \((R, L)\) along the localization map \(\phi_p : R \to R_p\), furnishing descent on \(X = \text{Spec}(R)\). More generally, we have the following pull-back of Lie-Rinehart algebroids.

Let \(\phi : Y \to X\) be a morphism of schemes, and \(L\) a Lie-Rinehart algebroid. We define the pull-back Lie-Rinehart algebroid \(\phi^*L\) to be the fibre product with respect to \(O_Y\)-module morphism \(\phi^*\rho : \phi^*L \to \phi^*T_X\) and the derivative \(d\phi : T_Y \to \phi^*T_X\) viz.

\[
\begin{array}{ccc}
\phi^*L & \xrightarrow{\rho^*} & \phi^*T_X \\
\downarrow \phi^* & & \downarrow \phi^*T_X \\
T_Y & \xrightarrow{d\phi} & \phi^*T_X
\end{array}
\]

The definition of the anchored module structure is predicated upon that of the underlying Lie-Rinehart algebras. The anchor, as above, is furnished by projection onto the second factor in the fibre product. This construction provides a functor from the category of Lie-Rinehart algebroids on \(X\) to the category of the same on \(Y\), given a schematic morphism \(\phi\).

A Lie-Rinehart algebra \((R, L)\) may be expressed in terms of a graded objects familiar in geometry. As a graded \(\kappa\)-vector space, this object is equal to
\[
\Lambda^\bullet L^\vee = \bigoplus_{p=0}^{\dim_R(L)} \Lambda^p L^\vee
\]
where \(L^\vee = \text{Hom}_R(L, R)\) is the \(R\)-module dual of \(L\) and \(\Lambda^0 L^\vee = R\). This object is denoted by \(\Omega^\bullet(L)\) and is called the \(L\)-\textit{de Rham complex}. One regards the \(p\)-th graded components as anti-symmetric \(R\)-linear maps from the \(p\)-fold product of \(L\) with itself to \(R\). As usual, \(\Omega^\bullet(L)\) is furnished with a degree 1 \textit{de Rham differential} \(d_L : \Lambda^p L^\vee \to \Lambda^{p+1} L^\vee\), where for
\[ \eta \in \bigwedge^p L^\vee \text{ and any } (\lambda_1, \ldots, \lambda_{p+1}) \in L^{\times(p+1)} \text{ the element } d_L(\eta) \in \bigwedge^{p+1} L^\vee \text{ is given by the formula} \]

\[
d_L(\eta)(\lambda_1, \ldots, \lambda_{1+p}) = \\
\sum_{i=0}^{p+1} (-1)^{i+1} \rho(\lambda_i)(\eta(\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \lambda_{p+1})) \\
+ \sum_{i<j} (-1)^{i+j} \eta([\lambda_i, \lambda_j], \lambda_1 \ldots \hat{\lambda}_i \ldots \hat{\lambda}_j, \ldots, \lambda_{p+1})
\]

Whenever the Lie bracket of the Lie-Rinehart algebra \((R, L)\) satisfies the Jacobi identity-which will always be the case for us-the de Rham differential is an actual differential for it satisfies the equation \(d_L^2 = 0\).

Observe the \(L\)-de Rham complex dualizes the anchor map to a morphism of differentially graded algebras \(\rho^\vee : \Omega_R \to \Omega^\bullet(L)\), where \(\Omega_R := \text{Hom}_R(\text{Der}_\kappa(R), R)\). In particular, one may recover the original Lie-Rinehart algebra structure from this co-anchor via \(d_L(r)(\lambda) = \rho(\lambda)(r)\) and, for an \(R\)-basis \(\{\lambda_i\}\) of \(L\), we have

\[
d_L(\eta_k)(\lambda_i, \lambda_j) = -\eta_k([\lambda_i, \lambda_j])
\]

where \(\lambda_i \in L\) and \(\eta_k \in L^\vee\). Therefore, the pair \((\Omega^\bullet(L), d_L)\) recapitulates the original Lie-Rinehart algebra \((R, L)\).

We shall encounter two distinct categories of anchored modules, so we adopt the following perspective on their representations. Given a type of anchored module (or its sheafification), we shall define modules over the same according to the following principle: if \(M\) is an anchored \(R\)-module, then a \(M\)-module is an \(R\)-module \(F\) such that \(M \oplus F\) is of the same type of anchored module as \(M\), it contains \(M\) as a sub-object in the appropriate category, and contains \(F\) as an abelian ideal in an appropriate sense. Furthermore, morphisms of \(M\)-modules according to this heuristic are pairs of morphisms \(M \oplus F \to M' \oplus F'\) mapping \(M\) to \(M'\) and \(F\) to \(F'\) satisfying certain compatibility conditions.
Immediately, one recognizes that with this heuristic, the definition of Lie-Rinehart algebra may be interpreted to mean $R$ is itself an $L$-module. Specifically, we will refer to this $L$-module $R \oplus L$ as the canonical $L$-module. Moreover, we can re-interpret Lie-Rinehart morphisms as morphisms of canonical Lie-Rinehart modules.

A key construction from the theory of Lie-Rinehart algebroids and Lie-Rinehart algebras is that of the universal enveloping algebra of the canonical $L$-module. This construction determines a functor from the category of Lie-Rinehart algebras to associative $\kappa$-algebras, and bears much the same relationship to representations of $(R, L)$ as the classical universal enveloping algebra bears to that of its underlying Lie algebra.

One defines for the $R$-module structure on the canonical $L$-module a semi-direct product Lie algebra structure with the Lie bracket

$$[r + \lambda, r' + \lambda'] := (\rho(\lambda)(r') - \rho(\lambda')(r)), [\lambda, \lambda']$$

Let $U(R \oplus L)$ denote its universal enveloping algebra in the sense of Lie algebras. We embed $R$ and $L$ under the maps $\iota_R$ and $\iota_L$ in $U(R \oplus L)$ in the standard way, viz.

$$\iota_R(r)\iota_L(\lambda) = \iota_L(r\lambda)$$

and

$$\iota_L(\lambda)\iota_R(r) - \iota_R(r)\iota_L(\lambda) = \iota_R(\rho(\lambda)(r))$$

These embeddings are injective by our hypothesis that $L$ is projective. Write $U_RL^-$ for the subalgebra generated by the image of these embeddings. Let $r_1 \in R$ and $r_2 + l \in R \oplus L$ and consider the two sided ideal $I_R \subseteq U_RL^-$ generated by the differences $(r_1(r_2 + l))^- - r_1^- (r_2 + l)^-$, where $-$ denotes projection from $U(R \oplus L)$ onto $U_RL^-$. Then we define the quotient

$$U_RL = U_RL^- / I_R$$

to be the universal enveloping algebra of the Lie algebroid $L$.

The universal enveloping algebra enjoys the following universal mapping property. If $Z$ is any unital $k$-algebra and $\nu_R : R \rightarrow Z$ is a homomorphism of $k$-algebras, $\nu_L : L \rightarrow Z$ a homomorphism of $\kappa$-Lie-algebras, respecting both the $R$-module structure and bracket, viz.
\[ \nu_R(r)\nu_L(\lambda) = \nu_L(r\lambda) \]

and

\[ \nu_L(\lambda)\nu_R(r) - \nu_R(r)\nu_L(\lambda) = \nu_R(\rho(\lambda)(r)) \]

then there exists a unique homomorphism of unital \(\kappa\)-algebras \(\Phi : \mathcal{U}_R L \to Z\) such that the obvious compatibilities of morphisms, \emph{id est} \(\Phi \circ \iota_R = \nu_R\) and \(\Phi \circ \iota_L = \nu_L\), hold. In particular, this shows there exists a unique homomorphism \(\Phi : \mathcal{U}_R L \to \text{Der}_k(R)\) extending the anchor map of \(L\).

A \(\mathcal{U}_R L\)-module \([BB]\) is an \(R\)-module \(M\) together with an action of \(L\), that is, a Lie algebra map \(L \to \text{End}(M)\) such that \(\rho(\lambda)(rm) = \rho(\lambda)(r)m + r(\lambda m)\) and \((r\lambda)m = r(\lambda m)\). It follows immediately \(\mathcal{U}_R L\)-modules are equivalent to anchored \(L\)-modules in the above sense by the universal mapping property of \(\mathcal{U}_R L\).

The associative \(k\)-algebra \(\mathcal{U}_R L\) is a filtered associative algebra. One assigns degrees 0 and 1 to the images of \(R\) and \(L\) under \(\iota_R\) and \(\iota_L\), respectively, in \(\mathcal{U}_R L\), and considers the filtration generated by \(n\)th-powers of the ideal \(\iota_L(L)\) as \(R\)-modules. Elements of \(\iota_L(L)^n\) are then assigned degree less than or equal to \(n\), viz.

\[ \iota_L(L)^0 = R \subset \iota_L(L)^1 = L \subset \ldots \subset \iota_L(L)^n \subset \ldots \]

We have \(\mathcal{U}_R L \cong \text{colim}_n \iota_L(L)^n\) with respect to this filtration. Rinehart has shown \([Rin]\) that under the hypothesis that \(L\) is a projective \(R\)-module \(\mathcal{U}_R L\) satisfies a PBW theorem, namely the commutative graded algebra associated to this filtration is isomorphic to the symmetrization of \(L\), viz.

\[ \text{gr}\mathcal{U}_R L = \bigoplus_{n \geq 0} \iota_L(L)^{n+1}/\iota_L(L)^n \cong \text{Sym}(L) \]

Next we consider the \(R\)-module dual of the universal enveloping Lie algebroid of \(L\), the so-called \(L\)-jets of \(R\), which are, by definition, \(\text{Hom}_R(\mathcal{U}_R L, R) := J_R L\). The \(L\)-jets of \(R\) exhibit the structure of a Hopf algebroid, which, geometrically, corresponds to a formal scheme by duality. Let us recall the Hopf algebroid structure of \(J_R L\).
A (pro-)Hopf algebroid \((R, \Gamma)\) over a commutative ring \(k\) is a cogroupoid object in the category of commutative \(k\)-algebras. This means a pair of \(k\)-algebras \((R, \Gamma)\), where \(\Gamma\) is allowed to be a pro-\(R\)-algebra, called objects and arrows respectively, with structure maps \((\sigma, \tau) : R \rightrightarrows \Gamma\), the source and target, \(\Delta : \Gamma \to \Gamma \otimes \Gamma\) the coproduct, \(\epsilon : \Gamma \to R\) the counit, and \(i : \Gamma \to \Gamma\), the inverse. The arrows \(\Gamma\) have the structure of both a left and a right \(R\)-module over the objects, and the tensor product is formed with respect to this bimodule structure. In case \(\Gamma\) is a pro-\(R\)-algebra, each of these structures maps is computed in the appropriate pro-category. These structure maps satisfy the following compatibilities.

The counit is a cosection of both the source and target, that is, \(\epsilon \circ \sigma = \epsilon \circ \tau = \text{Id}_R\). The coproduct with the identity of \(\Gamma\) is the identity, that is, \((\text{Id}_\Gamma \otimes \epsilon) \Delta = (\epsilon \otimes \text{Id}_\Gamma) \Delta = \text{Id}_\Gamma\). The coproduct is coassociative, \((\text{Id}_\Gamma \otimes \Delta) \circ \Delta = (\Delta \otimes \text{Id}_\Gamma) \circ \Delta\). Composition with the inverse exchanges source and target, that is, \(i \circ \sigma = \tau\) and \(i \circ \tau = \sigma\). Further, the inverse is idempotent, \(i^2 = \text{Id}_\Gamma\). Last, we require that the inverse is two sided, which is tantamount to the existence of maps inducing the commutativity of the following diagram

\[
\begin{array}{ccc}
\Gamma & \xleftarrow{\sigma} & \Gamma \\
\downarrow & & \downarrow \\
\Gamma \otimes_k \Gamma & \xleftarrow{\Delta} & \Gamma \\
\downarrow & & \downarrow \\
\Gamma \otimes_R \Gamma & \xleftarrow{\epsilon} & R \\
\end{array}
\]

A Hopf Algebroid can be viewed as the object that represents the functor from commutative \(k\)-algebras to groupoids such that the sets \(\text{Hom}_k(R, B)\) and \(\text{Hom}_k(\Gamma, B)\) are the objects and arrows, respectively, satisfying the axioms and compatibilities above for some test commutative \(k\)-algebra, \(B\). Again, when \(\Gamma\) is a pro-object, the corresponding functor is pro-representable instead.

Of course our main example of a Hopf algebroid is the pair \((R, J_R L)\) of \(R\) and its \(L\)-jets. As mentioned already, \(J_R L\) is a commutative, associative algebra structure with two \(R\)-module structures, \((s, t) : R \rightrightarrows J_R L\), regarded as the source and target maps, respectively, dual to
the left and right $R$-module structures of $U_R L$. These are $s : R \to J_R L$, $s(r) = (u \mapsto r \epsilon(u))$ and $t : R \to J_R L$, $t(r) = (u \mapsto u(r))$, for $r \in R, u \in U_R L$. There exists a comultiplication

$$
\Delta : J_R L \to J_R L \otimes_{s,R,t} J_R L
$$

with respect to $R$-$R$-bimodule structure furnished by $s$ and $t$; a counit

$$
\epsilon : J_R(L) \to R
$$

$$
\epsilon(\zeta) = \zeta(1)
$$

where "1" is the unit of $U_R L$, $\zeta \in J_R L$; and an antipode

$$
t : J_R(L) \to J_R(L)
$$

that exchanges the source and target maps.

These maps satisfy the following compatibilities

(I) $\Delta$ is coassociative

(II) $\epsilon \circ s = \epsilon \circ t = \text{Id}_R$

(III) for all $\zeta \in J_R L$, we have $\sum_\zeta (s \circ \epsilon)(\zeta(1))\zeta(2) = \zeta = \sum_\zeta (t \circ \epsilon)(\zeta(1))\zeta(2)$

(IV) for all $\Phi \in J_R L$, we have $\sum_\zeta t(\zeta(1))\zeta(2) = (t \circ \epsilon)(\zeta)$ and $\sum_\zeta \zeta(1)t(\zeta(2)) = (s \circ \epsilon)(\zeta)$

(V) $t_2 = \text{Id}_{J_R L}$

The increasing filtration induced by powers of the ideal of $L$ in $U_R L$ corresponds to an adic filtration with respect to the ideal $J_c$ given by the kernel of the counit map, which in turn is equivalent to the decreasing filtration given by $J_R^p = R \oplus J_c^n$. Consequently, as $L$ is projective, $J_R L \cong \lim_n J_R^p L$ is a complete topological ring with respect to the $J_c$-adic topology. The $J_c$-adic filtration justifies the dual formulation of the PBW theorem for $U_R L$, viz.,

$$
\text{gr}(J_R L) \cong \text{Sym}(L^\vee)
$$

Locally, since $L$ is projective, one may lift a basis of $J_c^1/J_c^2 \cong L^\vee$, say $v_1, \ldots, v_n$ to obtain an isomorphism
In this thesis, such a local isomorphism is treated as formal coordinates when \( L = T_X \), the tangent sheaf.

The universal enveloping algebra \( U_R L \) imparts even more structure to \( J_R L \) by determining connections. Write \( R_s \), respectively, \( R_t \), for the images of \( R \) under \( s \) and \( t \), then under these prescriptions we may regard \( J_R L \) as a \( R_s-R_t \)-bimodule. With respect to these structures, there are two distinct commuting actions of \( L \) itself by derivations, denoted \( \nabla^s \) and \( \nabla^t \), respectively. These are given by

\[
\nabla^s_\lambda(\zeta)(u) = \lambda(\zeta(u)) - \zeta(\lambda u)
\]

\[
\nabla^t_\lambda(\zeta)(u) = \zeta(u(\lambda))
\]

Both of these actions are flat connections on \( J_R L \) as \( L \)-modules with respect to either \( s \) or \( t \).

The connection \( \nabla^s_\lambda \) is referred to as the Grothendieck connection in the literature. It follows that \( J_R L \) is then a \( U_R L \)-module. These connections equip the corresponding geometric Lie algebroid \( V(L^\vee) \) with a Poisson structure, so that we may view this bundle as analogous to a \( D_X \)-module, where by analogy, \( U_R L \) plays the role of sheaf of differential operators. If one has \( L = \text{Der}_\kappa(R) \), that is, the global sections of the tangent sheaf \( T_X \), then these perspectives are equivalent. Furthermore, one can show that \( J_R(\text{Der}_\kappa(R)) := J_X \) is isomorphic to the structure sheaf of the completion \( X \times X \) along the diagonal \( \Delta(X) \) [CVdB], [Kow], thus rendering this construction equivalent to the standard construction of the jet scheme in the literature in chapter 3. In summation, these structures render

\[
\text{Spec}(J_R L) = \text{Jet}_X(\mathcal{L})
\]

with the structure of a groupoid object in the category of formal \( X \)-schemes, such that \( J_X \mathcal{L}_{\text{red}} \cong X \).

Two well known examples of Lie-Rinehart algebras are the tangent sheaf itself, \( T_X \), with anchor map the identity. Second, given a \( K \)-torsor \( \pi : S \to X \), the Atiyah bundle \( \text{At}(S) \) associated to the same is a Lie-Rinehart algebroid. The anchor \( \rho \) is taken to be the projection of horizontal lifts of vector fields to \( S \).
Now we turn to the definition of a Lie bialgebroid \([LWX]\) and inquire what is the structure of the double associated to it. First, observe that if \(L^\vee\) is itself furnished with a Lie-Rinehart algebra structure \((R, L^\vee)\) over \(R\), then we obtain *mutatis mutandi* its \(L^\vee\)-de Rham complex \(\Omega^\bullet(L^\vee)\), with de Rham differential denoted \(d_{\vee}\). We say that \((L, L^\vee)\) is a *Lie-Rinehart bialgebra* over \(R\) if \(L \cong L^\vee\) and if \(d_{\vee}\) is a derivation of the Lie bracket of \(\Omega^\bullet(L)\), viz.

\[
d_{\vee}[\lambda_1, \lambda_2] = [d_{\vee}(\lambda_1), \lambda_2] + [\lambda_1, d_{\vee}(\lambda_2)]
\]

for \(\lambda_1, \lambda_2 \in L\) and \(d_{\vee} : \bigwedge^p L \to \bigwedge^{p+1} L\) via the duality hypothesis.

One recalls that for \(R = \kappa\), that is, when \(L = g\) is an ordinary Lie algebra, the Drinfeld double \(g \oplus g^\vee\) corresponding to a Lie bialgebra \((g, g^\vee)\) is again a Lie algebra. Hence, the category of Lie algebras is closed under the formation of doubles. However, the situation is more complicated with respect to Lie-Rinehart bialgebras. Recognizing the structure on the naive analogue of the Drinfeld double of a Lie-Rinehart algebra was no longer still a Lie-Rinehart algebra was the original motivation for introducing the category of anchored sheaves called Courant algebroids *loc.cit*.

### 5.2 Courant-Dorfman Algebroids and their Representations

Let \((L, L^\vee)\) be a Lie-Rinehart bialgebra over \(R\). Consider then direct sum of \(R\)-modules \(L \oplus L^\vee\) associated to a Lie-Rinehart bialgebra \((L, L^\vee)\). It is the naive and, in a sense correct, computation of the Jacobi identity by its restriction to the submodules \(L\) and \(L^\vee\) as a semi-direct product of Lie algebras that fails to close this sum as a Lie-Rinehart algebra. Here, the correct analogy of the adjoint action of \(g^\vee\) on \(g\) is the Lie derivative of \(\eta \in L^\vee\) of the Shouten-Nijenhus bracket of \(\Omega^\bullet(L)\) according to the Lie-Rinehart bialgebra hypothesis. Indeed, the naive computation does not satisfy the actual Jacobi identity. The failure of the naive computation to define a Lie bracket of sections motivates the introduction of Leibniz-Rinehart algebras, which are defined by a Leibniz bracket satisfying a more relaxed Jacobi identity. We then refine the Leibniz structure to that of Courant-Dorfman algebroids,
as they are Leibniz-Rinehart algebras with additional axioms. The corresponding Jacobi identity for the Leibniz bracket will hold "up to homotopy." Given this, we adopt the above representation theoretic heuristic to define modules in such a way as to avoid the complexities associated with the homotopical setting.

**Definition 36.** Let $E$ be an $R$-module. We say $E$ is a Leibniz algebra over $R$ if $E$ is equipped with a $\kappa$-bilinear bracket

$$[\cdot, \cdot] : E \otimes_{\kappa} E \to E$$

such that

(Axiom I) for $e_1, e_2, e_3 \in E$, the bilinear bracket satisfies the Jacobi identity

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$$

(Axiom II) there exists a morphism of $R$-modules

$$\rho : E \to \text{Der}_{\kappa}(R)$$

which is called the anchor map.

(Axiom III) the anchor map is a morphism of $\kappa$-Lie algebras

$$\rho([e, e']) = [\rho(e), \rho(e')]$$

for any $e, e' \in E$ and where the bracket on the right hand side is the commutator in $\text{Der}_{\kappa}(R)$.

(Axiom IV) the anchor map satisfies the following compatibility between the $\kappa$-Lie algebra and $R$-module structures which exist upon $E$

$$[e, re'] = re'[e] + \rho(e')(r)e$$

We denote a Leibniz algebra over $R$ by the pair $(R, E)$
One should notice that AXIOM I is not a true Jacobi identity, and as such, has several alternative formulations in the literature. We prefer the above definition, as opposed to the definition emphasizing the homotopical nature of the Leibniz bracket. In light of this preference, one might be inclined to regard the Leibniz identity as the insistence that the adjoint operator is a derivation of a non-associative product attached to an $R$-module $E$, that is, an element of $\text{Der}_R(E, R)$.

We continue following the above heuristic used to describe representations of Lie-Rinehart algebra. A module over a Leibniz algebra $E$ should be an $R$-module $M$ such that the direct sum $M \oplus E$ is a Leibniz algebra and contains $E$ as a Leibniz subalgebra and $M$ is an abelian ideal. Thus, the Leibniz bracket on $M \oplus E$ restricts to the original bracket on $E$ and vanishes on $M$. Accordingly, the Leibniz bracket of $M \oplus E$ is the original bracket of $E$ when restricted to $E \otimes E$ and vanishes on $M \otimes M$. Therefore, the Leibniz bracket on $M \oplus E$ is determined by the values of $[m,e]$ and $[e,m]$, for $m \in M$ and $e \in E$. We have the following distillation of this remark.

**Definition 37.** Let $(R, E)$ be a Leibniz-Rinehart algebra over $R$. A module over $(R, E)$ is an $R$-module $M$ satisfying the following axioms.

(Axiom I) there exist both left and right actions $\mu^l \in \text{Hom}_R(E \otimes_R M, M)$ and $\mu^r \in \text{Hom}_R(M \otimes_R E, M)$

(Axiom II) the left and right actions are compatible as follows

\[
\begin{align*}
\mu^r[e, e'] &= \mu^r(e')\mu^r(e) + \mu^l(e)\mu^r(e') \\
\mu^r[e, e'] &= \mu^l(e)\mu^r(e') - \mu^r(e')\mu^l(e) \\
\mu^l[e, e'] &= \mu^l(e)\mu^l(e') - \mu^l(e')\mu^l(e)
\end{align*}
\]

for any $e, e' \in E$.

In particular, for any representation $\rho$ of $E$ on $M$, that is, a $R$-module homomorphism
preserving the Leibniz bracket $E \to \text{End}_R(M)$, then by setting $\mu^l = \rho$ and $\mu^r = -\rho$ a representation above is equivalent to the structure of a $(R, E)$ Leibniz module structure on $M$. Furthermore, as with Lie-Rinehart algebras, the definition of Leibniz algebra imparts an $E$-module structure to $R$ itself, namely, the canonical $E$-module, $R \oplus E$. However, unlike the canonical $L$-module, the canonical $E$-module is equipped with two anchors, by the above remark. As such, the canonical $E$-module $R \oplus E$ recapitulates the Courant-Dorfman algebra $(R, E)$. Specifically, $\mu^l(e \otimes r) = \rho(e)(r)$ and $\mu^r(r \otimes e) = -\rho(e)(r)$.

We also use canonical modules to define morphisms.

**Definition 38.** Let $(R, E)$ and $(R', E')$ be Leibniz algebras over $R$ and $R'$, respectively. A Leibniz morphism $(\varphi, \ell) : (R, E) \to (R', E')$ is a pair satisfying the following axioms.

(Axiom I) a $\kappa$-algebra morphism $\varphi : R \to R'$

(Axiom II) a $\kappa$-Leibniz algebra morphism $\ell : E \to E'$

(Axiom III) an additive map $\ell : E \to E'$ such that $\ell(re) = \varphi(r)\ell(e)$

(Axiom IV) $\varphi(\rho(e)(r)) = \rho'(\ell(e))(\varphi(r))$

Next, we present a refinement of this Leibniz algebras in the following definition.

**Definition 39.** A Courant-Dorfman algebra is a Leibniz algebra together with a symmetric $R$-bilinear form

$$\langle \cdot | \cdot \rangle : E \otimes \kappa E \to R$$

satisfying the following axioms for $r \in R$ and $e_1, e_2 \in E$

(Axiom I) $\rho(e_3)(\langle e_1 | e_2 \rangle) = \langle e_1 | [e_1, e_2] + [e_2, e_1] \rangle$

(Axiom II) $\rho(e_3)(\langle e_1 | e_2 \rangle) = \langle [e_3, e_1] | e_2 \rangle + \langle e_1 | [e_3, e_2] \rangle$

(Axiom III) a derivation $\partial : R \to E$

To express these objects as a category, we define their morphisms to be maps of canonical $E$-modules respecting the additional structure of the Courant-Dorfman algebras.
Definition 40. Let \((R, E)\) and \((R', E')\) be Courant-Dorfman algebras over \(R\) and \(R'\), respectively. A Courant-Dorfman morphism \((\varphi, \ell) : (R, E) \rightarrow (R', E')\) is a pair satisfying the following axioms.

(Axiom I) a \(\kappa\)-algebra morphism \(\varphi : R \rightarrow R'\)

(Axiom II) a \(\kappa\)-Leibniz algebra morphism \(\ell : E \rightarrow E'\)

(Axiom III) an additive map \(\ell : E \rightarrow E'\) such that \(\ell(re) = \varphi(r)\ell(e)\)

(Axiom IV) \(\varphi(\rho(e)(r)) = \rho'(\ell(e))(\varphi(r))\)

(Axiom V) \(\ell(\langle e|e'\rangle) = \langle \ell(e)|\ell(e')\rangle\)

(Axiom VI) \(\ell \circ \partial = \partial' \circ \ell\)

An automorphism of a Courant-Dorfman algebra is a bijective endomorphism in the above definition. We denote the group of automorphisms by \(\text{Aut}(E)\). There is an infinitesimal structure preserving morphism corresponding to automorphisms.

Definition 41. Let \((R, E)\) be a Courant-Dorfman algebra over \(R\). A derivation \(\delta\) of \(E\) consists of an endomorphism \(\delta\) and a vector field \(\xi \in \text{Der}_\kappa(R)\) such that the following axioms are satisfied.

(Axiom I) for \(e, e' \in E\)

\[
\delta[e, e'] = [\delta(e), e'] + [e, \delta(e')]
\]

(Axiom II) for \(\xi \in \text{Der}_\kappa(R)\)

\[
\xi\langle e|e'\rangle = \langle \delta(e)|e'\rangle + \langle e|\delta(e')\rangle
\]

There are a few important consequences of this definition. Firstly, the vector field \(\xi\) in the definition is determined by \(\delta\) according to the axioms of a Courant-Dorfman algebra. As such, the space of derivations forms a Lie algebra. Secondly, given that \(\xi\) is determined by \(\delta\), one has the identities \(\delta(re) = \xi(r)e + r\delta(e)\) and \(\rho([\delta(e)]) = [\xi, \rho(e)]\). Therefore, we
have a Lie algebra homomorphism $\Xi : \text{Der}_{CD}(E) \to \text{Der}_κ(R)$. Moreover, the assignment of $e \in E \mapsto \text{ad}_e \in \text{Der}(E)$ embeds $E$ in $\text{Der}(E)$. We say the image of this embedding is the subalgebra of inner derivations.

Our geometric approach to Courant-Dorfman algebras is precisely the same as our approach to the geometric interpretation of Lie-Rinehart algebras. We say a sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ over a ringed space $(X, \mathcal{O}_X)$ such that $(\mathcal{O}_X(U), \mathcal{E}(U))$ is a Courant-Dorfman algebra for any open set $U \subset X$ is a Courant-Dorfman algebroid. Again, for an affine scheme $X = \text{Spec}(R)$ the category of Courant-Dorfman algebras and the category of Courant-Dorfman algebroids are equivalent.

Returning to the problem of understanding a Lie-Rinehart double, let $(L, L^\vee)$ be a Lie-Rinehart bialgebra. The famous theorem of [LWX] below was the discovery which initiated both the definition and study of Courant algebroids loc.cit. Indeed, the definition given above was their solution to understanding the notion of Lie-Rinehart double analogous to that of Lie algebra double.

**Theorem 11.** Let $(L, L^\vee)$ be a Lie-Rinehart bialgebra, then the direct sum

$$E = L \oplus L^\vee$$

is a Courant-Dorfman algebra with

(I) the anchor given by $\rho = \rho_L + \rho_{L^\vee}$

(II) derivation $\partial = d_L + d_{L^\vee}$, where $d_L$ and $d_{L^\vee}$ are the differentials of $\Omega^\bullet(L)$ and $\Omega^\bullet(L^\vee)$, respectively.

(III) symmetric bilinear pairing $\langle \lambda_1 + \eta_1 | \lambda_2 + \eta_2 \rangle = \eta_2(\lambda_1) + \eta_1(\lambda_2)$

(IV) Leibniz bracket

$$[\lambda_1 + \eta_1, \lambda_2 + \eta_2] = [\lambda_1, \lambda_2] + \text{Lie}_{\lambda_1} \eta_2 - \iota_{\lambda_2} d_L \eta_1 + \iota_{\lambda_2} \iota_{\lambda_1} \hbar$$

where $\iota$ is the contraction operator of $\Omega^\bullet(L)$ and Lie is the Lie derivative of the same given by the standard Cartan homotopy formula, and $\hbar \in \Omega^3(L)$ is a $d_L$-closed 3-form.
So, given a Lie algebroid \( \mathcal{L} \) over some ringed space \((X, \mathcal{O}_X)\), then the formation of its double \( \mathcal{E} = \mathcal{L} \oplus \mathcal{L}^\vee \) is a Courant-Dorfman algebroid.

We say a Courant-Dorfman algebra \( E \) is transitive if the anchor map \( \rho : E \to \text{Der}_\kappa(R) \) is a surjection. Less generally, when \((R, E)\) fits into an exact sequence

\[
0 \to \Omega_R \to E \to \text{Der}_\kappa(R) \to 0
\]

we say \((R, E)\) is exact. Notice that the anchor map induces a morphism \( \rho^* : \Omega_R \to E^* \) by the universal derivation \( d_0 : R \to \Omega_R \). We say that \( \rho^* : \Omega_R \to E \) is the coanchor. In the case that \( E \) is the Courant-Dorfman algebra corresponding to a Lie-Rinehart bilgebra \((L, L^\vee)\), \( E \) is self-dual. Indeed, \( E \cong E^\vee \) by the symmetric bilinear pairing. As such, we identify \( \Omega_R \) with its image under the coanchor. One has the quotient \( E/\Omega_R \) is a Lie-Rinehart algebra and \( E/\partial(R) \) is a Lie algebra over \( \kappa \).

So far, exact Courant-Dorfman algebras and their relation to vertex algebras have been studied the most in the literature. In particular, the famous chiral de Rham complex is generated by the exact Courant-Dorfman algebra given by the sum of the module of derivations and Kahler differentials. The \( \beta\gamma \)-system is the subject of this construction in [GGW]. However, the literature is scant on examples of vertex algebras related to transitive Courant-Dorfman algebras, so it is our aim to consider an example of this below with respect to the transitive Courant-Dorfman algebra \( E = L \oplus L^\vee \) above. This more general example allows one to formulate constructions in geometry general enough to permit a geometric interpretation of weak modules over the same.

### 5.3 The Vertex \( \mathcal{O} \)-Algebra

In this section we construct of a vertex algebra from a transitive Courant-Dorfman algebra following [LiYam], referred to in this text as the vertex \( \mathcal{O} \)-algebra, owing to its emergence from geometry. Let \( R \oplus E \) be the canonical \( E \)-module for \( E = L \oplus L^\vee \), the Courant-Dorfman algebra corresponding to the Lie-Rinehart bialgebra \((L, L^\vee)\) now for the remainder of the
text and write $C = R \oplus E$. Define the $C[t^\pm] = (R \oplus E) \otimes \kappa[t^\pm]$ as a vector space over $\kappa$; we have the subspace embeddings $R \otimes \kappa[t^\pm] := R[t^\pm] \to C[t^\pm]$ and $E \otimes \kappa[t^\pm] = E[t^\pm] \to C[t^\pm]$. Both $R[t^\pm]$ and $E[t^\pm]$ are graded subspaces of $C[t^\pm]$, with homogeneous elements $r \otimes t^n$ and $e \otimes t^n$ of degrees $-n - 1$ and $-n$, respectively. With this convention, $C[t^\pm]$ is a graded $\kappa$-vector space

$$C[t^\pm] = \bigoplus_{n \in \mathbb{Z}} C[t^\pm]^n$$

where

$$C[t^\pm]^n = R \otimes \kappa t^{-n-1} \oplus E \otimes \kappa t^{-n}$$

Let $\mathfrak{d} = \partial \otimes 1 + \text{Id}_C \otimes \partial_t : R[t^\pm] \to C[t^\pm]$ be a partially defined linear operator of $C[t^\pm]$. Notice that $\mathfrak{d}$ is a linear operator of degree 1 with our grading convention.

Define a bilinear product $[,] : C[t^\pm] \times C[t^\pm] \to C[t^\pm]$ on $C[t^\pm]$ for $r, r_i \in R, e, e_i \in E$, as follows

$$[r_1 \otimes t^m, r_2 \otimes t^n] = 0$$

$$[r \otimes t^m, e \otimes t^n] = \mu^r(e)(r) \otimes t^{m+n}$$

$$[e \otimes t^m, r \otimes t^n] = \mu^l(e)(r) \otimes t^{m+n}$$

$$[e_1 \otimes t^m, e_2 \otimes t^n] = [e_1, e_2] \otimes t^{m+n} + m(e_1|e_2) \otimes t^{m+n-1}$$

We wish to show $\mathfrak{c} := C[t^\pm]/\text{Im} \mathfrak{d}$ the loop algebra is a Lie algebra with these definitions.

**Lemma 3.** The subspace $\text{Im} \mathfrak{d}$ is a two-sided ideal of the non-associative algebra $C[t^\pm]$.

**Proof.** For $r_1, r_2 \in R$, we have

$$[r_1 \otimes t^m, \mathfrak{d}(r_2 \otimes t^n)] = [r_1 \otimes t^m, \partial(r_2) \otimes t^n + nr_2 \otimes t^{n-1}] = \mu^r(\partial(r_2))(r_1) \otimes t^{m+n}$$

which is equal to zero by definition. Similarly,

$$[\mathfrak{d}(r_2 \otimes t^n), r_1 \otimes t^m] = [\partial(r_2) \otimes t^n + nr_2 \otimes t^{n-1}, r_1 \otimes t^m] = \mu^l(\partial(r_2))(r_1) \otimes t^{m+n}$$

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is also zero. Next, we have

\[
\mathcal{d}(r_2 \otimes t^n), e \otimes t^m] = [\partial(r_2) \otimes t^n + nr_2 \otimes t^{n-1}, e \otimes t^m] \\
= [\partial(r_2), e] \otimes t^{m+n} + n\langle \partial(r_2)|e\rangle \otimes t^{m+n-1} + n\mu^r(e)(r_2) \otimes t^{m+n-1}
\]

which again is zero by definition. Last, for

\[
[e \otimes t^m, \mathcal{d}(r_2 \otimes t^n)] = [e, \partial(r_2)] \otimes t^{m+n} + m\langle e|\partial(r_2)\rangle \otimes t^{m+n-1} + n[e, r_2] \otimes t^{m+n-1}
\]

we have the right hand side is equal to \(\mathcal{d}(\mu^l(e)(r_2) \otimes t^{m+n}) \in \text{Im}\mathcal{d}\). This completes the proof.

It now follows that \(\mathfrak{c}\) is a non-associative algebra.

**Lemma 4.** The bilinear product \([,]\) of \(C[t^\pm]\) restricted to \(\mathfrak{c}\) is skew-symmetric.

**Proof.** Observe,

\[
[r \otimes t^m, e \otimes t^n] + [e \otimes t^n, r \otimes t^m] = 0 \text{ mod}(\text{Im}\mathcal{d})
\]

is equivalent to

\[
(\mu^r(e)(r) + \mu^l(e)(r)) \otimes t^{m+n} \in \text{Im}\mathcal{d}
\]

but this follows from definition since \(\mu^l(e) = -\mu^r(e)\).

Next, to show

\[
[e_1 \otimes t^m, e_2 \otimes t^n] + [e_2 \otimes t^n, e_1 \otimes t^m] = 0 \text{ mod}(\text{Im}\mathcal{d})
\]

is equivalent to

\[
(e_1|e_2) \otimes t^{m+n} + (m + n)e_1|e_2 \rangle \otimes t^{m+n-1} \in \text{Im}\mathcal{d}
\]

by properties of the Courant-Dorfman algebra. This completes the proof.

Last, as a consequence of the fact the canonical \(E\)-module is a representation of the underlying Leibniz-Rinehart algebra, we have
Proposition 9. The non-associative quotient algebra $c$ together with its bilinear product $[,]$ is a Lie algebra.

Proof. This follows immediately from Axiom II for the canonical $E$-module $R \oplus E$. 

The grading on the subspaces $R[t^\pm]$ and $E[t^\pm]$ induce a $\mathbb{Z}$-grading on $c$. The decomposition

$$c = \bigoplus_{n \in \mathbb{Z}} c^n$$

is given by $c^n = R \otimes t^{-n-1} + E \otimes t^{-n}/\text{Im } \partial$. In particular, $c^0 \cong R \oplus E/\partial(R)$. This subspace may be furnished with the structure of a semi-direct product Lie algebra, as $E/\partial(R)$ is a Lie algebra with an obvious action on $R$.

The goal of constructing a vertex algebra from the canonical $E$-module $R \oplus E$ is now in view with the quotient map $\tau : C[t^\pm] \to c$ by

$$\tau(c \otimes t^n) = c \otimes t^n + \text{Im } \partial := c_{(n)}$$

for $c \in C$. This assignment allows us to form the generating series

$$Y(c, z) = \sum_{n \in \mathbb{Z}} c_{(n)} z^{-n-1}$$

which is a formal power series in $c[[z^\pm]]$. Given a $c$-module, say $V$, we shall abuse notation by also writing $Y(c, z) = \sum_{n \in \mathbb{Z}} c_{(n)} z^{-n-1} \in \text{End}(V)[[z^\pm]]$ for the corresponding formal power series. The bilinear product on the loop space $C[t^\pm]$ can be expressed in terms of these generating functions as

$$[Y(r_1, z), Y(r_2, w)] = 0$$

$$[Y(r, z), Y(e, w)] = w^{-1}\delta(z - w)Y([r, e], w)$$

$$[Y(e_1, z), Y(e_2, w)] = w^{-1}\delta(z - w)Y([e_1, e_2], w) + \delta((e_1|e_2), w)\partial_w w^{-1}\delta(z - w)$$

where $r, r_i \in R$ and $e, e_i \in E$.

The Lie algebra $c$ has a triangular decomposition

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\[ c = \sum_{n<0} c^n \bigoplus c^0 \bigoplus \sum_{n>0} c^n \]

and a grading decomposition of \( c \) obtained by the restriction of \( \tau \) to non-negative and negative powers of \( t \). We shall use the latter graded structure in conjunction with both the Lie structure and an application of the PBW-theorem to define our vertex algebra.

Observe, with the grading obtained by restriction of \( \tau \), \( c \) decomposes as

\[ c = c_+ \bigoplus c_- \]

where

\[ c_- = \tau(t^{-1}C[t^-]) = R(-1) \bigoplus E(-n) \]

and

\[ c_+ = \tau(C[t]) = \bigoplus E(n) \]

where one writes \( R(n) = \{ r(n) \mid r \in R \} \) and \( E(n) = \{ e(n) \mid e \in E \} \).

Let \( \kappa \) be the trivial one dimensional representation of \( c_+ \) and induce a \( c \)-module over \( k \), viz.

\[ \text{Ind}_{U(c_+)}^{U(c)} \kappa \]

where \( U \) denotes the universal enveloping algebra. Applying the PBW-theorem, we have

\[ \mathbb{V} := U(c_-) \]

which we call the vacuum module associated to \((R, E)\). Assign to \( \kappa \) degree zero, then the vacuum module is \( \mathbb{N} \)-graded as a \( c \)-module according to the embedding \( C \rightarrow C(-1) \subset \mathbb{V} \) where, \( c \mapsto r_{-1}1 + e_{-1}1 \), given by the PBW theorem viz.
\[ \mathbb{V} = \bigoplus_{n \in \mathbb{N}} \mathbb{V}^n \]

The vacuum module is a locally nilpotent \(c\)-module because \(c(n) \cdot v = 0\) for \(n\) sufficiently large by its \(\mathbb{N}\)-grading. Thus, the power series \(Y(c, z)\) associated to arbitrary \(c \in R \oplus E\) by this construction acts as a field on \(\mathbb{V}\). In particular, \(\mathbb{V}\) is strongly finitely generated by \(R \oplus E\). We have the following consequence of the reconstruction theorem.

**Theorem 12.** The assignment

\[ Y(\cdot, z) : \mathbb{V} \to \text{End}\mathbb{V}[[z^\pm]] \]

is a state-field correspondence, rendering \(\mathbb{V}\) as an \(\mathbb{N}\)-graded vertex algebra, strongly finitely generated by \(R \oplus E\), with vacuum the unit element of \(R\) and translation operator given by \(\mathcal{D}(c) = c(-2)\), for \(c \in R \oplus E\), extending the derivation \(\partial\) of \(R \oplus E\). In the reconstruction theorem, we have

\[ Y(c_{(-n_1)} \cdots c_{(-n_r)} |0\rangle, z) = \frac{1}{(n_1-1)! \cdots (n_r-1)!} : \partial_z^{-n_1} c_{\alpha_1}(z) \cdots \partial_z^{-n_r} c_{\alpha_m}(z) : \]

for \(c \in R \oplus E\). We say \(\mathbb{V}\) is the Vacuum module corresponding to the transitive Courant-Dorfman algebra \((R, E)\).

The PBW theorem gives us that the vacuum module is isomorphic to the following tensor product as a \(\kappa\)-vector space,

\[ \mathbb{V} \cong \text{Sym}(R(-1)) \otimes \text{Sym}(E(-n)) \]

One has such a map, for \(\mathbb{V}^0 \cong \text{Sym}(R(-1))\) via the PBW embedding of \(R\) into \(\mathbb{V}\). This module is rather large and unsuitable from the perspective of Harish-Chandra geometry to be interpreted geometrically according to the results in this text. However, the following fundamental theorem [LiYam] establishes the existence of a quotient of the vacuum module that is a vertex algebra small enough to be suitably treated geometrically under the auspices of localization by Harish-Chandra geometry and a second result of Li and Yamaskulna [LiYam2].
Theorem 13. Let \( \mathcal{V} \) be the vacuum module and \( I \) the \( c \)-submodule of \( \mathcal{V} \) generated by
\[
\langle 1 - 1_R, r_{(-1)}r' - rr', r_{(-1)}e - re \rangle
\]
then the quotient
\[
\mathcal{V}/I = \mathfrak{V}_E
\]
is an \( \mathbb{N} \)-graded vertex algebra, strongly finitely generated by its weight 0 and 1 spaces
\[
\mathfrak{V}_E^0 = R
\]
\[
\mathfrak{V}_E^1 = E
\]
and whose weight \( n \) spaces are
\[
\mathfrak{V}_E^n = \text{Span}\{e_1(-n_1) \cdots e_k(-n_k)1\}
\]
where \( e_i \in E, n_1 \geq \cdots n_k \geq 1 \) and \( \sum_i n_i = n \) for \( n \geq 2 \). We say that the quotient \( \mathfrak{V}_E \) is the vertex \( \mathcal{O} \)-algebra associated to the Courant-Dorfman algebra \( (R, E) \).

It is tempting to interpret this result to mean the vertex \( \mathcal{O} \)-algebra is generated by the structure sheaf of an affine scheme \( X = \text{Spec}(R) \) and the global sections of the \( \mathcal{O}_X \)-Courant algebroid \( \mathcal{E} \). However, there is a problem with the vertex \( \mathcal{O} \)-algebra \( \mathfrak{V}_E \) in this sense. First, we have not developed a formalism for vertex algebras over a commutative ring. The only reference which the author is familiar with is that of Borisov [Bor].

Secondly, granting for the moment such a formalism exists, then globally speaking, as a finitely generated projective module, elements of \( E \) can only be expressed uniquely as sums of elements in the direct sum decomposition \( L \oplus L^\vee \), but not uniquely as ring elements and module elements. Accordingly, the putative state-field correspondence \( \mathcal{Y} \) is not well-defined.

This second observation motivates the sense in which our construction is coordinate dependent. A straightforward consequence of the theorem allows us to work with \( \mathfrak{V}_E \) locally in coordinates, for locally the Courant-Dorfman algebra \( (R, E) \) is freely generated so that
the state-field correspondence is well-defined. We shall remedy this defect in the final section by globalizing this locally coordinate dependent construction by the methods of formal geometry. Indeed, such globalization maneuvers are the *raison d’etre* of formal geometry.

Let \( S \) be a multiplicatively closed set in \( R \) and \( S^{-1}R \) the corresponding localization. Further, denote by \( S^{-1}E \) the localization of the projective \( R \)-module at \( S \). By our hypothesis \( X = \text{Spec}(R) \) is a regular \( \kappa \)-scheme, locally etale, we can choose \( S \) such that there exists a regular sequence of generators of the unique maximal ideal in \( S^{-1}R \), say \( \{x_i\} \), in other words, a system of coordinates. Furthermore, \( S^{-1}E \) is locally free by our hypothesis that the Lie-Rinehart algebra \( L \) is projective. Therefore, let \( \{e_i\} \) be a locally free basis, which we interpret as local coordinates for \( E \). Notice a choice of such a basis is equivalent to the choice of a frame for \( E \). As the Lie-Rinehart algebra \( L \) satisfies descent by the pullback construction, consequently \( E \) satisfies descent by its construction, since localization commutes with direct sums. Hence \( (S^{-1}R, S^{-1}E) \) is a Courant-Dorfman algebra. Therefore, we have the following corollary of the theorem.

**Corollary 5.1.** Let \( \mathcal{V}_E \) be the vertex \( \mathcal{O} \)-algebra associated to the Courant-Dorfman algebra \( (R, E) \), and \( S \subset R \) a multiplicatively closed subset of \( R \). Define

\[
S^{-1}\mathcal{V}_E
\]

to be the vertex \( \mathcal{O} \)-algebra associated to the Courant-Dorfman algebra \( (S^{-1}R, S^{-1}E) \). Then \( S^{-1}\mathcal{V}_E \) is strongly finitely generated by the Courant algebroid \( (\mathcal{O}_X(U), E(U)) \), that is, by coordinates \( \{x_i, e_i\} \), where \( 1 \leq i \leq n \).

Unfortunately, this result is only local, and does not indicate whether a global object corresponds to \( S^{-1}\mathcal{V}_E \). To globalize this construction to a sheaf of pro-coherent \( \mathcal{O}_X \)-modules, we shall appeal to the localization technique of Harish-Chandra geometry with respect to a \( (\mathfrak{g}, K) \)-structure such that \( \mathcal{V}_E \) is a \( (\mathfrak{g}, K) \)-module.

In order to demonstrate that \( \mathcal{V}_E \) is a Harish-Chandra module, we must recall a results of Li and Yamaskulna [LiYam2], who show the grading preserving automorphisms of the vertex
Theorem 14. Let \( \mathfrak{V}_E \) be the vertex \( \mathfrak{O} \)-algebra. Then the grading preserving automorphisms of \( \mathfrak{V}_E \) isomorphic to \( \text{Aut}_{CD}(E) \), the group of automorphisms of the Courant-Dorfman algebra \( E \).

We end this section by recalling a result of [DLM], which states that locally nilpotent representations of the loop algebra \( \mathfrak{c} \) are in 1-1 correspondence with modules over the vertex algebra \( \mathfrak{V} \). Quotienting by the ideal \( I \) above, we have that locally nilpotent representations of \( \mathfrak{c} \) correspond to modules \( M \) over the vertex \( \mathfrak{O} \)-algebra, \( \mathfrak{V}_E \). In particular, a grading preserving automorphism of \( \mathfrak{V}_E \) corresponds to an automorphism of \( \mathfrak{c} \), which in turn extends to an action of \( \mathfrak{V}_E \) on \( M \). We shall use this result to localize representations of \( \mathfrak{V}_E \).

5.4 The Zhu Algebra of \( \mathfrak{V}_E \)

Recall the Zhu algebra \( A(V) \) of a vertex algebra \( V \) from chapter 4. We shall denote the Zhu algebra of \( \mathfrak{V}_E \) by \( A(E) \) in this section. Moreover, we denote the projection of \( \mathfrak{V}_E \) onto the ideal \( \mathfrak{V}_E \star \partial \mathfrak{V}_E \) by \( \pi_A \). Recall further that for a Lie-Rinehart algebra \( L \), we have its universal enveloping algebroid, \( U_R L \). The universal enveloping algebroid \( U_R L \) enjoys a universal mapping property that we shall employ to obtain a surjective map of associative \( \kappa \)-algebras between \( U_R L \) and \( A(E) \). Given the relationship between modules over \( A(E) \) and \( \mathfrak{V}_E \) outlined in chapter 4, we show that we may obtain a module whose weight 0 space is a module for the Lie-Rinehart algebra, \( L \). In general, there exist indecomposable modules for \( L \), and as such, the corresponding representation of \( \mathfrak{V}_E \) is weak.

**Proposition 5.2.** Let \( L \) be the Lie-Rinehart algebra in the formation of \( E \) as a Courant-Dorfman algebra. Equivalently, the \( R \)-submodule of \( \mathfrak{V}^1_E \). Given a module \( N' \) over \( A(E) \),
there exists a vertex $\mathfrak{O}$-algebra module $M$ over $\mathfrak{N}_E$ whose weight 0 space $M^0 = N$, where $N$ is the pull-back to $U_R L$ of $N'$, that is, an $L$-module.

**Proof.** Consider the Lie-Rinehart algebra $L$ in the hypothesis. There exists a surjective map of Lie-Rinehart algebras $L \to E/R\partial R \cong \mathfrak{N}_E^1/\mathfrak{N}_E^0 \star \mathfrak{N}_E^0$ by the construction. Observe, $d_L(R) \subset L$, so that

$$L \to L/d_L(R) \to E/d_L(R)$$

Since $d_L(R) \subset R\partial R$, we have by isomorphism theorems a projection

$$E/d_L(R) \to E/R\partial R$$

which gives the surjective map of Lie-Rinehart algebras. That the latter object is a Lie-Rinehart algebra is well-known. The isomorphism $E/R\partial R \cong \mathfrak{N}_E^1/\mathfrak{N}_E^0 \star \mathfrak{N}_E^0$ is demonstrated in [ACM]. Indeed, loc.cit shows $\mathfrak{N}_E^0 \star \mathfrak{N}_E^0 \cong R\partial R$, where $\partial = \mathfrak{D} + L_0$, and $L_0$ is the gradation operator. Therefore, since one has $\mathfrak{N}_E^0 \star \mathfrak{N}_E^0 \subset \mathfrak{N}_E^1 \star \mathfrak{N}_E^0$, there exists a $\kappa$-Lie algebra morphism preserving brackets

$$L \to \mathfrak{N}_E^1/\mathfrak{N}_E^0 \star \mathfrak{N}_E^0 \to \mathfrak{N}_E^1/\mathfrak{N}_E^0 \star \mathfrak{N}_E^0 = A(E)$$

Next, one observes by a calculation that $\mathfrak{N}_E^0 \cap \mathfrak{N}_E^1 \star \mathfrak{N}_E^0 = 0$ since $a \star \partial b$ does not have a degree 0 component for any $a, b \in \mathfrak{N}_E$. So the projection $\mathfrak{N}_E \to A(E)$ is injective on $\mathfrak{N}_E^0$ and is a morphism of $\kappa$-algebras. Consequently, by the universal mapping property of $U_R L$ there exists a map $\alpha : U_R L \to A(E)$. Since $\mathfrak{N}_E$ generated by $R \oplus E$ and $A(E)$ is generated by 1 and $\pi_A(R \oplus E) = \text{Span}\{\pi_A(r_e^1 \cdot e_{-n_1} \cdots e_{-n_k} | 0)\}$, where $r \in R, e^i \in E$ and $j \leq -2$, the morphism $\alpha$ is a surjection. The Zhu algebra $A(E)$ has a canonical filtration [Zhu] and [ACM] shows $\alpha$ is a morphism of filtered algebras.

Let $N'$ be a module over the Zhu algebra $A(E)$, and $N$ be the pullback of $N'$ under $\alpha$ to $U_R L$. Then $N$ is a $U_R L$-module, and so, by definition, an $L$-module. The quotient map $R \oplus L \to \mathfrak{c}_0 = R \oplus E/\partial R$ furnishes both a map $\kappa$-Lie algebras $L \to \mathfrak{c}_0 \subset \mathfrak{c}$ and $\kappa$-algebras $R \to \mathfrak{c}_0 \subset \mathfrak{c}$. Consequently, again, by the universal mapping property of $U_R L$, there
exists a morphism $\beta : U_L \rightarrow \mathfrak{c}$. Indeed, let $\mathfrak{c}_{\leq 0}$ act trivially for $\mathfrak{c}_{< 0}$, then induce to obtain $M' = \text{Ind}^{U(c)}_{U(\mathfrak{c}_{\leq 0})} N$. So, given the $L$-module $N$, we may be push $N$ forward to a $\mathfrak{c}$-module $M'$. Moreover, the action of $\mathfrak{c}$ is locally nilpotent since it annihilates the ideal $I_R \subset U_R L$ \cite{LiYam}.

There is a one-to-one correspondence between $\mathcal{V}$-modules $M'$ and locally nilpotent $\mathfrak{c}$-modules \cite{DLM}. Consider $W = \text{Span}\{a(\nu) a \in I, \nu \in N\}$ in $M'$. Set

$$M = M'/U(\mathfrak{c})W$$

then $M$ is an $\mathbb{N}$-graded $\mathfrak{Y}_E$-module, since $\mathfrak{Y}_E$ is obtained as the quotient of $\mathcal{V}$ by $I$. Furthermore, $M^0 = N$. This last statement is proven in \cite{LiYam}.

So, given an indecomposable $L$-module $N$, we obtain a weak module $M$ over $\mathfrak{Y}_E$ by the proposition. We shall use this in the final section to aver weak modules may be globalized as sheaves of vertex algebra modules over $X$.

### 5.5 The Vacuum Space and the Vertex Bundle $\mathcal{V}_X(\mathcal{E})$

In this final section, we construct a vertex bundle $\mathcal{V}_X(\mathcal{E})$ by the localization construction applied to $\mathfrak{Y}_E$ with respect to an extension of a transitive structure on $\text{Cor}_X$. Further, we obtain a bundle of indecomposable modules $\mathcal{M}_X$ by the same construction. One may rightly ask at this point why formal geometry and its extensions are involved in this construction at all. Beside using formal geometry in chapter three to illustrate an example of Harish-Chandra geometry, there is nothing obviously significant about this example up until now to explain why it should arise in an associated bundle construction of this nature over, say, any other example of Harish-Chandra geometry. The answer to that question, which will be demonstrated in this section, is that there exists a transitive Harish-Chandra pair which is an extension by the tautological Harish-Chandra pair associated to the ring of $\mathfrak{gl}(E)$-valued formal power series, where $\mathfrak{gl}(E)$ is the Lie algebra of derivations of $E$ as an $R$-module. We
show the extension \((g, K)\) of the transitive pair acts by grading preserving vertex algebra automorphisms and derivations of \(V_E\). Accordingly, the vertex \(\mathcal{O}\)-algebra is a \((g, K)\)-module.

Granting the extension \((g, K)\) exists, one then defines a corresponding \((g, K)\)-structure in a manner analogous to the torsor of formal coordinates. After all, this structure is an extension of the same. As the aforementioned extension \((g, K)\) acts on both the \((g, K)\)-structure and the vertex \(\mathcal{O}\)-algebra, we can therefore construct the localization \(V_X(\mathcal{E})\) of \(V_E\) to \(X\). Thus, one can appreciate the special role that formal geometry plays in our construction, after all.

First we recall the construction of the \(J_X\)-module \(J_X(\mathcal{E})\) of jets of a finitely generated projective \(\mathcal{O}_X\)-module, \(\mathcal{E}\). Let \(\Delta : X \to X \times X\) be the diagonal embedding and \(p_i : X \times X \to X\) be the projections onto the \(i\)-th factors. Furthermore, as above, let \(I_{\Delta}\) be the ideal sheaf of the diagonal embedding.

Define the \(k\)-th sheaf of \(\mathcal{E}\)-jets to be

\[
\mathcal{J}_X^k(\mathcal{E}) = (p_1)_*(\mathcal{O}_{X \times X}/I_{\Delta}^{k+1} \otimes p_2^{-1}\mathcal{O}_X p_2^{-1}\mathcal{E})
\]

One has that the \(k\)-th sheaf of \(\mathcal{E}\)-jets is a union of its fibres over points \(x \in X\), where the fibre of \(\mathcal{J}_X^k(\mathcal{E})\) is given by \(\mathcal{E}_x/m_x^k = \mathcal{J}_x^k(\mathcal{E})\) and \(m_x\) is the ideal sheaf of the point \(x \in X\). The definition shows that \(\mathcal{J}_X^k(\mathcal{E})\) is a \(\mathcal{J}_X^k\)-module. According to the embedding of \(\mathcal{O}_X\) in its jet algebra

\[
s^* : \mathcal{O}_X \to \mathcal{J}_X^k \to \mathcal{J}_X
\]

described above, one has by tensoring this arrow with \(\mathcal{E}\) over \(\mathcal{O}_X\) that one may expand sections of \(\mathcal{E}\) as \(k\)-jets of sections. Observe, given the isomorphism

\[
\mathcal{J}_X^k \otimes_{\mathcal{O}_X} \mathcal{E} \cong \mathcal{J}_X^k(\mathcal{E})
\]

the tensored arrow \(s_k^* : \mathcal{E} \to \mathcal{J}_X^k(\mathcal{E})\) sends a section of \(\mathcal{E}\) to its \(k\)-jet. In particular, this furnishes a morphism \(\mathcal{J}_X^k(\mathcal{E}) \to \mathcal{J}_X^{k-1}(\mathcal{E})\). Alternatively, one has the exact sequence

\[
0 \to \text{Sym}^k(\Omega_X^1) \otimes \mathcal{E} \to \mathcal{J}_X^k(\mathcal{E}) \to \mathcal{J}_X^{k-1}(\mathcal{E}) \to 0
\]

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A more geometric perspective, perhaps, of the expansion of sections in \( k \)-jets is to recognize that the morphism \( s_k \) corresponds to an evaluation map \( X \times \mathcal{V}(E) \to \mathcal{J}_X^k(E) \) as a morphism of \( \mathcal{O}_X \)-modules. However, we have not yet defined \( \mathcal{J}_X^k(E) \) as any kind of scheme.

Next, to define jets of sections, we have that, since powers of the ideal sheaf \( I_\Delta \) of the diagonal form a projective system of \( \mathcal{O}_{X \times X} \)-modules, the \( k \)-th jet sheaves \( \mathcal{J}_X^k(E) \) form a projective system of \( \mathcal{O}_X \)-modules. Hence, as usual, we define the sheaf of \( E \)-jets to be the projective limit

\[
\mathcal{J}_X(E) = \lim_k \mathcal{J}_X^k(E)
\]

of this projective system.

The sheaf of \( E \)-jets is filtered by order of vanishing of sections in its fibres. Denote the filtration by \( F^k \mathcal{J}_X(E) \), so that with this notation quotients of the \( E \)-jets by the \( k \)-th filtrant recover the \( k \)-th sheaf of \( E \)-jets, that is,

\[
\mathcal{J}_X(E)/F^k \mathcal{J}_X(E) \cong \mathcal{J}_X^k(E)
\]

These quotients exhibit \( \mathcal{J}_X(E) \) as a \( \mathcal{J}_X \)-module and \( \mathcal{J}_X(E) \) as a linearly compact vector space. Typically, the sheaf of \( E \)-jets is treated geometrically as a topological \( \mathcal{O}_X \)-module, which in our case we would treat as a linearly compact object. We, however, shall not use it in this way, but in a way proceeding by analogy with the usage of \( \mathcal{J}_X \) in chapter three to define the formal scheme \( \mathcal{J}_{et_X} \) and \( \mathcal{C}_{or_X} \) thereafter.

So, given the recollection of \( \mathcal{J}_X(E) \) as a pro-module, we shall construct an extension of a transitive Harish-Chandra pair by observing a compatibility condition between derivations of its fibres and formal vector fields, in the special case \( E \) is a Courant-Dorfman algebroid.

We construct this extension of Harish-Chandra pairs as an extension of a transitive pair by a tautological pair. Further, we obtain a transitive structure on \( \mathcal{C}_{or_X} \) adopted to the latter. Our last piece of geometry is therefore to construct a \((g,K)\)-structure adopted to the extension, for it will be this Harish-Chandra pair that supports the vertex \( \mathcal{O} \)-algebra as a Harish-Chandra module. Ergo, we may localize \( \mathfrak{V}_E \) along the same.
Let us describe the extension of Harish-Chandra pairs. Let \( \mathcal{E} \) be the Courant-Dorfman algebroid corresponding to the Lie-Rinehart bialgebroid \((\mathcal{L}, \mathcal{L}^\vee)\), as above. Consider the Lie algebra \( \mathfrak{w}_E \) of derivations of \( \mathcal{J}_x(\mathcal{E}) \) consisting of derivations \( \chi \) such that the image of bracketed sections of \( \mathcal{E} \) interpolates the derivation, viz.

\[
\chi([e, e']) = [\chi(e), e'] + [e, \chi(e')]
\]

and a formal vector field \( \xi \in \mathfrak{w} \) interpolating the symmetric bilinear form of sections of \( \mathcal{E} \), viz.

\[
\xi(\langle e | e' \rangle) = \langle \chi(e) | e' \rangle + \langle e | \chi(e') \rangle
\]

This consideration is analogous to considering the derivations of \( \mathcal{J}_x \), that is, the Lie algebra \( \mathfrak{w} \).

The formal vector field \( \xi \) is uniquely determined by \( \chi \) by the definition of the derivation of a Courant-Dorfman algebra and so we obtain the following compatibility condition of such formal vector fields and such derivations of \( \mathcal{J}_x(\mathcal{E}) \), viz.

\[
\chi(fe) = \xi(f)e + f\chi(e)
\]

for any \( \mathcal{O}_x \)-linear morphism \( \chi : \mathcal{J}_x(\mathcal{E}) \to \mathcal{J}_x(\mathcal{E}) \) interpolating the bracket of sections and \( \xi \in \mathfrak{w} \) interpolating the bilinear form.

Let \( \mathfrak{w}_J \subset \mathfrak{w} \) be the Lie subalgebra of formal vector fields consisting of elements \( \xi \in \mathfrak{w} \) satisfying this compatibility condition. This Lie algebra is specifically obtained as the image of the the morphism of Lies algebra of derivations of the Courant-Dorfman algebroid \( \mathfrak{X} : \text{Der}_{CD}(\mathcal{E}) \to T_X \) together with the map of \( T_X \) to its fibre, for then one has \( T_x \to \mathfrak{w} \) by the construction of \( \mathfrak{w} \) in chapter three.

Hence, the Lie algebra \( \mathfrak{w}_E \) to consist pairs of \( \mathcal{O}_x \)-linear morphisms \((\xi, \chi)\) satisfying the above compatibility condition. This Lie algebra is obtained as a semi-direct product of the Lie algebras \( \mathfrak{w}_J \) and \( \mathcal{O}_x \otimes \mathfrak{gl}(E) := O(\mathcal{E}) \), viz.

\[
\mathfrak{w}_E = O(\mathcal{E}) \ltimes \mathfrak{w}_J
\]
The semi-direct product is formed by the action given by formal vector fields acting on formal \( \mathfrak{gl}(E) \)-valued power series \( O(e) \), viz.

\[
[\xi_1 + \chi_1, \xi_2 + \chi_2] = [\xi_1, \xi_2] + [\chi_1, \chi_2] + \xi_2(\chi_1) - \xi_1(\chi_2)
\]

where \( \chi_i \in O(e) \) and \( \xi_i \in \mathfrak{w} \).

Notice, such pairs do not necessarily preserve the ideal \( \mathfrak{m} \otimes \mathcal{E}_x \subset \mathcal{J}_x(E) \), as \( \mathfrak{w}_J \) does not necessarily preserve \( \mathfrak{m} \). Restricting to \( \mathfrak{w}_0 \cap \mathfrak{w}_J = \mathfrak{w}_{J,0} \subset \mathfrak{w}_J \), such pairs preserve the ideal \( \mathfrak{m} \otimes \mathcal{E}_x \), as \( \mathfrak{w}_{J,0} \) preserves \( \mathfrak{m} \). Denote the Lie subalgebra that preserves \( \mathfrak{m} \otimes \mathcal{E}_x \) by \( \mathfrak{w}_{E,0} \). This Lie subalgebra is an ideal since \( \mathfrak{w}_0 \) is fundamental. Moreover, it is a pronilpotent Lie subalgebra for it is obtained by restriction of \( \mathfrak{w}_J \) to \( \mathfrak{w}_0 \). Denote the corresponding pro-unipotent group by \( Aut(E, O) \).

The group \( Aut(E, O) \) consists of automorphisms of the fibre \( \mathcal{J}_x(E) \) preserving the ideal \( \mathfrak{m}_x \otimes \mathcal{E}_x \) that are compatible with invertible \( \mathcal{O}_x \)-linear endomorphisms \( \psi_x \) of \( \mathcal{J}_x(E) \) and elements of \( Aut(O) \), viz.

\[
\psi_x(fe) = \phi_x(f)\psi_x(e)
\]

for \( f \in \mathcal{O}_x \) and \( e \in \mathcal{J}_x(E) \). By construction, we have the following lemma.

**Lemma 5.3.** The pair \((\mathfrak{w}_J, Aut(J))\) form a transitive Harish-Chandra pair.

**Proof.** Since \( \mathfrak{w}_{J,0} = \mathfrak{w}_0 \cap \mathfrak{w}_J \subset \mathfrak{w}_J \) is a transitive Lie algebra, we have by the realization theorem a morphism of Lie algebras \( \mathfrak{R} : \mathfrak{w}_J \to \mathfrak{w} \) such that \( \mathfrak{R}^{-1}(\mathfrak{w}_0) = \mathfrak{w}_{J,0} \). Denote by \( Aut(J) \) the corresponding pro-unipotent group corresponding to \( \mathfrak{w}_{J,0} \). Then the morphism of Harish-Chandra pairs to \((\mathfrak{w}, Aut(O))\) is furnished simply by the inclusion, and is transitive by definition.

As our aim is to geometrize the vertex \( \mathcal{O} \)-algebra \( \mathfrak{H}_E \) by a lift of the torsor of formal coordinates followed by an extension, we recognize that the lemma determines a transitive structure on \( Cor_X \). Indeed, define the space \( GS_X \) to be the reduction of the structure group of \( Cor_X \) to the subgroup \( Aut(J) \subset Aut(O) \). Then, by the lemma, \( GS_X \) is a transitive structure.
on $C_X$. This gives us the transitive structure on $C_X$ required in our set-up. Last is to obtain an extension of $G_S_X$, but first we require both a Harish-Chandra pair and morphism of the same to $(w_J, Aut(J))$. This point is the content of the following theorem.

**Theorem 15.** The pair $(w_E, Aut(O, E))$ is a Harish-Chandra pair and the sequence of Harish-Chandra pairs

$$1 \rightarrow (O(e), Aut(O(e)) \rightarrow (w_E, Aut(O, E)) \rightarrow (w_J, Aut(J)) \rightarrow 1$$

is exact.

**Proof.** The only thing to prove is that the pair $(w_E, Aut(O, E))$ is an extension, as the pair forms a Harish-Chandra pair by its construction. To this end, one notes that since pairs in $w_{E,0}$ are $O_x$-linear and preserve the ideal $m \otimes E \subset J_x(E)$, therefore they act on the quotient $J_x(E)/m \otimes E_x \cong O_x$ by derivations. Moreover, the Lie subalgebra $w_{E,0}$ is of finite codimension in $w_E$ since $w_E/w_{E,0} \cong gl_n \oplus gl(E)$. Hence, by the realization theorem, there exists morphism of Lie algebras

$$\mathcal{R}: w_E \rightarrow w$$

such that $\mathcal{R}^{-1}(w_{J,0}) = w_{E,0}$, as the realization morphism factors through $w_J$. Thus, the kernel of $\mathcal{R}$ is $O(e)$.

Since $\mathcal{R}^{-1}(w_{J,0}) = w_{E,0}$, we obtain a corresponding morphism of pro-unipotent groups $\Phi : Aut(O, E) \rightarrow Aut(J)$ with kernel $Aut(O(e))$. Therefore, have the sequence

$$1 \rightarrow (O(e), Aut(O(e)) \rightarrow (w_E, Aut(O, E)) \rightarrow (w_J, Aut(J)) \rightarrow 1$$

is exact. In particular, we obtain a morphism of Harish-Chandra pairs

$$\Phi_{\mathcal{R}} : (w_E, Aut(E, O)) \rightarrow (w_J, Aut(J))$$

as desired. □

The last item of geometry now is to define a space adopted to the Harish-Chandra pair of the theorem, in other words, a $(w_E, Aut(O, E))$-structure. To this end, we define an analogue
of formal coordinates with respect to the the topological module \( \mathcal{J}_X(\mathcal{E}) \) and show that there exists an \( X \)-scheme parameterizing such formal coordinates.

Consider an open neighborhood \( U \subset X \) of \( x \in X \) and a system of formal coordinates \( \phi_X : \mathcal{J}_X(U) \cong \mathcal{O}_X(U)[[v_1, \ldots, v_n]] \) on \( X \). We give an analogue of systems of formal coordinates on \( X \) with respect to the sheaf of \( \mathcal{E} \)-jets. So, consider in addition a choice of frame for \( \mathcal{E} \) locally, say, \( \{e_1, \ldots, e_n\} \), so that \( \mathcal{J}_X(\mathcal{E}) \) is trivialized, viz.

\[
\psi_E : \mathcal{J}_X(\mathcal{E})(U) \cong \mathcal{O}_X(U)[[v_1, \ldots, v_n]] \otimes \bigoplus_{j=1}^n \kappa e_i
\]

and \( \psi_E \) is compatible with \( \phi_X \) as a morphism of \( \mathcal{J}_X \)-modules, that is, \( \psi_E(fm) = \phi_X(f)\psi_E(e) \) for \( f \in \mathcal{J}_X \) and \( e \in \mathcal{J}_X(\mathcal{E}) \). We say \( \psi_E \) is a system of formal \( \mathcal{E} \)-coordinates on \( X \). Given the above theorem, we define the \((\mathfrak{g}, K)\)-structure of systems of formal \( \mathcal{E} \)-coordinates adopted to the Harish-Chandra extension \((\mathfrak{w}_E, Aut(E, O))\), as follows. Denote by \( \text{Vor}_X \) the \( X \)-scheme of pairs \( (\phi_x, \psi_x) \) identifying \( \mathcal{J}_x \) with \( \mathcal{O}_x \) and \( \mathcal{J}_x(\mathcal{E}) \) with \( \mathcal{O}_x \otimes \bigoplus_{j=1}^n \kappa e_j \) such that \( \psi_x \) is compatible with \( \phi_x \) as an \( \mathcal{O}_x \)-module morphism. We refer to \( \text{Vor}_X \) as the vacuum space. We have the following main theorem.

**Theorem 16.** The \( X \)-scheme \( \text{Vor}_X \) is a \((\mathfrak{w}_E, Aut(E, O))\)-structure extending the transitive structure \( \mathcal{G}\mathcal{S}_X \) on \( \text{Cor}_X \)

**Proof.** First, we prove that \( \text{Vor}_X \) indeed represents an \( X \)-scheme by first elaborating upon the formal scheme corresponding to \( \mathcal{J}_X(\mathcal{E}) \) given above. We appropriately denote this formal scheme by \( \mathcal{J}et_X(\mathcal{E}) \). Our aim is to construct \( \mathcal{J}et_X(\mathcal{E}) \) in a manner consistent with the construction of \( \mathcal{J}et_X \) in chapter 3. In order to demonstrate this, we recall some important results on the projective completion of coherent \( \mathcal{O}_X \)-modules from [FL].

First, consider the geometric vector bundles \( \mathbb{V}(\mathcal{L}) \) and \( \mathbb{V}(\mathcal{L}^\vee) \) corresponding to the locally free sheaves underlying the Lie-Rinehart algebroids \( \mathcal{L}^\vee \) and \( \mathcal{L} \), respectively. Consider the \( X \)-scheme \( \mathbb{V}(\mathcal{E}) \) given by the pull back of the diagram.
We want to recall the embedding of $X$ in $V(\mathcal{E})$.

So, let us consider the associated projective bundle $P_X(\mathcal{E})$ associated to the coherent $\mathcal{O}_X$-module $\mathcal{O}_X \oplus \mathcal{E}$, that is, $P_X(\mathcal{E}) := \text{Proj}(\text{Sym}(\mathcal{O}_X \oplus \mathcal{E}))$, together with its structure morphism $\pi : P_X(\mathcal{E}) \to X$. In this special case, $P_X(\mathcal{E})$ is said to the the projective completion of $\mathcal{E}$. We have the universal exact sequence on $P_X(\mathcal{E})$

$$0 \to \mathcal{Q} \to \pi^*(\mathcal{O}_X \oplus \mathcal{E}) \to \mathcal{O}(1) \to 0$$

and the projection $\mathcal{O}_X \oplus \mathcal{E} \to \mathcal{O}_X$, which determines the zero section $z : X \to P_X(\mathcal{E})$ by the universal mapping property of the associated projective bundle since $\mathcal{O}_X$ is a line bundle on $X$. This embedding is regular since $\mathcal{E}$ is locally free and $z^*(\mathcal{Q}) = \mathcal{E}$ on $X = z(X)$. The latter equality follows from dualizing the universal exact sequence on the left \textit{loc.cit}.

The projective completion $P_X(\mathcal{E})$ decomposes into the disjoint union of $\text{Proj}(\mathcal{E}) \cup V(\mathcal{E})$. Moreover, the zero section $z(X) \subset P_X(\mathcal{E})$ is disjoint from the hyperplane at infinity, $\text{Proj}(\mathcal{E})$, so that $z(X) \subset V(\mathcal{E})$. Since the embedding is regular, its conormal sheaf is given by $I_z/(I_z)^2 \cong \mathcal{E}$, where $I_z$ is the ideal sheaf of the regular embedding or zero section. More generally, $z^{-1}\mathcal{O}_V(\mathcal{E})/I_z^{k+1} \cong \text{Sym}^{\leq k}(I_z/(I_z)^2) \cong J^k_X(\mathcal{E})$ as defined above. Taking limits, we have

$$\prod_{k=1}^{\infty} (I_z)^k/(I_z)^{k+1} \cong \prod_{k=0}^{\infty} \text{Sym}^{\leq k}(I_z/(I_z)^2)$$

and define $J_{et}X(\mathcal{E})$ to be the formal scheme associated to $(X, \text{Sym}^{\leq k}(I_z/(I_z)^2))$. In other words,

$$J_{et}X(\mathcal{E}) = \text{colim}_k \text{Spec}(\text{Sym}^{\leq k}(I_z/(I_z)^2))$$
and

\[ \mathcal{J}_X(\mathcal{E}) \cong \prod_{k=0}^{\infty} \text{Sym}^{\leq k}(I_z/(I_z)^2) \]

\[ \cong \hat{\text{Sym}}(\mathcal{E}) \]

In particular, \( \mathcal{J}et_X(\mathcal{E}) \to \mathcal{V}(\mathcal{E}) \to X \) renders \( \mathcal{J}et_X(\mathcal{E}) \) as an \( X \)-scheme and \( \mathcal{J}_X(\mathcal{E}) \) as an \( \mathcal{O}_X \)-algebra. This is a feature the structure sheaf of the completion of \( \mathcal{V}(\mathcal{E}) \) along the zero section \( z \) does not necessarily possess. Nonetheless, we can still compare this construction to that of the construction of \( \mathcal{J}et_X \) as the completion of the square \( X \times X \) along the regular diagonal embedding \( \Delta : X \to X \times X \).

Observe, \( \mathcal{O}_{\mathcal{V}(\mathcal{E})} \) is \( I_z \)-adically filtered, so that the structure sheaf of the completion of \( \mathcal{V}(\mathcal{E}) \) along the zero section is \( \lim_k \mathcal{O}_{\mathcal{V}(\mathcal{E})}/I_z^k := \mathcal{O}_z \), which in turns can be described in the lc-topology as the infinite direct product of the associated graded components with respect to the \( I_z \)-adic filtration, viz.

\[ \prod_{k} I_z^k / I_z^{k+1} \]

Assume

\[ o : \hat{\text{Sym}}(\mathcal{E}) \to \mathcal{O}_z \]

is an isomorphism, then the associated graded homomorphism of \( o \) is given by \( \text{Sym}(\mathcal{E}) \cong \bigotimes_{k} I_z^k / I_z^{k+1} \) if and only if its first graded component \( \text{gr}_o \) is the canonical isomorphism

\[ \mathcal{O}_{\mathcal{V}(\mathcal{E})}/I_z \oplus I_z/I_z^2 \to \mathcal{O}_X \oplus \mathcal{E} \]

cf. [Cal]

Conversely, since \( X \) and \( \mathcal{V}(\mathcal{E}) \) are both affine, we have that there exists a filtered algebra isomorphism of \( \hat{\text{Sym}}(\mathcal{E}) \) and \( \mathcal{O}_z \) such that the first graded component is the canonical isomorphism \( \text{loc.cit.} \). So indeed, one may regard \( \mathcal{J}et_X(\mathcal{E}) \) as the analogue of \( \mathcal{J}et_X \) when the
regular embedding of the diagonal is replaced by the regular embedding of the zero section in the geometric vector bundle $\mathbb{V}(\mathcal{E})$.

Recall, by [EGAIV], formal schemes over $X$ are closed under fibre products, so define $\mathcal{V}_{or_X} \subset (\tilde{\mathcal{E}}_{or_X})^{\times n}$ as the open subscheme of the $n$-fold fibre product defined by a condition similar to that of the defining condition of $\mathcal{C}_{or_X}$ above. Namely, power series automorphisms that are equivalent to $n$-tuples of power series $\phi_i$ such that the determinant of their Jacobian is a unit. Such tuples are sequences in $(\tilde{\mathcal{E}})^{\times n}$, where $\tilde{\mathcal{E}}$ is the completion of $I_z$ in $\hat{\text{Sym}}(\mathcal{E})$. For such an $n$-tuple, we have

$$(\tilde{\mathcal{E}}^{\times n})^{\times n} \to (\tilde{\mathcal{E}}) / (\tilde{\mathcal{E}})^2 \cong (\mathcal{E})^{\times n} \to^{\text{det}} \mathcal{E}$$

Now, as a fine moduli space, $\mathcal{V}_{or_X}$ represents the functor on $X$-schemes defined by pairs

$$(X, \phi) : (\mathcal{O}_{X}[v_1, \ldots, v_n]) \otimes \bigoplus_{i=1}^{n} \kappa e_i, \phi(\mathcal{E}) = \mathcal{m} \bigoplus_{i=1}^{n} \kappa e_i.$$  

[Gwil] proves isomorphisms $\phi$ are equivalent to pairs $(\phi, \psi)$. Indeed, given such a pair, one has

$$\mathcal{J}_{X}(\mathcal{E}) \cong \mathcal{J}_{X} \cong \mathcal{J}_{X} \otimes \mathcal{E} \cong \mathcal{J}_{X} \otimes \mathcal{E} \cong \mathcal{O}_{X}[v_1, \ldots, v_n] \otimes \mathcal{E}$$

as base change commutes with taking free algebras. Restricting to fibres, we have $\mathcal{V}_{or_X}$ indeed represents the above functor on the category of $X$-schemes. The lifting of $\mathcal{G}_{S_X}$ is furnished simply by forgetting $\psi$, that is,

$$\mathcal{V}_{or_X} \to \mathcal{G}_{S_X}$$

maps $(x, \phi, \psi)$ to $(x, \phi)$.
It is obvious that $\mathcal{V} or_X$ is a $(\mathfrak{w}_E, \text{Aut}(E, O))$-structure by construction, for the fibres of $\mathcal{V} or_X \to X$ are given by

$$\prod_{k=0}^{\infty} \text{Sym}^{\leq k}(E_x) \cong \prod_{k=0}^{\infty} \text{Sym}^{\leq k}(\mathcal{O}_{X,x}/\mathfrak{m}_x) \oplus \mathcal{E}_{X,x} \cong \mathcal{O}_x \otimes \oplus_{j=1}^{n} \kappa e_j.$$  

The importance of this space and how it relates to the geometric perspective on vertex algebras is the following theorem.

**Theorem 17.** Let $\mathfrak{V}_E$ be the vacuum algebra associated to the canonical $E$-module $R \oplus E$. Then $\mathfrak{V}_E$ is a $(\mathfrak{w}_E, \text{Aut}(O, E))$-module.

**Proof.** We have to show there are structure preserving maps

$$\mathfrak{w}_E \to \text{Der}_{VA}(\mathfrak{V}_E)$$

and

$$\text{Aut}(E, O) \to \text{Aut}_{VA}(\mathfrak{V}_E)$$

satisfying Axiom I in the definition. The intuition is fairly straightforward with [LimYam2] in mind. To this end, we restrict ourselves to the grading preserving automorphisms of $\mathfrak{V}_E$, which are given by automorphisms of the Courant-algebroid $E$ according theorem 12. Specifically, the algebraic group denoted $\text{Aut}_{CD}(E)$. Then, it follows from Weierstrass’ preparation theorem [Lang] that a compatible $\phi_x$ corresponds to a unit of $R$. Furthermore, an $\mathcal{O}_x$-linear isomorphism induces an automorphism of $E$ as an $R$-module, by a choice of a free basis. Taken together, one has a pair of an automorphism of $R$ and an $R$-module automorphism of $E$ and such a pair furnishes an automorphism of the Courant-Dorfman
algebra $E$, that is, an element of $Aut_{CD}(E)$, a grading preserving automorphism of $\mathfrak{Y}_E$. Therefore, one has $Aut(E, O) \to Aut(E) \subset Aut_{VA}(\mathfrak{Y}_E)$. We can make this more precise by way of the Atiyah algebroid of an $R$-module and its relationship to the derivations of a Courant-Dorfman algebra. Indeed, in some ways, our extension $(\mathfrak{m}_E, Aut(O, E))$ of Harishi-Chandra pairs could be thought of the linearly compact expression of the Atiyah algebroid.

Let $Der_{CD}(E)$ be the Lie algebra of Courant-Dorfman algebra derivations corresponding to $Aut(E)$. This is the Lie algebra consisting of $R$-module derivations $D$ of $E$ satisfying $D[e, e'] = [D(e), e'] + [e, D(e')]$ such that there exists a unique derivation of $R$, say $\tau$, such that $\tau(\langle e|e' \rangle) = \langle D(e)|e' \rangle + \langle e|D(e') \rangle$. It follows that $\tau$ is uniquely determined from $D$, that is, $\rho(D) = \tau$, where by abusing notation we use the anchor notation to indicate the induced map of Lie algebras $Der_{CD}(E) \to Der(R)$. As such, one has $D(fe) = \tau(f)e + fD(e)$, for $e \in E$ and $f \in R$.

Now let us recall further the Atiyah sequence associated to a finitely generated projective module, $E$. Denoted $\mathcal{A}t(E)$, this module consists of first order differential operators of order 1 with scalar symbol. More specifically, $\mathcal{A}t(E) \subset \mathfrak{gl}(E) \otimes \text{Der}_\kappa(R)$ projecting to the identity in $\text{Id}_E \otimes \text{Der}_\kappa(R) \subset \mathfrak{gl}(E) \otimes \text{Der}_\kappa(R)$. Accordingly, $\mathcal{A}t(E)$ is positioned in the exact sequence

$$0 \to \mathfrak{gl}(E) \to \mathcal{A}t(E) \to \text{Der}_\kappa(R) \to 0$$

and from this perspective, consists of pairs $(D, \tau)$ such that, $D(fe) = \tau(f)e + fD(e)$. Moreover, as above in the geometric setting, $\mathcal{A}t(E)$ is a Lie-Rinehart algebra with obvious anchor map. In particular, $\mathcal{A}t(E)$ is a $\kappa$-Lie algebra.

Observe, there is a Lie algebra homomorphism $\mathfrak{m}_E \to \mathcal{A}t(E)$, by $Aut(O, E)$-equivariance. So, we can apply this to $\mathfrak{Y}_E$ by noticing that $\mathcal{A}t(E) = Der_{CD}(E)$ by definition. For this, we construct a morphism from $\mathcal{A}t(E) \to \text{End}(R \oplus E)$ compatible with the action of $\mathcal{A}t(E)$. This is a consequence of the short five lemma applied to the following diagram, viz.
Consequently, a morphism from $At(E) \to At(R \oplus E)$ is equivalent to a morphism of Lie algebras $At(E) \to \text{End}(R \oplus E)$ compatible with the action of $At(E) = \text{Der}_{CD}(E)$ [BB]. Since $\mathfrak{V}_E$ is strongly finitely generated by $R \oplus E$, we have, together with the results of Li and Yamskulna [LiYam2] a Lie algebra morphism

$$w_E \to \text{Der}_{VA}(\mathfrak{V}_E)$$

Similarly, restricting to $w_E, o$ and exponentiating, we obtain a morphism of algebra groups $Aut(O, E) \to Aut_{CD}(E)$. Indeed, applying Exp to the above morphism of infinitesimal objects, one has, viz.

$$0 \to Aut(R \oplus E) \to Aut_{CD}(R \oplus E) \to Aut(X) \to 0$$

Again, the morphism of algebraic groups $Aut_{CD}DE \to Aut_{CD}(R \oplus E)$ furnishes a morphism $Aut(O, E) \to Aut_{VA}(\mathfrak{V}_E)$ by loc.cit.

The compatibility of these actions is obvious since it is obtained simply by restriction, which is precisely the Harish-Chandra condition. Therefore, $\mathfrak{V}_E$ is a $(w_E, Aut(O, E))$-module, as desired.

Collectively, the above list of results imply the following theorem.
Theorem 18. Let $\mathfrak{V}_E$ be the vertex $\mathcal{O}$-algebra and $\mathfrak{V}_{\text{or}}_X$ the $(\mathfrak{w}_E, \text{Aut}(O, E))$-structure defined above. Given the Harish-Chandra module structure on the vacuum algebra in the above theorem, one has that the associated vector bundle

$$\mathfrak{V}_X(\mathcal{E}) := \mathfrak{V}_{\text{or}}_X \times_{\text{Aut}(E, O)} \mathfrak{V}_E$$

is a pro-finite $\mathcal{O}_X$-module. Moreover, it is flat according to the existence of the connection given by the principal $\mathfrak{w}_E$-structure on $\mathfrak{V}_{\text{or}}_X$.

Proof. Obvious by the formalism of Harish-Chandra geometry. In particular, the localization requires the result of Li and Yamskulna [LiYam2].

Finally, we have our main result in the following theorem.

Theorem 19. Let $M$ be a module over the vacuum algebra $\mathfrak{V}_E$ and $(\mathfrak{w}_E, \text{Aut}(O, E))$ the Harish-Chandra pair defined above. Then $M$ is a Harish-Chandra module and the associated vector bundle

$$\mathcal{M}_X(\mathcal{E}) := \mathfrak{V}_{\text{or}}_X \times_{\text{Aut}(E, O)} M$$

is a pro-finite $\mathcal{O}_X$-module. Moreover, it is flat according to the existence of the connection given by the principal $\mathfrak{w}_E$-structure on $\mathfrak{V}_{\text{or}}_X$.

Proof. Let $M$ be a module over the vacuum algebra $\mathfrak{V}_E$. Then $M$ is by construction a restricted module over the Lie algebra $\mathfrak{c}$. Given an automorphism of the canonical $E$-module $R \oplus E$, one has an induced endomorphism of the loop algebra $\mathfrak{c}$. In this manner, the action of $\text{Aut}(O, E)$ on $\mathfrak{V}_E$ is extended to a module action on $M$. A similar remark applies to the Lie algebra. Therefore, $M$ is a Harish-Chandra module over $(\mathfrak{w}_E, \text{Aut}(E, O))$. Accordingly, one is able to form the associated vector bundle $\mathcal{M}_X(\mathcal{E})$ as one does in the case of the vertex $\mathcal{O}$-algebra.

In particular, given the relationship between modules $M$ over the Zhu algebra $A(E)$ of $\mathfrak{V}_E$ and $U_R L$ the universal enveloping algebroid of $L$, one has that, in general, such repre-
sentations are not conformal. The sheaf $M_X$ therefore provides a geometric interpretation of representations of vertex algebras without an action of the Virasoro algebra, as desired.
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