Photometric flux in EXONEST

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Photometric Flux in EXONEST

by

Steven K. Young

A Thesis
Submitted to the University at Albany, State University of New York
in Partial Fulfillment of
the Requirements for the Degree of
Master of Science

College of Arts & Sciences
Department of Physics
May 2017
To my parents, sister, and two good friends. Your constant encouragement and support made this all possible.
Abstract

As a planet orbits its parent star, the amount of light that reaches Earth from that system is dependent on the dynamics of that star system. Known as photometric variations, these slight changes in light flux are detectable by the Kepler Space Telescope and must be fully understood in order to properly model the system. There are four main factors that contribute to the photometric flux: reflected light from the planet, thermal emissions from the planet, doppler boosting in the light being emitted by the star, and ellipsoidal variations in the star. The total observed flux from each contribution then determines how much light will be seen from the star system to be used for analysis. Previous studies have normalized the photometric variation fluxes by the observed flux emitted from the star. However, normalizing data inherently and unphysically skews the result which must then be taken into account. Additionally, when the stellar flux is an unknown it is impossible to normalize the photometric variation fluxes with respect to it. This paper will preliminarily attempt to improve upon the existing studies by removing the source of the deviation for the flux results, i.e. the stellar flux. The fluxes found from each photometric variation factor will then be incorporated into EXONEST, an algorithm using Bayesian inference, that will be implemented for characterizing extrasolar systems.
Acknowledgement

I would like to sincerely thank my adviser Dr. Kevin Knuth. I am pleased to be able to conduct research in Astronomy, a field I have always wanted to work in since elementary school. Working with the exoplanet group has no doubt taught me how to perform research in both theory and computational methods.

I would also like to deeply thank my colleagues, Jenn Carter, Tony Gai, and Bryan D’Angelo for their help in running me through the exoplanet project. Since I was a late starter in the group, they have tirelessly assisted me with getting up to speed. They were very eager to aid me in resolving my problems and they were never annoyed with my myriad of trivial questions.

Finally, I would like to thank the members of my committee: Dr. Vivek Jain and Dr. Ariel Caticha. While I have been learning how to write code for research, I’ve been to Dr. Jain numerous times throughout graduate school to ask him questions about programing. In Dr. Caticha’s General Relativity class, I saw his enthusiasm for theoretical physics which motivated me to continue developing my own personal interest in astronomy.
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Chapter 1

Introduction

1.1 Exoplanets

The term exoplanet is the general name given to any planet that exists outside our solar system. There is a provisional definition, created by the IAU’s Working Group on Extrasolar Planets, that an extrasolar object must satisfy in order for it to be considered a planet [1]. The upper limit for the true mass of objects with solar metallicity (the fraction of mass, resembling our Sun, that is neither H nor He) must be below $13 \, M_{\text{Jup}}$ – which is the limiting mass for thermonuclear fusion of deuterium to occur. The minimum mass should be the same as that used for our solar system. It does not matter how these objects are formed but they must be in orbit around at least one star or stellar remnant to be a “planet”. Substellar objects with true masses above the limiting mass for thermonuclear fusion of deuterium are considered “brown dwarfs” despite how they formed or where they are located. Finally, substellar objects with masses below the limiting mass that are free-floating in young star clusters are “subbrown dwarfs” [1].

The first time an exoplanet was detected was in 1992 from precisely timing the pulsar, PSR1257+12 [2]. Two planets orbit the pulsar, between 0.33 AU and 0.5 AU, which are approximately three times more massive than Earth. It was not until 1995 that Mayor and
Queloz discovered an exoplanet around a main sequence star [3]. They were able to observe 51-Pegasi, a Sun-like star that had a Jupiter-mass companion. The planet was found from measuring the periodic variations in the radial velocity of the star. From the line-of-sight velocity, the team had determined the minimum mass of the planet had to be about half of Jupiter’s mass. The minimum mass was reported since the mass is coupled with the inclination of the orbit. This is because a small planet in an edge-on view orbit may give the same line-of-sight velocity as a larger planet in an inclined orbit. A larger planet with a nearly edge-on orbit gives the greatest Doppler effect. Thus, this method of detection is biased toward larger planets with smaller inclined orbits.

1.2 Methods of Detection and Characterization

1.2.1 Transit Method and Kepler Space Telescope

The process known as transiting or primary eclipse occurs when an exoplanet passes in front of its host star as seen from Earth. Occultation or secondary eclipse occurs when the planet passes behind the star. Transit surveys searching large numbers of star systems discover transiting planets by observing the characteristic dip in star light that reaches Earth during primary and, more rarely, secondary transits. In addition to the main decreases in light during transit, a companion object will also affect the total observed flux in other ways. Although these photometric variations are small compared to the effects of the transit, they are still detectable. There is a larger probability of observing a transit for planets with short orbital periods. Larger planets will also correspond to larger photometric variations. These two facts combined means the method of studying transits is biased towards larger planets with shorter orbital periods.

The space-based telescope, Kepler, was the largest contributor to discovering exoplanets using the transit method. From March 2009 to May 2013, Kepler had monitored the flux of over 100,000 individual stars. The photometer within Kepler has the capability to detect
a change in a star’s brightness equal to 20 ppm for stars that are >250 times fainter than what the naked eye can see [4]. The detection of exoplanets by Kepler is also made more accurate since a positive detection requires it to record three or more transits with a consistent period, brightness change, and duration [4]. The data used in this work are obtained from the Kepler database [5].

Figure 1.1: During a transit, the planet blocks out a fraction of the star light. As the planet comes out of transit, the planet’s day-side progressively comes into view and the total flux increases. During the secondary eclipse, when the planet passes behind the star, the flux dips again and, when the planet comes out from behind, the flux returns to maximum but begins to decrease until the planet exits the primary transit. The first to fourth contact points are shown by the dashed circles. The total transit duration $t_T$ lasts from first to fourth contact while $t_F$ is from second to third contact. (Modeled after Perryman figure 6.9 [1].)

The CHaracterising ExOPlanet Satellite (CHEOPS) is a future mission, scheduled for
launch in 2017, that will improve on the photometric variation precision [6]. The goal is to bring the precision into the low ppm range of <10 ppm in flux detection. CHEOPS will review the star systems that have previously confirmed exoplanets to produce more detailed characterization and identification of those systems. It will have a bandpass ranging from 0.4 to 1.1 microns which spans wavelengths from blue visible light into the infrared. Another mission, which will have the photometric precision to search for earthlike planets orbiting Sun-like stars, is the European Space Agency’s, PLAnetary Transits and Oscillations (PLATO) [7]. Planned to be launched in 2024, in addition to looking for habitable planets, PLATO will also study the effects of stellar oscillations to provide more information on the star’s mass, size, and age.

1.2.2 Other Methods

Microlensing and direct imaging comprise two additional methods of exoplanet detection. Microlensing tries to look for the slight perturbations in starlight flux due to gravitational lensing by planets traversing in front of their star [8]. Direct imaging, on the other hand, attempts to see the exoplanet directly by applying a mask to block out the intense light from the star [1]. A future NASA observatory that will incorporate both microlensing and direct imaging techniques is the Wide Field Infrared Survey Telescope (WFIRST). It will be a 2.4 m space telescope that is set to begin observations in the mid-2020’s. Included in the telescope is a coronagraph that will allow for direct imaging and spectroscopy measurements of exoplanets and debris disks around stars [9].

In terms of studying the exoplanets themselves, the Spitzer Space Telescope had characterized exoplanet atmospheres using spectroscopy [10]. The James Webb Space Telescope is also set to perform future characterizations of exoplanetary atmospheres [11].
Chapter 2

Orbital Mechanics

As the planet revolves around its host star, its position in relation to its star and Earth will affect the modeling of the photometric effects.

2.1 Orbits in Central Force Fields

In order to study transiting exoplanets, the motion of the planet around its host star must first be understood. Consider two masses, $m_1$ and $m_2$ revolving about a common center of mass of some reference frame. Let their respective positions be $\vec{r}_1$ and $\vec{r}_2$. The Langrangian for such a system, with total mass $M_{tot} = m_1 + m_2$, is then

$$L = \frac{1}{2}M_{tot}\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r}) \tag{2.1}$$

where $\vec{r}' = |\vec{r}_1 - \vec{r}_2|$ is the relative position of the two masses, $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass, $V(\vec{r})$ is the gravitational potential at $\vec{r}$, and $\dot{\vec{R}}$ is the velocity of the center of mass relative to the origin of the reference frame. If the center of mass motion is assumed to be zero (i.e. $\dot{\vec{R}} = 0$), this two-body problem reduces to the motion of a single particle with
mass $\mu$ and Lagrangian

$$L = \frac{1}{2} \mu \dot{r}^2 - V(r)$$

$$= \frac{1}{2} \mu (r^2 + r^2 \dot{\theta}^2) - V(r)$$

(2.2)

in polar coordinates. From this Lagrangian, the two resulting equations of motion are:

$$\mu r^2 \dot{\theta} = l$$

(2.3)

and

$$\mu \ddot{r} = \mu r \dot{\theta}^2 - \frac{\partial V(r)}{\partial r}.$$  

(2.4)

The equation (2.3) describes the angular motion, where $l$ is the angular momentum and equation (2.4) describes the radial motion of the particle. Solving the radial motion equation will give the radial separation distance between the two masses as a function of angle $\theta$. The angular motion equation gives $\dot{\theta} = \frac{l}{\mu r^2}$. Substituting $\dot{\theta}$ into (2.4) gives Newton’s second law

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} - \frac{\partial V(r)}{\partial r}.$$  

(2.5)

The right-hand side of (2.5) represents the net force acting on the system resulting from the inertia of the particle working to keep it moving in a straight line and the force of gravity between the two masses trying to prevent each other from escaping. Now writing the gravitational potential energy as $V(r) = k/r$, where $k$ is the gravitational parameter, $Gm_1$, the equation becomes

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} - \frac{k}{r^2}.$$  

(2.6)
From \( \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \dot{\theta} + \frac{dr}{d\theta} \ddot{\theta} \) so

\[
\mu \left[ \frac{d^2r}{d\theta^2} \left( \frac{l}{\mu r^2} \right)^2 + \frac{dr}{d\theta} \left( -\frac{2l^2}{\mu^2 r^5} \right) \left( \frac{dr}{d\theta} \right) \right] = \frac{l^2}{\mu r^2} + \frac{k}{r^2} \tag{2.7}
\]

\[
\frac{d^2r}{d\theta^2} \frac{l^2}{\mu^2 r^4} - \frac{2l^2}{\mu^2 r^5} \left( \frac{dr}{d\theta} \right)^2 - \frac{l^2}{\mu^2 r^3} = \frac{k}{\mu r^2} - \frac{1}{r} \left( \frac{dr}{d\theta} \right)^2 - \frac{1}{r} \left( \frac{dr}{d\theta} \right) = \frac{k}{\mu r^2}.
\]

Letting \( s = \frac{1}{r} \), then \( \frac{dr}{d\theta} = -\frac{1}{s^2} \frac{ds}{d\theta} \) and \( \frac{d^2r}{d\theta^2} = \frac{2}{s^3} \left( \frac{ds}{d\theta} \right)^2 - \frac{1}{s^2} \frac{d^2s}{d\theta^2} \),

\[
\frac{l^2}{\mu s^4} \left[ \frac{2}{s^3} \left( \frac{ds}{d\theta} \right)^2 - \frac{1}{s^2} \frac{d^2s}{d\theta^2} - 2s \left( - \frac{1}{s^2} \frac{ds}{d\theta} \right)^2 - \frac{1}{s} \right] = \frac{ks^2}{\mu} - \frac{l^2}{\mu^2} \left[ 2s \left( \frac{ds}{d\theta} \right)^2 - s^2 \frac{d^2s}{d\theta^2} - s^3 \right] = \frac{ks^2}{\mu} - \frac{l^2}{\mu^2} \left[ \frac{d^2s}{d\theta^2} + s \right] = \frac{k}{\mu}
\]

Finally, the general solution is

\[
s = \xi \cos(\theta - \theta_0) + \frac{k \mu}{l^2} = \frac{k \mu}{l^2} (1 + e \cos(\theta - \theta_0)) \tag{2.9}
\]

with \( e = \frac{\xi l^2}{k \mu} \). Given that \( r = 1/s \), this produces the position of mass \( m_2 \) relative to mass \( m_1 \)

\[
r(\theta) = \frac{l^2}{k \mu} \frac{1}{1 + e \cos(\theta - \theta_0)}. \tag{2.10}
\]
where $\frac{l^2}{k\mu} = \frac{l^2}{G(m_1 + m_2)}$ is known as the *semilatus rectum*\(^1\) [12]. Considering only elliptical motion for the planet, the *semilatus rectum* becomes $a(1 - e^2)$ so

\[
    r(\theta) = \frac{a(1 - e^2)}{1 + e \cos(\theta - \theta_0)} = \frac{a(1 - e^2)}{1 + e \cos \nu}
\]

with $a$ being the semi-major axis, $e$ being the eccentricity of the orbit, and $\theta_0$ being an arbitrary starting point, normally taken to be zero. The angle $\nu = \theta - \theta_0$ is known as the true anomaly.

### 2.2 Orbital Anomalies

Assuming that the planet is moving along a circular orbit with constant angular velocity, the mean anomaly, $M(t)$, is the angular distance between the position of the planet and its periastron (the location of the planet’s closest approach to its star). In other words, $M(t)$ is the location of the planet in its orbit and the rate of change of $M(t)$ describes the mean motion of the system, which is the angular velocity needed to complete one orbit. The mean anomaly for circular orbit is

\[
    M(t) = M_o + \frac{2\pi t}{T} = \frac{2\pi}{T} (t - t_p)
\]

where $M_o$ is the initial mean anomaly, $T$ is the orbital period of the planet, and $t_p$ is the time elapsed after the planet passes the periastron.

Kepler’s equation,

\[
    E(t) = M(t) + e \sin E(t),
\]

\(^1\)“Latus rectum” is a compound of the Latin words *latus*, meaning “side,” and *rectum*, meaning “straight.” “Semilatus rectum” is half the latus rectum.
provides the relation between the polar coordinates of the planet and the time elapsed for it to revolve from an initial position. In the equation, \( E(t) \), is the eccentric anomaly which is the angular separation between periastron and the projection of the planet’s position onto an auxiliary circle that circumscribes the orbit. Kepler’s equation is a transcendental equation meaning \( E(t) \) cannot be solved for directly given some \( M(t) \) and must be solved for numerically.

The relation between the true, \( \nu(t) \), and the eccentric, \( E(t) \), anomaly is then

\[
\tan \frac{\nu(t)}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E(t)}{2}.
\]  

Figure 2.1: The eccentric, \( E(t) \), and true, \( \nu(t) \), anomalies of a planet. Here, the periastron is labeled as \( P \) and the auxiliary circle is labeled as \( O \). (Source: Placek, 2014, used with permission)

Therefore, given the eccentric and true anomalies, the distance between the star and planet is

\[
r(t) = a(1 - e \cos E(t)) = \frac{a(1 - e^2)}{1 + e \cos \nu(t)}.
\]  

(2.15)
2.3 Euler Angles

Since our universe is three-dimensional, there are three angles, called Euler angles, that can be used to describe any rotation about the Cartesian coordinate axes. As a result, the position of a revolving planet can be determined using Euler angles. These three angles are denoted by $\phi$, $\theta$, and $\psi$ which represent rotations with respect to each of the three Cartesian axes, $z$, $x$, and $y$, respectively. Now, $R(\phi)$ is a rotation of angle $\phi$ about the $z$-axis, $R(\theta)$ is a rotation of angle $\theta$ about the $x$-axis, and $R(\psi)$ is a rotation of angle $\psi$ about the $y$-axis.

In orbital mechanics, in order to obtain the positional information of a planet and the orientation of its orbit with respect to the observer, the Euler angles are redefined in terms of the orbital elements $i$, $\omega + \nu$, and $\Omega$. Measurements of the orbital elements are taken with respect to the reference plane. The orbital inclination, $i$, is the angle between the reference plane and the orbital plane. In general, orbital inclination is measured from the $z$-axis. A planet with a face-on orbit has an inclination of $0^\circ$ while a planet with an edge-on orbit has an inclination of $90^\circ$. The argument of periapsis, $\omega$, is the angle between the periastron, or the point of closest approach by the planet to its star, and the point at which the orbital plane intersects the reference plane. The longitude of ascending node, $\Omega$, is the orbital rotation in the reference plane about the line of sight. The longitude of ascending node is usually set equal to zero when studying exoplanets. This is because the total light signature coming from a system remains the same when rotating the orbit in the reference plane. Figure 2.2 depicts each of the three orbital elements.

In order to obtain accurate studies of the photometric effects, an accurate prediction of the position of the planet, as a function of time, must be found. Given the three orbital elements – orbital inclination, argument of periapsis, and longitude of ascending node, the
position of a planet can be found in Cartesian coordinates at any time $t$:

$$
\begin{pmatrix}
    x(t) \\
    y(t) \\
    z(t)
\end{pmatrix}
= r(t)
\begin{pmatrix}
    \cos \Omega \cos(\omega + \nu(t)) - \sin \Omega \sin(\omega + \nu(t)) \cos i \\
    \sin \Omega \cos(\omega + \nu(t)) + \cos \Omega \sin(\omega + \nu(t)) \cos i \\
    \sin(\omega + \nu(t)) \sin i
\end{pmatrix}
$$

(2.16)

Equation 2.16 can be simplified to

$$
\begin{pmatrix}
    x(t) \\
    y(t) \\
    z(t)
\end{pmatrix}
= r(t)
\begin{pmatrix}
    \cos(\omega + \nu(t)) \\
    \sin(\omega + \nu(t)) \cos i \\
    \sin(\omega + \nu(t)) \sin i
\end{pmatrix}
$$

(2.18)

by setting $\Omega$ equal to zero since the observed flux variation is not affected by the rotation.
of the orbit in the reference plane. The position of the planet is given by the vector
\( \vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \) so that the unit vector \( \hat{r}(t) = \frac{\vec{r}(t)}{r(t)} \) points from the star to the planet. The phase angle \( \theta(t) \) of the planet is

\[
\theta(t) = \arccos(\hat{r}(t) \cdot \hat{r}')
= \arccos\left(\frac{z(t)}{r(t)}\right)
= \arccos(\sin(\omega + \nu(t)) \sin i) \tag{2.19}
\]

with \( \hat{r}' = \hat{z} \) being the line-of-sight.
Chapter 3

EXONEST

EXONEST is an exoplanet hunting algorithm developed by Placek, Knuth, and Angerhausen [13] that uses Bayesian model testing to model transits and the four photometric effects.

3.1 Bayesian Methods

This research was performed using Bayesian Methods. Bayes’ Theorem is

\[
P(x|D, M) = \frac{P(x|M)P(D|x, M)}{P(D|M)} \tag{3.1}
\]

where \( M \) is the hypothesized model, \( D \) is the recorded data, and \( x \) is the set of parameters (3.1) associated with model \( M \). Studying the right-hand side of Bayes’ Theorem first, in the numerator, the prior probability, \( P(x|M) \), is multiplied by the likelihood function, \( P(D|x, M) \), while in the denominator exists the evidence, \( P(D|M) \). The prior probability constitutes all the information we know about the parameter \( x \) before considering the data \( D \). The likelihood function is the probability that the model and its hypothesized parameter can produce the data. The evidence is the probability that the data is produced by the model \( M \), regardless of the model’s parameters; so it allows for the assessment of
the probability that the model describes the data. Moving to the left-hand side of Bayes’ Theorem is the posterior probability, \( P(x|D, M) \), which is the probability distribution that represents the updated beliefs about the parameter after considering the data. Essentially, Bayes’ Theorem takes what is known about a system before considering the data (prior probability) and then adjusts it with the ratio of two data-dependent terms (likelihood and evidence) to yield the posterior probability. The posterior probability is then used to estimate the model parameter values \( x \).

To find the evidence, integrate or marginalize over all model parameter values as shown:

\[
P(D|M) = Z = \int P(x|M)P(D|x, M)dx.
\]  

(3.2)

Therefore, the evidence is often known as the marginal likelihood and is considered to be a prior-weighted average of the likelihood. The more parameters a model has, the lower its uniform prior probability will be. This is because the unit probability is spread over a larger volume. With equal prior probabilities assigned to each model, meaning that no one model is biased over another, the Bayesian evidence will naturally favor the simpler models. Thus, if two models equally describe the same data, then the simpler model should be the one that is chosen.

To understand Bayes’ theorem better, consider the example where there are two models, \( M_1 \) and \( M_2 \), that should be compared and tested by applying them to a data set. Compute the posterior odds ratio by dividing the posterior probability of one model with the posterior probability of the other

\[
\frac{P(M_1|D, I)}{P(M_2|D, I)} = \frac{P(M_1|I)P(D|M_1)}{P(M_2|I)P(D|M_2)} = K \frac{P(M_1|I)}{P(M_2|I)}
\]

(3.3)

to determine which model is favored by the data. Here, \( I \) is the prior information that one has about the problem and \( K \) is the Bayes’ factor which represents the ratio of the model
evidences. Usually, the ratio of the model priors are set to imply no prior preference for either model. This means that the evidence quantifies the probability of a model when given the data such that the Bayes’ factor allows for the comparison of the probabilities of the two models.

Parameter Estimation

3.2 Nested Sampling

Nested Sampling is a calculation method, developed by John Skilling [14], which uses Markov Chain Monte Carlo (MCMC) methods to study probability space and find a posterior distribution. Bayes’ Rule relates the prior $\pi$, likelihood $L$, evidence $Z$, and posterior $P(x)$ giving

$$L(x) \times \pi(x) = Z \times P(x).$$

Normalisation must hold for the prior and posterior. Now, integration over all the parameters returns unity

$$\int \int \cdots \int \pi(x) dx = 1$$

and

$$\int \int \cdots \int P(x) dx = 1.$$ (3.6)

The aforementioned normalization conditions define a posterior and given Bayes’ Rule,

$$P(x) = \frac{L(x)\pi(x)}{Z},$$

the evidence becomes

$$Z = \int \int \cdots \int L(x)\pi(x) dx.$$ (3.8)
For systems with many parameters, the advantage of using nested sampling is that it allows the study of the parameter space without having to evaluate every possible outcome. The method works by using $N$ objects within parameter space $x$ that are randomly distributed within the prior $\pi$ and then applying the constraint $L(x) > L^*$. From the remaining probability space, a new object is then randomly selected that satisfies the constraint on the likelihood. This new object replaces the worst object from the previous iteration. Thus, due to the constraint, there will be an improved likelihood $L(x) > L^*$ and there will be a natural contraction towards regions with higher likelihood. As the process continues, the algorithm collapses the probability space at an approximate rate of $e^{1/N}$ where $N \neq 0$. The parameter space is explored more densely with increasing number of samples but this also increases the computation time. Currently, EXONEST uses Multinest which is a nested sampling algorithm that can efficiently sample multimodal and highly complex distributions [15]. Samples can be generated and then used to estimate parameters by taking the mean and standard deviation of the samples. This is one of the major advantages of nested sampling.

### 3.2.1 Evidence

The main goal of nested sampling is to stochastically integrate the uniform prior times the likelihood. The normalization factor or average of the likelihood over the prior is the evidence. It is the integral as a function of the prior that satisfies the nested sampling constraint. The evidence is thus expressed as

$$Z = \int_0^1 L(x_i) dx_i$$  \hspace{1cm} (3.9)
where the integral goes from 0 to 1 because the integration is over the prior. This can be numerically approximated using a weighted sum and the trapezoid rule, giving

\[
Z = \sum_{i=1}^{m} L_i \times \frac{1}{2}(x_{i-1} - x_{i+1})
\]  

(3.10)

with \( m \) being the number of iterations that the likelihood is summed over. An \( N \) number of points are randomly selected from the full prior, \( \pi(x) \), which will serve as the initial set of active samples. The volume of the initial prior is \( x_0 = 1 \) and the samples are sorted according to their likelihood values. The sample with the lowest likelihood is omitted and a new sample is randomly selected from the remaining prior that satisfies the constraint dictating that the new likelihood value must be larger than the previous worst sample \( L_i > L_{i-1} \). The new volume of the prior is defined as \( x_1 = t_1 x_0 \), with \( t_1 \) being a randomly chosen variable from the distribution \( Pr(t) = Nt^{N-1} \), and the volume is contained within an iso-likelihood contour. The volume shrinks at an average rate of \( \log x_i \approx -(i \pm \sqrt{i/N}) \).

The process stops by choosing the evidence to a specified precision. When regions of high probability are being discovered, initially, the likelihood will increase at a faster rate compared to the rate that the width of the prior is decreasing. Eventually, the likelihood will plateau and the decreasing width of the prior will overtake the likelihood. At that point, the evidence will no longer change significantly and the computation can be terminated.
Chapter 4

Photometric Variations

When a planet is orbiting its host star, there are four photometric effects that are detectable. Two of these effects, reflection of stellar emitted light and thermal emission of energy absorbed by the planet, are determined by studying the exoplanet. The other two, doppler boosting and ellipsoidal variations, occur from the influence of the planet on its host star.

To begin, first set up the geometry of the star system that is being studied relative to the observer. Let \( \hat{z} \) be the line-of-sight of the observer, perpendicular to the plane of the sky. Now the coordinate \( z \) describes the position of the planet with respect to the plane of the sky. Set \( z = 0 \) to be the location of an imaginary plane that passes through the center of the star. This imaginary plane is perpendicular to the line-of-sight of the observer and parallel to the plane of the sky. Any positions where \( z < 0 \) are positions that lie between \( z = 0 \) and the observer while any positions where \( z > 0 \) are positions that place \( z = 0 \) between the planet and the observer, as shown in Figure 4.1.

The model for a transiting planet is taken to be an eclipsing of a spherical star by an opaque and dark sphere. From Mandel and Agol [16], the ratio of the obscured to unobscured starlight flux for a uniform source is

\[
F^e(p, z) = 1 - \lambda^e(p, z) \quad (4.1)
\]
Figure 4.1: Geometry of the star system being studied relative to the observer. The line-of-sight of the observer is \( \hat{z} \) and the dashed line, bisecting the star represents the imaginary plane at \( z = 0 \).

where, for \( z < 0 \),

\[
\lambda^e = \begin{cases} 
0 & 1 + p < z \quad (\text{no obstruction}) \\
\frac{1}{\pi} \left[ p^2 \kappa_0 + \kappa_1 - \sqrt{\frac{4z^2 - (1 + z^2 - p^2)^2}{4}} \right] & |1-p| < z \leq 1 + p \quad (\text{partial obstruction}) \\
p^2 & z \leq 1 - p \quad (\text{full transit}) \\
1 & z \leq p - 1 \quad (\text{full transit with } R_p > R_\star) 
\end{cases}
\]  

(4.2)

and is the fractional area of the star obscured by the planet. Setting \( d \) to be the center-to-center distance between the star and the planet, \( R_p \) to be the radius of the planet, and \( R_\star \) to be the radius of the star, then \( z = d/R_\star \) is the normalized separation of the centers, \( p = R_p/R_\star \) is the size ratio,

\[
\kappa_1 = \arccos \left( \frac{1 - p^2 + z^2}{2z} \right),
\]

(4.3)

and

\[
\kappa_0 = \arccos \left( \frac{p^2 + z^2 - 1}{2pz} \right).
\]

(4.4)
For \( z > 0 \),

\[
\lambda^e = \lambda^{se} = \begin{cases} 
0 & 1 + p < z \quad \text{(no occultation)} \\
\frac{1}{p^2} \left[ p^2 \kappa_0 + \kappa_1 - \sqrt{\frac{4z^2 - (1 + z^2 - p^2)^2}{4}} \right] & |1 - p| < z \leq 1 + p \quad \text{(partial occultation)} \\
1 & p - 1 < z \leq 1 - p \quad \text{(full occultation)} \\
\frac{1}{p^2} & z \leq p - 1 \quad \text{(full transit with } R_p > R_*)
\end{cases}
\] (4.5)

and is the fractional area of the planet being occulted by the star.

The plasma at different locations of a star have different temperatures. When the star is observed at its center, the measured temperature will be the greatest since the observation penetrates into the hotter layers of the star, closer to the stellar core. However, as the observation moves towards the limb of the star, the measured temperature decreases due to observations limited only to the cooler plasma on the surface. Since the outer edge of the star is cooler than its center, it will appear darker compared to the center. This effect is known as limb-darkening [15].

Currently, EXONEST is using the quadratic limb-darkening model for studying transits. Let \( \theta \) be defined as the angle between the observer and the normal to the stellar surface. Then the normalized radial coordinate on the disk of the star is \( \mu = \cos \theta = \sqrt{1 - r^2} \), \( 0 \leq r \leq 1 \). The quadratic limb-darkening law has the quadratic in \( \mu \) and is described by the function

\[
I(r) = 1 - \gamma_1(1 - \mu) - \gamma_2(1 - \mu)^2
\] (4.6)

with \( I(r) \) being the intensity as a function of \( r \) or \( \mu \). Here, \( \gamma_1 + \gamma_2 < 1 \). During a transit, there are eleven different geometrical cases, as shown in Table 4. The light curve for quadratic limb-darkening is

\[
F = 1 - (4\Omega)^{-1} \left[ (1 - c_2) \lambda^e + c_2 [ \lambda^d + \frac{2}{3} \Theta(p - z) ] - c_4 \eta^d \right]
\] (4.7)
Table 4.1: Limb-Darkened Occultation

<table>
<thead>
<tr>
<th>Case</th>
<th>( p )</th>
<th>( z )</th>
<th>( \lambda^d(z) )</th>
<th>( \eta^d(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>((0, \infty))</td>
<td>([1 + p, \infty))</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>((0, \infty))</td>
<td>(\left[\frac{1}{2} + \left</td>
<td>p - \frac{1}{2}\right</td>
<td>, 1 + p\right))</td>
</tr>
<tr>
<td>III</td>
<td>((0, \frac{1}{2}))</td>
<td>((p, 1 - p))</td>
<td>(\lambda_2)</td>
<td>(\eta_2)</td>
</tr>
<tr>
<td>IV</td>
<td>((0, \frac{1}{2}))</td>
<td>(1 - p)</td>
<td>(\lambda_5)</td>
<td>(\eta_2)</td>
</tr>
<tr>
<td>V</td>
<td>((0, \frac{1}{2}))</td>
<td>(p)</td>
<td>(\lambda_4)</td>
<td>(\eta_2)</td>
</tr>
<tr>
<td>VI</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{3} - \frac{4}{9\pi})</td>
<td>(\frac{3}{32})</td>
</tr>
<tr>
<td>VII</td>
<td>(\frac{1}{2}, \infty)</td>
<td>(p)</td>
<td>(\lambda_3)</td>
<td>(\eta_1)</td>
</tr>
<tr>
<td>VIII</td>
<td>(\frac{1}{2}, \infty)</td>
<td>(\left[\left</td>
<td>1 - p\right</td>
<td>, p\right))</td>
</tr>
<tr>
<td>IX</td>
<td>((0, 1))</td>
<td>(\left(0, \frac{1}{2} - \left</td>
<td>p - \frac{1}{2}\right</td>
<td>\right))</td>
</tr>
<tr>
<td>X</td>
<td>((0, 1))</td>
<td>0</td>
<td>(\lambda_5)</td>
<td>(\eta_2)</td>
</tr>
<tr>
<td>XI</td>
<td>((1, \infty))</td>
<td>([0, p - 1])</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

where \(c_1 = c_3 = 0\), \(c_2 = \gamma_1 + 2\gamma_2\), and \(c_4 = -\gamma_2\). Now, assigning \(k = \sqrt{(1 - a)/(4zp)}\), \(q = p^2 - z^2\), and using the complete elliptic integrals of the first, second, and third kind \((K(k), E(k), \text{and } \Pi(n, k) \text{ respectively})\), the various functions that define \(\lambda^d\) and \(\eta^d\) are

\[
\begin{align*}
\lambda_1 &= \frac{1}{9\pi \sqrt{p^2}} \left[(1 - b)(2b + a - 3) - 3q(b - 2)\right] K(k) + 4pz(z^2 + 7p^2 - 4) E(k) - \frac{3q}{a} \Pi \left(\frac{a - 1}{a}, \frac{1}{k}\right), \\
\lambda_2 &= \frac{2}{9\pi \sqrt{1 - a}} \left[(1 - 5z^2 + p^2 + q^2) K(k^{-1}) + (1 - a)(z^2 + 7p^2 - 4) E(k^{-1}) - \frac{3q}{a} \Pi \left(\frac{a - b}{a}, \frac{1}{k}\right)\right], \\
\lambda_3 &= \frac{1}{3} + \frac{16p}{9\pi} (2p^2 - 1) E(1/2k) - \frac{(1 - 4p^2)(3 - 8p^2)}{9\pi p} K(1/2k), \\
\lambda_4 &= \frac{1}{3} + \frac{2}{9\pi} [4(2p^2 - 1) E(2k) + (1 - 4p^2) K(2k)], \\
\lambda_5 &= \frac{2}{3\pi} \cos^{-1}(1 - 2p) - \frac{4}{9\pi} (3 + 2p - 8p^2), \\
\lambda_6 &= -\frac{2}{3} (1 - p^2)^{3/2}, \\
\eta_1 &= (2\pi)^{-1} [\kappa_1 + 2\eta_2\kappa_0 - \frac{1}{4}(1 + 5p^2 + z^2) \sqrt{(1 - a)(b - 1)}], \\
\eta_2 &= \frac{p^2}{2} (p^2 + 2z^2).
\end{align*}
\]
The flux from the primary transit is

\[ F_{\text{primary}} = -F_\star \lambda^e \]  

(4.9)

where \( F_\star \) is the stellar flux received at Earth. The primary transit includes a negative sign which indicates that \( F_{\text{primary}} \) reduces the total observed photometric flux and also takes into account, \( \lambda^e \), the fractional area of the star that is blocked out by the planet.

It was common practice to normalize with respect to the stellar flux [15], however, the value of \( F_\star \) is not known. This method of normalizing by an unknown parameter is inherently risky. Doing so skews the estimation of the other parameters because a source of systematic error is basically being ignored. Thus, the deviation from the actual result is an unphysical artifact which must be taken into account when performing further studies. Here, \( F_\star \) is taken to be a parameter to be estimated or marginalized out and will not be used to normalize the following photometric fluxes.

### 4.1 Reflected Light

The first effect of a planet orbiting its star is the reflection of stellar emitted light by the planet’s surface and/or atmosphere. The amount of reflected light that the observer receives depends on the orbital position of the planet with respect to its star and the observer. This means that the observer will see the characteristic phases (New, Crescent, Full, etc.) as the planet moves.

To model reflected light, it is first assumed that the star radiates light isotropically and that the planet is a sphere with Lambertian reflectance. The total light that is reflected from the planet is then the integral over the illuminated surface as seen from the line-of-sight.
The infinitesimal luminosity, with $A_{\text{eff}}$ as an effective albedo, is

$$dL_p = A_{\text{eff}} F_0 \hat{n} \cdot \hat{r} dA,$$  \hspace{1cm} (4.10)$$ reflected from a surface element, $\hat{n} dA$, on a planet. $\hat{n} = \sin \alpha \cos \beta \hat{x} + \sin \alpha \sin \beta \hat{y} + \cos \beta \hat{z}$ is the unit normal vector to the planetary surface and $\hat{r} = \sin \theta(t) \hat{x} + \cos \theta(t) \hat{y}$ is the unit orbital radius vector. The surface area element on the planet is $dA = R_p^2 \sin \alpha d\alpha d\beta$ and the incident stellar flux is

$$F_0 = \frac{L_\star}{4\pi r(t)^2}$$ \hspace{1cm} (4.11)$$ where $L_\star$ is the stellar luminosity and $r(t)$ is the time-varying orbital separation between the planet and its star. Taking the dot-product of the normal vector to the planet surface with the unit orbital radius vector,

$$\hat{n} \cdot \hat{r} = \sin \theta(t) \sin \alpha \cos \beta + \cos \theta(t) \sin \alpha \sin \beta$$

$$= \sin \alpha \sin(\theta(t) + \beta)$$ \hspace{1cm} (4.12)$$ and then integrating with respect to $dL_p$,

$$L_p = \int dL_p$$

$$= \int A_{\text{eff}} F_0 \hat{n} \cdot \hat{r} dA$$

$$= \int A_{\text{eff}} F_0 \sin \alpha \sin(\theta(t) + \beta) R_p^2 \sin \alpha d\alpha d\beta$$ \hspace{1cm} (4.13)$$

$$= A_{\text{eff}} F_0 R_p^2 \int_0^\pi \sin^2 \alpha d\alpha \int_{\theta}^{\pi-2\theta} \sin(\theta(t) + \beta) d\beta,$$

the luminosity of the planet is

$$L_p(t) = \frac{A_{\text{eff}} F_0 \pi R_p^2}{2} (1 + \cos \theta(t)).$$ \hspace{1cm} (4.14)$$
Substituting in the expression for incident stellar flux, (4.11), the luminosity of the planet becomes

\[
L_p(t) = \frac{A_{\text{eff}}}{8} \frac{R_p^2}{r(t)^2} L_\star (1 + \cos \theta(t)). \tag{4.15}
\]

The flux of the planet observed from Earth is

\[
F_{\text{refl}}(t) = \frac{L_p(t)}{4\pi d^2} \frac{A_{\text{eff}}}{8} \frac{R_p^2}{r(t)^2} L_\star (1 + \cos \theta(t)) \tag{4.16}
\]

where \(d\) is the distance from the planet to Earth.

The albedo, \(A_{\text{eff}}\), can be written as

\[
A_{\text{eff}} = g A_s \tag{4.17}
\]

where \(A_s\) is the spherical albedo and \(g\) is a correction factor that is used to account for anisotropic scattering from the planet. In terms of the geometric and spherical albedos, this correction factor is \(g = \frac{4A_g}{A_s}\). Then substituting back into (4.17) gives

\[
A_{\text{eff}} = 4A_g \tag{4.18}
\]

which yields the final result for the reflected planetary flux component

\[
F_{\text{refl}}(t) = \frac{A_g}{2} \frac{R_p^2}{r(t)^2} \frac{L_\star}{4\pi d^2} (1 + \cos \theta(t)) \tag{4.19}
\]

\[
= \frac{A_g}{2} \frac{R_p^2}{r(t)^2} F_\star (1 + \cos \theta(t)).
\]

### 4.2 Thermal Emissions

The second effect is the thermal emission of radiation from a planet’s surface or atmosphere. Thermal radiation arises from stellar radiation which is absorbed by the planet. As this
starlight is absorbed, the surrounding area heats up and this heat is reemitted as radiation. Thermal effects therefore depend heavily on the day and night temperatures of the planet with the temperature varying in a way that is similar to reflected light. According to Placek [15], reflected light and thermal emissions vary sinusoidally in circular orbits and it was found that planets with eccentricities less than $e \sim 0.3$ produced light curves where these two photometric effects are nearly identical. There are opportunities, however, to distinguish between these two for sufficiently eccentric orbits. This is because the reflected light curve has the potential to significantly deviate from a sinusoid, since it takes into account the time-varying orbital separation between the planet and its star, $r(t)$. Thus, the ability to separate thermal emissions from reflected light depends on the eccentricity of the orbit.

Given that the thermal flux from the dayside of a planet is $F_p(T_d)$, the infinitesimal thermal luminosity from a surface element on the dayside is

$$dL_{Th,d} = F_p(T_d)\hat{n} \cdot \hat{r}dA. \quad (4.20)$$

Integrating over the visible dayside of the planet, the thermal luminosity of the dayside becomes

$$L_{Th,d}(t) = \int dL_{Th,d}$$

$$= F_p(T_d)R_p^2 \int_0^\pi \sin^2 \alpha d\alpha \int_{-\theta}^{\pi-2\theta} \sin(\theta(t) + \beta)d\beta$$

$$\quad = F_p(T_d)\pi R_p^2 \frac{1 + \cos \theta(t)}{2}.$$  \hspace{1cm} (4.21)

This means the flux received on Earth from the thermal luminosity is

$$F_{Th,d}(t) = \frac{L_{Th,d}(t)}{4\pi d^2}$$

$$\quad = \frac{F_p(T_d)R_p^2}{8d^2}(1 + \cos \theta(t)). \quad (4.22)$$
The Kepler bandpass detects the range of stellar flux

\[ F_* = \int B(T_{\text{eff}})K(\lambda) d\lambda \]  \hspace{1cm} (4.23)

found by integrating the product of the spectral radiance of a blackbody

\[ B(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \]  \hspace{1cm} (4.24)

evaluated at the effective temperature of the star \( T = T_{\text{eff}} \) and the Kepler response function \( K(\lambda) \).

The planetary thermal flux detected by the Kepler bandpass is

\[ F_{Th} = F_{Th,d} + F_{Th,n} \]  \hspace{1cm} (4.25)

using the dayside and nightside temperatures \( T_d \) and \( T_n \). The thermal flux from the dayside is

\[ F_{Th,d}(t) = \frac{R_p^2}{8d^2} (1 + \cos(\theta(t))) \int B(T_d)K(\lambda) d\lambda \]  \hspace{1cm} (4.26)

and from the nightside is

\[ F_{Th,n}(t) = \frac{R_p^2}{8d^2} (1 + \cos(\theta(t) - \pi)) \int B(T_n)K(\lambda) d\lambda. \]  \hspace{1cm} (4.27)

Thus, the thermal flux component becomes

\[ F_{Th} = \frac{R_p^2}{8d^2} \left[ (1 + \cos(\theta(t))) \int B(T_d)K(\lambda) d\lambda + (1 + \cos(\theta(t) - \pi)) \int B(T_n)K(\lambda) d\lambda \right]. \]  \hspace{1cm} (4.28)
4.3 Doppler Boosting

As the planet and its host star orbit a common center of mass, there is a relativistic Doppler effect that occurs with the emitted stellar light. An observer will see an increase in stellar flux when the star is approaching the observer and a decrease in flux when the star is receding. Doppler boosting varies at the same frequency as the above two effects except being 90° out of phase. The photometric effect arising from Doppler Boosting follows from the derivation given by Rybicki and Lightman [17].

The energy in frame $S$ moving at velocity $-v$ with respect to a particle is

$$dW = \gamma dW' .$$  \hfill (4.29)

The time interval is

$$dt = \gamma dt'$$  \hfill (4.30)

where $dt'$ is the proper time of the particle. Now, the total power emitted in frames $S$ and $S'$ are

$$P = {dW \over dt} \quad \text{and} \quad P' = {dW' \over dt'}$$  \hfill (4.31)

which means

$$P = P' .$$  \hfill (4.32)

In the instantaneous rest frame of the particle, consider an amount of energy $dW'$ that is emitted into the solid angle $d\Omega' = \sin \theta' d\theta' d\phi'$ about the direction at angle $\theta'$ to the $x'$ axis. Introduce

$$\mu = \cos \theta \quad \text{and} \quad \mu' = \cos \theta'$$  \hfill (4.33)

such that

$$d\Omega = d\mu d\phi \quad \text{and} \quad d\Omega' = d\mu' d\phi' .$$  \hfill (4.34)
Energy and momentum form a four-vector. The transformation of the energy of the radiation is

\[ dW = \gamma (dW' + v dP') = \gamma (1 + \beta \mu') dW' = \gamma (1 + \beta \cos \theta') dW' . \] (4.35)

Next, the aberration of light is

\[ \mu = \frac{\mu' + \beta}{1 + \beta \mu'} \] (4.36)

which describes the apparent motion of astronomical objects about their true positions depending on the velocity of the observer. Differentiating \( \mu \) with respect to \( \mu' \) gives

\[
d\mu = \frac{d\mu'}{(1 + \beta \mu')^{-1} + (\mu' + \beta)(-1)(1 + \beta \mu')^{-2} \beta d\mu'} = \frac{\beta(\mu' + \beta)}{(1 + \beta \mu')^2} d\mu' = \frac{d\mu'}{1 + \beta \mu'} = \frac{d\mu'}{\gamma^2(1 + \beta \mu')^2} \] (4.37)

and because \( d\phi = d\phi' \),

\[ d\Omega = \frac{d\Omega'}{\gamma^2(1 + \beta \mu')^2} . \] (4.38)

Therefore,

\[ \frac{dW}{d\Omega} = \gamma (1 + \beta \mu') dW' \cdot \frac{\gamma^2(1 + \beta \mu')^2}{d\Omega'} = \gamma^3 (1 + \beta \mu')^3 \frac{dW'}{d\Omega'} . \] (4.39)

The power \( P' \) emitted in the rest frame \( S' \) is found by dividing \( dW' \) by the time interval \( dt' \). However, in the frame of the stationary observer, \( S \), there are two possible time intervals that can be used to divide \( dW \).

The first time interval, \( dt = \gamma dt' \), is the interval during which the emission occurs in
frame $S$. From this, the emitted power in frame $S$ is

\[
P_e = \frac{dW}{dt} = \gamma (1 + \beta \mu') \frac{dW'}{dt'} = (1 + \beta \mu') P' \quad (4.40)
\]

The second time interval, $dt_A = \gamma (1 - \beta \mu) dt'$, is the interval of the radiation that is received by the stationary observer in $S$. The moving source produces an extra factor of $(1 - \beta \mu)$ which is the retardation effect. From this, the received power in frame $S$ is

\[
P_r = \frac{dW}{dt_A} = \frac{\gamma (1 + \beta \mu') dW'}{\gamma (1 - \beta \mu) dt'} = \frac{1 + \beta \mu'}{1 - \beta \mu'} \frac{dW'}{dt'} = \frac{1 + \beta \mu'}{1 - \beta \mu'} P' \quad (4.41)
\]

so

\[
dP_r = \frac{1 + \beta \mu'}{1 - \beta \mu'} dP'. \quad (4.42)
\]
Thus, noting that flux is the amount of energy per unit time per unit solid angle (or power per unit solid angle),

\[ dF = \frac{dP_x}{d\Omega} = \frac{1 + \beta \mu'}{1 - \beta \mu} dP' \cdot \frac{\gamma^2 (1 + \beta \mu')^2}{d\Omega'} \]

but substituting in the equation for \( \mu' \),

\[
\begin{align*}
1 - \frac{\beta \mu' + \beta}{1 + \beta \mu'} & \quad 1 - \frac{\beta \mu' + \beta^2}{1 + \beta \mu'} \\
1 + \beta \mu' - \frac{\beta \mu' - \beta^2}{1 + \beta \mu'} & \quad \frac{1 - \beta^2}{1 + \beta \mu'} \\
\frac{1}{\gamma^2 (1 + \beta \mu')} & \quad (1 + \beta \mu') \gamma^2 (1 + \beta \mu') \frac{\gamma^2 (1 + \beta \mu')^2}{d\Omega'} \\
& \quad = (1 + \beta \mu')^4 dP' \cdot \frac{\gamma^2 (1 + \beta \mu')^2}{d\Omega'} \\
& \quad = \gamma^4 (1 + \beta \mu')^4 dP' \\
& \quad = \gamma^4 (1 + \beta \mu')^4 dF'.
\end{align*}
\]

Going back to (4.36) and solving for \( \mu' \),

\[
\begin{align*}
\mu & = \frac{\mu' + \beta}{1 + \beta \mu'} \\
1 + \beta \mu' & = \frac{\mu' + \beta}{\mu} \\
\mu' (\beta \mu - 1) & = \beta - \mu \\
\mu' & = \frac{\beta - \mu}{\beta \mu - 1}.
\end{align*}
\]
Substituting $\mu'$ back into (4.43), the final result is

$$dF = \gamma^4 (1 + \beta \mu')^4 dF'$$

$$= \gamma^4 \left( 1 + \beta \frac{\beta - \mu}{\beta \mu - 1} \right)^4 dF'$$

$$= \gamma^4 \left( \frac{\beta^2 - 1}{\beta \mu - 1} \right)^4 dF'$$

$$= \gamma^4 \left( \frac{-\gamma^{-2}}{-(1 - \beta \mu)} \right)^4 dF'$$

$$= \frac{1}{\gamma^4 (1 - \beta \mu)^4} dF'. \quad (4.45)$$

$P_r$ is the power which is actually measured by an observer. $P_r$ also has the expected symmetry property for yielding the inverse transformation by interchanging the primed and unprimed variables, along with a sign change of $\beta$. If the radiation is isotropic (or nearly isotropic) in the star’s frame, then the angular distribution in the observer’s frame will be highly peaked in the forward direction where $\theta \approx 0$. For the star moving in a radial, nonrelativistic velocity $v_r$ such that $\beta_r = \frac{v_r}{c} \ll 1$ so $\gamma \approx 1$, relative to the observer will have an observed flux of

$$F = F_* \left( \frac{1}{\gamma^4 (1 - \beta_r \cos \theta)^4} \right) \approx F_* \left( \frac{1}{(1 - \beta_r)^4} \right)$$

$$F^{(0)}(\beta_r) = \frac{1}{(1 - \beta_r)^4}, \quad F^{(1)}(\beta_r) = (F^{(0)}(\beta_r))' = \frac{4}{(1 - \beta_r)^5}$$

$$\approx F_* \left( \frac{F^{(0)}(0) \beta_r^0 + F^{(1)}(0) \beta_r^1}{0!} \right)$$

$$\approx F_* \left( 1 + 4 \beta_r \right)$$

$$\approx F_* \left( 1 + 4 \frac{v_r}{c} \right). \quad (4.46)$$

As a result, the Doppler boosted flux is

$$F_{\text{boost}} = -4F_* \beta_r. \quad (4.47)$$
with the negative sign accounting for how the coordinate system was defined. Recall that
\( z < 0 \) is when the planet is orbiting around the front hemisphere of the star and \( z > 0 \)
is when the planet is orbiting around the back hemisphere of the star. In this way, the
addition or subtraction of the boosted flux from the total flux is determined by the sign of
the radial velocity of the star. When the radial velocity is negative, the star is approaching
the observer and the boosted flux attains a positive sign, meaning the total flux receives a
contribution from the boosted flux. When the radial velocity is positive, the star is receding
from the observer and the boosted flux remains a negative, meaning the total flux decreases
by the boosted flux.

\[ \text{4.4 Ellipsoidal Variations} \]

As the planet orbits its star, it is gravitationally pulling on the star as well. The resulting
tidal forces then warp the stellar surface into an ellipsoid. “To first order, the star is
shaped like a prolate spheroid with the semi-major axis pointing approximately toward the
planet” [15]. Ellipsoidal variations occur at twice the frequency of the reflected/thermal
variations.

\[ \text{4.4.1 BEER} \]

The BEER model for the observed flux from ellipsoidal variations is currently used in
EXONEST. This method was developed by Faigler & Mazeh, 2011 [18]. BEER stands for
BEaming, Ellipsoid, and Reflection and is a model that is used to describe the periodic
deviations that occur in observed fluxes of a star due to a close proximity planet. After
the data has been cleaned, where the jumps and outliers are removed and the long-term
variation of the lightcurve are subtracted, it is then fitted with a model that includes
the ellipsoidal effect for the period, \( P_{\text{orb}} \), of the planet’s orbit. The ellipsoidal effect is
approximated by a cosine function, relative to phase zero when the planet is in front of the
star at $t_{\text{conj}}$.

$$F_{\text{ellipse}}(t) = -F_\star \beta \frac{M_p}{M_*} \left( \frac{R_\star}{r(t)} \right)^3 \sin^2(i) \cos \left( \frac{4\pi}{P_{\text{orb}}} \hat{t} \right)$$  \hspace{1cm} (4.48)

where $r(t)$ is the distance from the star to the planet, $\hat{t} = t - t_{\text{conj}}$, and

$$\beta = 0.15 \frac{(15 + u)(1 + g)}{3 - u}$$  \hspace{1cm} (4.49)

with $u$ being the limb-darkening coefficient and $g$ being the gravity-darkening coefficient. The limb-darkening and gravity-darkening coefficients are found by modeling the metallicity and effective temperature of the star. The amplitude of $F_{\text{ellipse}}$ is modified by accounting for the inclination of the orbit, $i$, with respect to the observer. Equation (4.48) becomes

$$F_{\text{ellipse}}(t) = -F_\star \beta \frac{M_p}{M_*} \left( \frac{R_\star}{r(t)} \right)^3 \sin^2(i) \cos(2\theta)$$  \hspace{1cm} (4.50)

after the orbital period is replaced with $2\pi/T$, which is the parameter EXONEST uses.

### 4.4.2 Kane & Gelino (2012)

Kane and Gelino [19] proposed a different model for ellipsoidal variation which accounts for changes in the observed flux by using the phase angle $\theta$. The flux equation for this model is

$$\frac{F_{\text{ellipse}}(t)}{F_\star} = \beta \frac{M_p}{M_*} \left( \frac{R_\star}{r(t)} \right)^3 \left[ \cos^2(\omega + \nu(t)) + \sin^2(\omega + \nu(t)) \cos^2 i \right]^{\frac{1}{2}}$$  \hspace{1cm} (4.51)

with the amplitude being the same as in BEER. Note the normalization by $F_\star$.  

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4.4.3 Modified Kane & Gelino

A modified version of the Kane and Gelino (2012) model was proposed by Placek, Knuth, and Angerhausen [13], noting the normalization by $F_\star$ again.

$$\frac{F_{\text{ellipse}}(t)}{F_\star} = \beta \frac{M_p}{M_\star} \left( \frac{R_\star}{r(t)} \right)^3 \left[ \cos^2(\omega + \nu(t)) + \sin^2(\omega + \nu(t)) \cos^2 i \right]$$  \hspace{1cm} (4.52)

Here, the discontinuity in the first derivative of the original model is gone by removing the square root. $M_p$ is the mass of the planet, $M_\star$ is the mass of the star, $R_\star$ is the radius of the star, $r(t)$ is the distance from the star to the planet, $\nu(t)$ is the true anomaly, $\omega$ is the argument of periapsis, and $\beta$ is the gravity-darkening exponent given by

$$\beta = \frac{\log \left( \frac{GM_\star}{R_\star^2} \right)}{\log T_{\text{eff}}}$$  \hspace{1cm} (4.53)

with $T_{\text{eff}}$ being the effective temperature of the star [15]. Gravity-darkening arises from ellipsoidal distortions of the stellar surface by the orbiting planet. Plasma on the surface of the star that is pulled farther from the core will be cooler. This primarily occurs at the tidal bulges where there is the greatest gravitational interaction between the planet and the star. Since the stellar surface there is at a greater distance from the center of the star, there is lower surface gravity and lower effective temperature at those regions. This means the bulges are, in fact, dimmer than the rest of the stellar surface.

Stellar rotation also causes the star to become an oblate spheroid. As the star rotates about a symmetry axis, the star will be squished at its poles. Since the poles of the spheroidal star are closer to the core, the plasma at the poles will be hotter than the plasma at the equator. Therefore, the poles will be brighter than the equator.

According to Placek [15], one can approximate the amplitude of the variations in equation (4.52) through considering tidal acceleration of a small portion of a host star at distance
\( r - R_\star \) from the planet.

\[
a_T = G M_p \left( \frac{1}{(r - R_\star)^2} - \frac{1}{r^2} \right)
\]

\[
= \frac{G M_p}{r^2} \left( \frac{1}{(1 - R_\star/r)^2} - 1 \right) \tag{4.54}
\]

Expanding the above equation to first order in \( R_\star/r \),

\[
a_T = \frac{G M_p}{r^2} \left( 1 - 2 \frac{R_\star}{r} + \ldots - 1 \right)
\]

\[
\approx \frac{2 G M_p R_\star}{r^3} \tag{4.55}
\]

The ratio of the tidal acceleration \( a_T \) to the surface gravity on the star \( a_g = G M_\star/R_\star^2 \) is then

\[
\frac{a_T}{a_g} \propto \frac{M_p}{M_\star} \left( \frac{R_\star}{r} \right)^3 \tag{4.56}
\]

Due to the dependence of boosting and ellipsoidal variations on both true anomaly, \( \nu(t) \), and \( \omega \), the argument of periapsis, these effects can be modeled in both the cases of circular and eccentric orbital patterns.

### 4.4.4 EVIL-MC

EVIL-MC stands for Ellipsoidal Variations Induced by a Low-Mass Companion. The model was developed by Jackson [20] and works by projecting a grid onto the surface of a star which can then be used to predict the deviation from sphericity of the star caused by the planet for each projected stellar grid point.

On the surface of a star, the gravitational potential is

\[
U = \frac{G M_S}{R_S} + \frac{G M_p}{(A^2 - 2 R_S A \cos \psi + R_S^2)^{1/2}} - \frac{G M_p}{A^2} R_S \cos \psi + \frac{1}{2} \omega_S^2 R_S^2 (1 - \cos^2 \lambda) \tag{4.57}
\]

where \( G \) is the gravitational constant, \( M_S \) is the stellar mass, \( R_S \) is the distance from the center of the star to a point on the photosphere, \( M_p \) is the mass of the planet, \( A \) is the
distance from the star to the planet, \( \omega_S \) is the stellar rotation rate, \( \cos \psi = \mathbf{R}_S \cdot \hat{\mathbf{A}} \), and 
\( \cos \lambda = \mathbf{R}_S \cdot \hat{\omega}_S \). Set the stellar rotation to zero \( (\omega_S = 0) \).

The first term in (4.57) is the potential supplied by the star and the second term is the potential contributed by the planet. Taylor expanding the second term in \( R_S/A \), the first term will become proportional to \( \cos \psi \). The second term represents the force that keeps the star in orbit around the system’s orbital center of mass (barycenter) and is constant. Because this force does not change, it does not contribute to the tidal interaction between the star and the planet. The third term is therefore included in the equation in order to remove that force. Lastly, the fourth term is the potential due to centrifugal acceleration from the star’s rotation.

To find the deviation from sphericity, \( \delta R \), first normalize \( U \) to get the new potential \( \Phi \):

\[
\Phi \equiv U \left( \frac{R_0}{GM_S} \right) = \frac{1}{R} + \frac{q}{(a^2 - 2aR \cos \psi + R^2)^{1/2}} - \frac{q}{a^2} R \cos \psi \quad (4.58)
\]

where \( R = R_S/R_0 \), \( q = M_p/M_S \), and \( a = A/R_0 \). When \( \mathbf{R}_S \perp \hat{\mathbf{A}} \), define \( |R_S| \equiv R_0 \). The stellar radius can now be rewritten as \( R = (1 + \delta R)R_0 \). The surface of the star is defined by the isopotential contour, \( \Phi = \text{const} \) and the potential at \( \mathbf{R}_0 \) (when \( \cos \psi = 0 \)) is

\[
\Phi_0 = 1 + \frac{q}{(a^2 + 1)^{1/2}}. \quad (4.59)
\]

If \( \delta R \) is small, then \( \Phi \) can be expanded

\[
\Phi = \frac{1}{1 + \delta R} + \frac{q}{(a^2 - 2a(1 + \delta R) \cos \psi + (1 + \delta R)^2)^{1/2}} - \frac{q}{a^2} (1 + \delta R) \cos \psi \\
\approx (1 - \delta R) + \frac{q}{(a^2 - 2a \cos \psi + 1)^{1/2}} - \frac{q}{a^2} \cos \psi \quad (4.60)
\]
dropping second-order and higher terms. Finally, setting Equation 4.59 equal to Equation 4.60 and then solving for $\delta R$ yields

$$\delta R = q \left( [a^2 - 2a \cos \psi + 1]^{-1/2} - [a^2 + 1]^{-1/2} - \frac{\cos \psi}{a^2} \right) - \frac{\omega^2}{2a^3} \cos^2 \lambda. \quad (4.61)$$

The EVIL-MC algorithm uses quadratic limb-darkening, described at the beginning of Chapter 3. The observed ellipsoidal variation is computed using

$$F_{\text{ellipse}} = 1 - \frac{F_{\text{sphere}}}{F_{\text{star}}} \quad (4.62)$$

with $F_{\text{sphere}}$ being the flux from a spherical star and $F_{\text{star}}$ being the flux from the ellipsoidal star. It should be noted, however, that equation (4.62) does not explicitly accommodate the planet’s phase.
Chapter 5

Early Results

5.1 Total Observed Flux

In order to compute the net photometric variation of a modeled exoplanetary system, first start with calculations of the orbital position $(r(t), \theta(t))$, with respect to time, for the given planet or planets. Combining this with the model parameters that are used to describe the star and planet, the four components of the photometric flux can then be computed. The four photometric fluxes include the light reflected from the planet, the thermal emission from the dayside and nightside of the planet, Doppler boosting of the starlight, and the ellipsoidal variations in the shape of the star. A predictive model of the observed photometric flux variations can finally be generated by summing these photometric flux contributions

$$F_{\text{tot}} = F_* + F_{\text{primary}} + (F_{\text{Th}} + F_{\text{refl}})(1 - \lambda^{se}) + F_{\text{ellipse}} + F_{\text{boost}}.$$  

Here, the sum of the thermal flux, $F_{\text{Th}}$, and reflection flux, $F_{\text{refl}}$, is also multiplied by $(1 - \lambda^{se})$ because it takes into account how much of the planet is being occulted by the star. This affects how much of the dayside is able to be observed during a secondary transit.
In this total flux, $F_{primary}$ is from Eq. 4.9, $F_{refl}$ is from Eq. 4.19, $F_{Th}$ is from Eq. 4.28, $F_{boost}$ is from Eq. 4.47, and $F_{ellipse}$ is from Eq. 4.50.

5.2 Preliminary Findings

In this preliminary study of the method to use the total photometric flux without normalizing with respect to $F_\star$, it was found that there are two main problems arising from this method. The first problem involves the term $F_\star$ which is approximately five orders of magnitude greater than the photometric fluxes of interest. Since $F_\star$ is summed with the other fluxes, their contributions to the total flux become overwhelmed. The second problem pertains to $F_{Th}$ being divided by $d^2$ (or the distance from the star to Earth squared) since this means the distance to Earth must be estimated in addition to the other model parameters.

Previously [15], the total flux was normalized by $F_\star$ in order to eliminate the first stated problem. However, the $F_\star$ that the total flux is being divided by is actually an average flux. It is an average of the stellar light fluctuations due to including the stellar and planetary photometric variations as well as stellar variability. As such, it is not the true apparent magnitude that Kepler receives if the star does not have stellar activity and there were no other objects in the star system. Instead, the true value of $F_\star$ is $F_\star + C$ where $C$ is a term to account for $F_\star$ not being the average. Chapter 6 will provide a more in-depth discussion on this issue and what will be attempted to solve it.
Chapter 6

Ongoing Work

6.1 Finding True $F_*$

As stated in Section 5.2, the current stellar flux is an average flux, now labelled as $F'_*$, which is

$$F'_* = F_* + C$$

(6.1)

where $F_*$ is the genuine non-averaged stellar flux and $C$ is a term that adjusts $F_*$ to fit the data. At present, $C$ is defined to be some constant that accounts for the overall shift from $F_*$ due to transit photometric effects. As such, it cannot account for individual events pertaining to stellar effects and planetary effects. The next immediate step to improve this work is to begin estimating the value of $F_*$. This will provide an original baseline for the stellar flux and will allow for $F_*$ to be marginalized from the total photometric flux.

One possible strategy would be to use the secondary transit to approximate $F_*$ since only stellar effects are observed when the planet is behind the star. More specifically, only the ellipsoidal variation in the star is observed during a secondary transit. There are two points when the ellipsoidal variation is at a minimum. These are when the planet is in new and full phase. During a secondary transit, the planet is always undergoing the full phase which means the secondary transit always yields a minimum ellipsoidal variation.
flux. Thus, the observed flux at a secondary transit is likely to be the best representation of the genuine stellar flux and if $F_{\text{ellipse}}$ can be properly modeled, it can be subtracted from the secondary transit dip to produce the baseline stellar flux. A second possible strategy for approximating $F_\star$ would be to consider the blackbody of the effective temperature of the star and then integrating over the wavelengths that are observable by Kepler.

### 6.2 Marginalizing $F_\star$

$F_\star$ is taken to be some nuisance parameter. It can thus be marginalized so that it does not need to be taken into account. The total photometric flux is

$$F_{\text{tot}} = F_\star + F_{\text{primary}} + (F_{\text{Th}} + F_{\text{refl}})(1 - \lambda^{se}) + F_{\text{ellipse}} + F_{\text{boost}}$$

$$= F_\star + F_{\star} \left[ -\lambda^e + \frac{A_g R_p^2}{2 r(t)^2} (1 + \cos \theta) - \frac{A_g R_p^2}{2 r(t)^2} (1 + \cos \theta(t)) \lambda^{se} \right.$$  

$$- \beta \frac{M_p}{M_\star} \left( \frac{R_\star}{r(t)} \right)^3 \sin^2(i) \cos(\theta) - 4 \beta_r \] + F_{\text{Th}} - F_{\text{Th}} \lambda^{se}$$

$$= F_\star + F_{\star} \left[ -\lambda^e + \frac{A_g R_p^2}{2 r(t)^2} (1 + \cos \theta(t)) - \frac{A_g R_p^2}{2 r(t)^2} (1 + \cos \theta(t)) \lambda^{se} \right.$$  

$$- \beta \frac{M_p}{M_\star} \left( \frac{R_\star}{r(t)} \right)^3 \sin^2(i) \cos(\theta) - 4 \beta_r \] + \frac{R_p^2}{8 d^2} \left[ (1 + \cos \theta(t)) \int B(T_d) K(\lambda) d\lambda + (1 + \cos \theta(t) - \pi) \int B(T_n) K(\lambda) d\lambda \right]$$

$$- \frac{R_p^2}{8 d^2} \left[ (1 + \cos \theta(t)) \int B(T_d) K(\lambda) d\lambda + (1 + \cos \theta(t) - \pi) \int B(T_n) K(\lambda) d\lambda \right] \lambda^{se}$$

Beginning with the Gaussian likelihood,

$$P(D_i|F_\star, F, I) = \left(\frac{1}{\sigma \sqrt{2\pi}}\right)^N \exp \left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{N} (D(t_i) - F_{\text{tot}})^2 \right]$$

(6.3)
and then using Bayes’ theorem, the posterior probability density function is

$$P(F, D_i, I) \propto P(D_i | F, I) \times P(F | I)$$

$$= \left( \frac{1}{\sigma \sqrt{2\pi}} \right)^N \exp \left[ -\frac{1}{2\sigma^2} \sum_{i=1}^{N} (D(t_i) - F_{tot})^2 \right] P(F | I).$$ \hspace{1cm} (6.4)

It can be seen that $F_\star$ can be marginalized from the prior $P(F_\star | I)$. $F_\star$ should range from 0 to $h$ where $h$ is the brightest star in the sky. From this prior knowledge, $F_\star$ should have a uniform distribution. Thus, the prior probability is

$$P(F_\star | I) = \frac{1}{h - 0} = \frac{1}{h}.$$ \hspace{1cm} (6.5)

Let $A = -\lambda^e + \frac{A_\theta}{2} \frac{R_p^2}{r(t)^2} (1 + \cos \theta(t)) - \frac{A_e}{2} \frac{R_p^2}{r(t)^2} (1 + \cos \theta(t)) \lambda^e - \beta \frac{M_p}{M_\star} \left( \frac{R_\star}{a} \right)^3 \sin^2(i) \cos(\theta) - 4\beta$, and $B = F_{Th} - F_{Th} \lambda^e$ so that (6.2) becomes

$$F_{tot} = F_\star + F_\star A + B.$$ \hspace{1cm} (6.6)

Substituting (6.6) into the exponential in (6.4), the exponential becomes

$$\exp \left[ -\frac{1}{2\sigma^2} (D - F_\star(1 + A) - B)^2 \right].$$ \hspace{1cm} (6.7)
Finally marginalizing $F_\star$, the marginal likelihood for a single datum is

$$P(D|F,I) = \int_0^h P(D|F_\star, F,I) P(F_\star|I) \, dF_\star$$

$$= \frac{1}{h\sigma\sqrt{2\pi}} \int_0^h e^{-\frac{1}{2\sigma^2}(D-F_\star(1+A-B))^2} \, dF_\star$$

$$= \frac{1}{h\sigma\sqrt{2\pi}} \int_0^h e^{-\frac{1}{2\sigma^2}(-D+F_\star(1+A)+B)^2} \, dF_\star$$

$$u = \frac{F_\star(1+A) - D + B}{\sqrt{2}\sigma}, \quad du = \frac{1+A}{\sqrt{2}\sigma} \, dF_\star$$

$$= \frac{1}{h\sigma\sqrt{2\pi}} \frac{\sqrt{2}\sigma}{1+A} \int_{-D+B}^{D+h(1+A)+B} e^{-u^2} \, du$$

$$= \frac{1}{h\sigma\sqrt{2\pi}} \frac{\sqrt{2}\sigma}{1+A} \sqrt{\pi} \text{erf}(u)\bigg|_{-D+B}^{D+h(1+A)+B}$$

$$= \frac{1}{h\sigma\sqrt{2\pi}} \frac{\sigma}{\sqrt{2}(1+A)} \text{erf}\left(\frac{F_\star(1+A) - D + B}{\sqrt{2}\sigma}\right)\bigg|_0^h$$

$$= \frac{1}{h\sigma\sqrt{2\pi}} \left[ \frac{\sigma}{\sqrt{2}(1+A)} \text{erf}\left(\frac{h(1+A) - D + B}{\sqrt{2}\sigma}\right) - \frac{\sigma}{\sqrt{2}(1+A)} \text{erf}\left(\frac{0 - D + B}{\sqrt{2}\sigma}\right) \right]$$

$$= \frac{1}{h\sigma\sqrt{2\pi}} \left[ \frac{\sigma}{\sqrt{2}(1+A)} \text{erf}\left(\frac{h(1+A) - D + B}{\sqrt{2}\sigma}\right) - \frac{\sigma}{h\sqrt{2}(1+A)} \text{erf}\left(\frac{B - D_i}{\sqrt{2}\sigma}\right) \right]$$

The marginal likelihood for a set of data is thus

$$P(D_i|F,I) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \times \sum_{i=1}^N \left[ \frac{\sigma}{h\sqrt{2}(1+A)} \text{erf}\left(\frac{h(1+A) - D_i + B}{\sqrt{2}\sigma}\right) - \frac{\sigma}{h\sqrt{2}(1+A)} \text{erf}\left(\frac{B - D_i}{\sqrt{2}\sigma}\right) \right]$$

(6.9)
meaning the logarithm of the marginalized likelihood is

\[
\log P(D_i|F,I) = -\frac{N}{2} \log(2\pi \sigma^2) + \sum_{i=1}^{N} \log \left[ \sigma \sqrt{\pi} \frac{\sqrt{2}(1 + A)}{h \sqrt{2}(1 + A)} \left( \text{erf} \left( \frac{h(1 + A) - D_i + B}{\sqrt{2}\sigma} \right) \right) \right] 
\]

which will be incorporated into EXONEST.

It should be noted again that this marginalized likelihood was derived from the most general case where the value of \( F_* \) ranges from zero to \( h \). As of now, the most simple solution to find physical values for \( F_* \) would be to calculate it using

\[
F_* = \frac{L_*}{4\pi d^2} \tag{6.11}
\]

at different \( d \)'s to produce a rudimentary bound for \( F_* \). The stellar luminosity is given by

\[
L_* = 4\pi R_*^2 \sigma_{SB} T_{\text{eff}}^4 \tag{6.12}
\]

with \( \sigma_{SB} \) being the Stefan-Boltzmann constant. The resulting model will change depending on the value of \( h \) in the prior probability.

### 6.3 Pixel Response Function

A starting point to begin working through the model and better understanding the issues surrounding the value of \( F_* \) is to study Kepler photometry. Since Kepler was designed as a high-precision photometer and not as an imaging experiment, the observed flux from planet transits do not yield the incident stellar flux, \( F_* \), as an apparent magnitude. The Pixel Response Function (PRF) is a super-resolution representation of how starlight interacts with Kepler’s pixels [21]. Its output is represented as a continuous piece-wise polynomial...
function of sub-pixel position on each pixel [21]. Therefore, given a star’s pixel and sub-
pixel positions, it provides a continuous representation that enables the prediction of the
star’s flux value on any pixel [21]. Understanding the PRF computation is not meant to
be a solution, rather it will act as a good jump to provide insight into approximating $F_\star$.

### 6.4 De-Trending Kepler Data

In a separate but concurrent project, work must be done to de-trend the data obtained from
the Kepler Database. The light curve data pulled from the database contains time-series
trends. In order to study the data correctly, the light curve must be uniform, aside from
the periodic variations that are intrinsic to the curve itself. Any unnatural variation from
systematic errors, dominated by differential velocity aberration, thermal/focus drift, and
spacecraft motion, will skew the results used for analysis. Testing will be done on data
from Kepler-8b, which is a system with very crisp transit and photometric curves.

#### 6.4.1 PyKE

PyKE is an open source project, written in python, that is a pipeline developed to reduce
time-series trends that are systematic to Kepler [25]. Kepler pixel data is reduced and
analyzed according to the user’s specific research goals. The main uses of PyKE are for the
re-extraction of light curves from manually-chosen pixel apertures and co-trending and/or
de-trending data to reduce or remove systematic noise artifacts by using methods that
conform with the users’ specific requirements.
Bibliography


Appendices
Appendix A

Figure A.1: Permission grant from the original author of [15] to reproduce Figures 2.2.1 and 2.3.1 as Figures 2.1 and 2.2 in this paper.