1-1-2017

**Mathematics discourse and its relationship to the development of collective and individual student knowledge in the context of limits**

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Mathematics Discourse and Its Relationship to the Development of Collective and Individual Student Knowledge in the Context of Limits

by

Patterson Rogers

A Dissertation
Submitted to the University at Albany, State University of New York
In Partial Fulfillment of
The Requirements for the Degree of
Doctor of Philosophy

School of Education
Department of Educational Theory and Practice
2017
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Abstract

Observing a group discussion, it is not possible to infer an individual student’s understanding from the dialogue. When students discuss, they share their individual ideas to develop a collective understanding which in turn informs students’ individual understanding. Clarifying the relationship between collective and individual understanding provides insight into how individuals construct mathematical thinking. This study presented an embedded case study to answer three questions regarding the role discourse plays in student development of mathematical knowledge: (1) how individual students construct mathematical knowledge through discourse, (2) the nature of discourse which supports the development of mathematical knowledge, and (3) the relationship between the collective and individual students’ mathematical knowledge.

The setting of the study was the first semester of a year-long college calculus course. Case studies of six students were conducted along with a case study of the calculus class the students were enrolled. Sociocultural Theory (Vygotsky, 1978, 1986; Wertsch, 1985) and Harré model (Harré, 1983) provided the theoretical framework for this study. Multiple methods of data collection were employed. Audio and video recordings and field notes of each class meeting were collected along with student artifacts produced during the seven weeks the students covered limits and limit based topics. Participants completed a mathematical background and self-concept survey. Additionally, in-depth interviews were conducted with the six case study participants and the course professor. A qualitative longitudinal analysis was utilized.

The findings of this study revealed students’ pathway for construction of knowledge of limits. Students clung to their initial conceptions and required active guidance from their professor to build on this conception towards a mathematical understanding of limits. Even after
students constructed personal meaning, it was found that this understanding was tentative and
students easily reverted to collective narrative about limits. Discourse which resulted in zones of
proximal development and reflection helped to support student’s construction of knowledge.
Moreover, the study found several factors that influenced individual construction of knowledge:
imagery presented by the teacher, how the imagery aligned with student’s initial conceptions,
student’s perceptions of their classmates, and of themselves.
Dedication

In memory of my grandmother Valri P. Sandoe (December 31, 1934 - March 08, 2015). This work would not have been possible without her love and unconditional support. Whenever this journey felt too hard or unattainable she was there. She listened when I needed to vent and believed in me when I did not. Through the example she set I learned the resolve and determination needed to complete this lifelong ambition.
Acknowledgements

I want to thank all the participants in this study, the instructor and students, for welcoming me and making me a part of their class. It was a true joy to attend class each day. I learned so much from the experience, not just in terms of this research, and I can only hope they benefited from my presence as much as I benefited from the experience. An extra thank you to the six case study participants for sharing their pathway through knowledge. Their honesty and openness about math, learning, and life was invaluable.

I want to thank my dissertation committee, past and present, for their guidance for my construction of knowledge. I was very fortunate to have a number of people guide me through this dissertation. I want to thank Dr. Vicky Kouba for helping me begin this journey. I am deeply grateful for the long conversations about education, research, teaching, and life that have been instrumental in shaping my thinking of myself as a researcher and teacher. Even though it is unrecognizable, my work with Dr. Kouba became the foundation for what would eventually become this research. I want to thank Dr. Arthur Applebee, my initial chair, for his wealth of knowledge and experience mentoring students. He always seemed to have the right question or suggestion to guide my thinking as this study developed from a notion to a reality. He had a way of letting me come to terms with ideas on my own and steering me on the right path when I needed it. Reflecting on the final work, I realize how well he prepared me to conduct research on my own. I hope to emulate his mentoring style with my own students. I am eternally grateful to Dr. Alan Oliveira for sticking with me through this journey. He made sure I did not slip through the cracks and stepped in to chair my committee. His practical guidance and feedback have been instrumental for completing this research. I am thankful to Dr. Abbe Herzig for her guidance and friendship. Whenever I doubted myself, Abbe was there to give me words of encouragement and
gently push me when I needed it. Her honesty and perspective on this process helped me to finish this study. Lastly, I want to thank Dr. Caro Williams-Pierce for agreeing to sit on this committee. It was not easy to come in over midway through this process, never having met or worked together. Your feedback has been invaluable, helping me to reflect, clarify and improve this work.

To my husband, Joseph Bocanegra, my deepest thanks. You moved hundreds of miles to the land of ice and snow so I could pursue this dream. You gave me a sounding board and helped me to think through ideas when I got stuck. You held me tight through the tears and loss, and celebrated each triumph with me. You constantly reassured and encouraged me. I am eternally grateful for everything you did to help me finish this research. You have been my love and strength through this whole process.
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Chapter 1 – Introduction

Traditionally college mathematics courses have relied on a lecture format (Barker et al., 2004). In the tradition lecture format the instructor predominantly talks at students, occasionally initiating a question, eliciting a brief reply from students, and evaluating the reply (called IRE) (Mehan, 1979). The expectation is that students will take the concepts presented in class to independently understand the mathematics. This requires mathematical novices to make sense of nuanced mathematical concepts with minimal chance to clarify and revise their understanding with the professor or peers. It also perpetuates the practice of mathematics studied in isolation without collaboration or joint construction of knowledge. This environment is one factor that contributes to college students’ struggle with mathematics, especially abstract reasoning, conceptual knowledge, and communication skills (Barker et al., 2004).

Discourse in the College Mathematics Classroom

Current research points to dialogic discourse—oral communication between members (student or instructor) in the same class—as a means to support mathematical thinking and conceptual understanding (Cobb, Boufi, McClain, & Whitenack, 1997; Goos, 2004; Huang, Normandia, & Greer, 2005; Yackel, Cobb, & Wood, 1991). Discourse can help students organize and clarify their understanding of content, troubleshoot solutions, reflect on their learning, and construct mathematical objects (Cobb et al., 1997; Yackel, Cobb, & Wood, 1991; Sfard, 2000). Moreover, students demonstrate higher-level thinking during instruction which supports discourse (Wood, Williams, & McNeal, 2006). At the collegiate level, small-group learning in Science, Technology, Engineering, and Mathematics has been linked to more positive attitudes towards learning and greater academic achievement (Springer, Stanne, & Donovan, 1999). As a result, educational reforms in collegiate mathematics have begun to stress the use of discourse to
help support the development of students’ understanding, reasoning, and critical thinking about mathematics. This shift towards discourse is reflected in the 2004 Curriculum Guide for undergraduate mathematics of the Mathematical Association of America (MAA) which recommends that:

Every course should incorporate activities that will help all students progress in developing analytical, critical reasoning, problem-solving, and communication skills and acquire mathematical habits of the mind. More specifically these activities should be designed to advance and measure students’ progress in learning to … read mathematics with understanding and communicate mathematical ideas with clarity and coherence through writing and speaking (Barker et al., p. 13).

The curriculum guidelines suggest using, among other instructional practices, classroom conversations, informal oral presentations, and group work to help students develop mathematical thinking and communication skills.

However, merely facilitating student discussions in the mathematics class may not necessarily lead to student understanding (Cobb, Boufi, McClain, & Whitenack, 1997; Remillard, 2010). Moreover, the types of discourse which successfully help grade school students construct mathematical understanding may not be effective in the college setting as the mathematics becomes more abstract. For example, Remillard (2010) observed that college students’ discourse about proofs helped to clarify student understanding, but did not lead to the more sophisticated type of discourse needed to help students make sense of abstract concepts. Although students did not take the leap to more abstract discourse, opportunities arose which would allow the instructor to steer the discourse towards more abstract discussion. Unfortunately, even when college instructors want to incorporate discourse, the instructors feel
less competent at leading discussions than using a traditional lecture format (Nunn, 1996) and the MAA Curriculum Guide provides little practical guidance on how to incorporate discussion effectively into the college mathematics curriculum.

It is important to understand the nature of classroom discourse which helps to develop students’ mathematical knowledge – knowledge of mathematical facts, definitions, principles, and generalizations, including understanding of relationships between concepts (Baroody, Feil, & Johnson, 2007). Understanding this relationship involves understanding how student communications during in-class discussions (either small group or whole-class) influence the development of mathematical knowledge. However, exploring how discourse supports the development of mathematical knowledge is a complex task.

When students work together on a mathematical problem, they construct a dialogue to make sense of the given task, negotiate the group dynamics, and understand the mathematical concepts. Although students are trying to construct understanding individually, they are also trying to construct understanding as part of the group. The mathematical understanding constructed as a result of this group interaction is called collective mathematical understanding (Martin, Towers, & Pirie, 2006). When one observes a group discussion, it is difficult to infer an individual student’s understanding from the collective’s understanding (Cobb, Boufi, McClain, & Whitenack, 1997) since the two are developing reciprocally (Vygotsky, 1986), as student discourse informs the collective understanding which in turn informs student’s individual understanding which is then shared again with the group becoming collective understanding (Harré, 1983). One problem is that some students do not necessarily synthesize the collective’s understanding to construct their own individual understanding (Harré, 1983; Tall & Vinner, 1981). Knowing more about how mathematical discourse supports the construction of collective
and individual understanding may elucidate what types of discourse are productive in developing mathematical thinking and what implications this may have for teaching and learning.

**Purpose of the Study**

The purpose of this study is to determine the role discourse plays in student development of mathematical knowledge. Moreover, this study seeks to explore how in-class discourse (whole class or small groups) helps the collective to make meaning of concepts, and also influence individual understanding. The goal is to answer the following research questions:

1. How do individual students construct mathematical knowledge through discourse?
2. What is the nature of discourse which supports the development of mathematical knowledge?
3. What is the relationship between the collective and individual students’ mathematical knowledge?

To answer these questions, this study will be situated in the context of a calculus course. Calculus is an ideal course for exploring discourse which supports the development of mathematical knowledge. First, it is an introductory college mathematics course which historically students have struggled to understand (Ohland, Yuhasz, & Sill, 2004; Eng, Li, & Julaihi, 2009). One report by the MAA (2015) found that that over half of students received a grade of C or lower and that students’ mathematical confidence significantly dropped after completing calculus. Second, calculus is the first time students experience the concepts of a limit, derivative, and integral which means students do not have pre-established mathematical meanings for these concepts making it an ideal context for understanding students’ construction of knowledge.
This study will specifically focus on the concept of the limit of a function. In addition to being the first new major concept taught in a traditional calculus sequence (Larson & Edwards, 2014; Stewart, 2015), limits are fundamental for defining many calculus concepts including continuity, derivatives, and integrals. Students need a strong foundational understanding of limits for success in calculus (Sofronas, 2011); however, students struggle with the concept of a limit (Tall & Vinner, 1981; Monaghan, 1991; Williams, 1991, 2001; Bezuidenhout, 2001). The aim of this study is to explore the nature of discourse which supports the development of mathematical knowledge, how individual students construct mathematical knowledge through discourse, and the relationship between the collective and individual students’ mathematical knowledge within the context of limits.

**Significance of Study**

This study seeks to understand the role discourse plays in the development of students’ mathematical knowledge. This is not a novel concept, but studies on discourse and mathematical understanding have focused on the construction of knowledge by the collective or individual student. This study proposes to use an approach which will not only consider collective and individual understanding, but also the relationship between knowledge constructed through classroom discourse and the individual’s personal understanding of said knowledge.

Research on mathematical discourse has focused primarily in a K-12 setting (Cobb, Boufi, McClain, & Whitenack, 1997; Goos, 2004; Huang, Normandia, & Greer, 2005; Yackel, Cobb, & Wood, 1991; Wood, Williams, & McNeal, 2006; Sfard, 2000). Although integrating discourse in collegiate mathematics is part of the MAA guidelines, there is little research on discourse in collegiate mathematics courses. As a result, there are no guidelines on practices that support integration of classroom discourse into a collegiate mathematics course. It takes time and
effort to create a productive discourse community; without a developed body of research on the nature of classroom discourse to provide faculty support for instructional improvement, it is unlikely that discourse will become a widespread practice in collegiate mathematics.

Research on students’ understanding of limits has concentrated on misconceptions and why these misconceptions form (Tall & Vinner, 1981; Williams, 1991, 2001; Tall, 1993; Bezuidenhout, 2001; Jaffar & Dindyal, 2011; Oehrtman, 2009; Bressoud et al., 2016), but there is a scarcity of research which seeks to understand how students’ conceptions of limits develop. As a result, the significance of this study is to examine the nature of discourse that impacts students’ development of mathematical knowledge and how this discourse supports the construction of collective and individual student’s mathematical knowledge. A byproduct of this research on discourse is insight into the teaching and learning of calculus in a way that supports conceptual understanding.

**Definition of Terms**

*Discourse* - oral communication between members (student or instructor) in the same class as a means to support mathematical thinking (Cobb, Boufi, McClain, & Whitenack, 1997).

*Conceptual Knowledge* – knowledge of facts, definitions, principles, and generalizations (Baroody, Feil, & Johnson, 2007).

*Analytical Procedure* – an algorithm which utilizes symbolic manipulation to solve a mathematical problem, for example using the Quadratic Formula to find the roots of a quadratic equation.¹

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¹ Common mathematical usage
**Numeric Procedure** – an algorithm which utilizes a numerical approximation to solve a mathematical problem, for example using Newton’s Method to approximate the roots of a quadratic formula\(^2\).

**Procedural Knowledge** – knowledge of mental manipulations, algorithms, and rules and strategies for solving tasks (Hiebert & Lefevre, 1986; Baroody, Feil, & Johnson, 2007), both numeric and analytical procedures are types of procedural knowledge.

**Understanding** – knowledge which is complete and connected, characterized by accurate representation of procedural and conceptual knowledge, how concepts and procedures are connected and of relationships between concepts (Baroody, Feil, & Johnson, 2007).

**Chapter 2 – Literature Review**

The current study examines the nature of discourse that supports the development of mathematical knowledge and to identify how that knowledge is constructed. This work hinges on the belief that discourse is necessary for constructing knowledge (Vygotsky, 1978, 1986; Wertsch, 1985). In this chapter, an overview of the sociocultural perspective that serves as the lens for this study is provided along with a discussion of the Vygotsky Space framework (Harré, 1983; Gavelek & Raphael, 1996) which relates social interaction and knowledge construction. The cornerstone of Vygotsky Space is interaction, both with others and oneself. For this reason a general discussion of discourse and mathematical learning is provided followed by a detailed discussion of the role of the zone of proximal development in the construction of mathematical knowledge. Since the mathematical knowledge focused on in this study is the limit of a function, the review concludes with a summary of research on students’ conceptions of the limit of a function.

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\(^2\) Common mathematical usage
Discourse and Mathematical Learning

Mathematics has traditionally been viewed from a platonic belief that mathematical objects exist in their own realm, a math world, outside of the social or individual realm. This view positions mathematics as absolute truth and objectivity. However, Restivo (1993) posits mathematics as a social construction. The foundation of the argument lies in the thought that “all talk is social; the person is a social structure; and the intellect is a social structure” (Restivo, 1993, p. 248). He furthers the argument by discussing mathematical thought associated with different cultures. Mathematical objects were created by humans and “take their meaning from the history of their construction and usage inside and outside of mathematics, and the network of ideas that they are part of” (Restivo, 1993, p. 270), so it is impossible for a boundary to exist between math worlds and social worlds. By adopting the notion of mathematics as constructed through social interaction, it becomes necessary to consider the relationship between social interaction and learning mathematics.

The underlying assumption for this study is that discourse (this includes discourse with people, but also with texts) is necessary for the construction of mathematical knowledge. However, not all discourse necessarily supports students’ conceptual development. This section explores the nature of discourse that supports the development of students’ mathematical knowledge. Yackel, Cobb, and Wood (1991) found that small-group discussions about solving mathematical problems provide opportunities for learning. The teacher designed classroom discourse so that students were expected to explain and justify their thinking. In addition, students needed to resolve conflicts, clarify, and correct solutions. The researchers showed the value of collaborative activities by illustrating three types of learning opportunities that arose during observation: using another group member’s solution to inspire a student to develop her
own solution; assisting another group member whose solution was erroneous; and making sense of a group member’s solution to reach consensus. Wood, Williams, and McNeal (2006) examined five elementary mathematics classes (four reform-based second-grade classes and one conventional third-grade class) for differences in the types of mathematical thinking expressed by students. This research supports Yackel et al.’s claim that small-group work provides learning opportunities which support mathematical development, not inherent in traditional mathematics classrooms. Wood et al. found that regardless of instructional approach, students expressed higher-level thinking (beyond recognition and recall, including analyzing, synthesizing, and evaluating) only during lessons in which the pattern of interaction between the teacher and students was dialogic. This type of interaction did not occur during conventional IRE. Moreover, higher-level thinking occurred most frequently when the activity involved students working to develop a shared understanding and consensus.

Cobb, Boufi, McClain, and Whitenack (1997) looked at the role of reflective discourse, mathematizing discourse so that what the students are doing becomes the focus of discussion, and collective reflection, “joint or communal activity of making what was previously done in action an object of reflection” (p. 258) in two episodes from a first-grade classroom. The researchers found that participation in reflective discourse creates conditions for mathematical conceptual development, although does not guarantee learning. In addition, it was found that students participate in collective reflection, but the researchers could not infer individual children’s abstractions or conceptualizations from the collective’s.

Much of the research focuses on communication about mathematics, but Sfard (2000) contends discourse can lead to the construction of mathematical objects. Studying an episode on statistics from a seventh-grade class, Sfard observed that the students constructed terms and
tools, as well as meaning for these terms/tools to allow for communication about solving the presented problem. She defined symbolic mediation as this mathematical discourse which utilized symbolic objects to construct conceptual understanding. According to Sfard, symbolic mediation is the tool for creating new mathematical objects and therefore new mathematical meaning. The construction of these mathematical objects arose out of the students’ need to communicate their ideas about the problem, not the other way around.

**Sociocultural Perspective**

This research is grounded in a sociocultural perspective. Sociocultural theory expands on the theoretical work of Vygotsky. Wertsch (1985) identified three underlying principles in Vygotsky’s theories which lay the foundation for sociocultural theory.

(1) “A reliance on a genetic or developmental method” (Wertsch, 1985, p. 14); Vygotsky (1978) argued that to understand mental processes one needed to understand how and where these processes occur. “We need to concentrate not on the product of development but on the very process by which higher functions are established” (Vygotsky, 1978, p. 64).

(2) “The claim that higher mental processes in the individual have their own origin in social processes” (Wertsch, 1985, p. 14) is the basis for the notion that social interaction is necessary for the development of mental functions. Social interaction is necessary for individual development, specifically “every function in the child’s development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological)” (Vygotsky, 1978, p. 57). The process of moving from interpsychological to intrapsychological, known as internalization, does not necessarily occur instantaneously. There is a series of events, which differ in length and type for each individual (meaning all individuals do not experience the same series of events nor does the process take
the same amount of time), which leads to learners internalizing knowledge. Not only does this process differ from person to person, but from concept to concept. Since social interaction is pivotal for students’ development, exploring how students internalize knowledge created through a shared activity helps in understanding how students learn. It is important to note that just because social interaction is a necessary condition for internalization, it is not a sufficient condition, meaning social interaction does not necessarily lead to internalization.

(3) “The claim that mental processes can be understood if we understand the tools and signs that mediate them” (Wertsch, 1985, p. 14) stems from Vygotsky’s (1986) belief that “thought is not merely expressed in words; it comes into existence through them” (p. 218). Language (along with symbols, gestures, writing, etc.), then, is a tool for mediating thought. Understanding how thoughts are mediated by tools and signs allows for insight into the relationship between social and individual development. Moreover, Wertsch (1985) emphasizes that each of these themes is interrelated so that one cannot fully understand one theme without considering the other themes.

Building on Vygotsky’s notion of mediation, “language enables new modes of representation and communication, and thereby fosters the emergence of the mediated mind” (Homer, 2004, p. 236). Nelson (1996) highlights how the interaction between a child’s experience and social interaction with its culture develops the child’s cognition. However, Nelson pushes this relationship further by rejecting the idea that language is an impartial medium through which concepts are created and conveyed. Language is the primary tool for culture to enter the mind of the child, but the child modifies this language to shape her thoughts. The child’s internalization of the language results in words carrying meaning for the child, which can be quite different from other individuals’ conceptions of the same information. Since individuals
must internalize words to create thought, the language is not neutral in the formation of meaning. Although Nelson emphasizes language, I claim that her ideas can be extended to mathematics because mathematics is itself a symbolic language for representing and expressing thought. This is an important point because it means children’s individual mathematical conceptions and use of symbols are unique based on their experiences, interactions, and how they mediate language.

Zone of proximal development.

Discourse has been shown to help students make meaning of mathematics, construct mathematical objects, and develop understanding. Discourse serves two functions; the first is to convey meaning and the second is to generate meaning (Perissini & Knuth, 1998). For discourse to be productive, all members of the discourse must be able to make meaning of the communication. This is why a univocal discourse, in which information is sent from the sender to the received, is not always productive; many individuals are unable to decode and make meaning of the information they are receiving in a way which aligns with the sender’s intended meaning. The same holds true for a dialogic discourse in which individuals are simultaneously sending and receiving information (Perissini & Knuth, 1998); if individuals in a conversation talk past each other, meaning they are unable to create a shared meaning, then the conversation adds nothing to each individuals’ understanding (Vygotsky, 1986). However, when dialogic discourse involves individuals working together, it acts as a “thinking device;” allowing all students to make meaning of mathematical concepts, raising the understanding of all individuals involved. This section looks at how discourse can help to construct mathematical knowledge through the creation of a zone of proximal development (ZPD).

As discussed earlier, internalization is a process through which learners experience ideas through social interaction in the external world which is transformed into personal understanding
in the mind. Vygotsky viewed ZPD as a space where this transformation could occur. Vygotsky distinguishes between learning processes experienced through social interaction and internal learning processes and proposes that the space where this transformation of the former into the latter occurs is the ZPD. “The zone of proximal development defines those functions that have not yet matured but are in the process of maturation, functions that will mature tomorrow but are currently in an embryonic state” (Vygotsky, 1978, p. 86). The ZPD exists between the learner’s level of actual developmental and the level of potential development which is the level of understanding the learner can obtain through guidance or collaboration with a more advanced partner. However, Forman and McPhail (1993) found that students of comparable expertise can help each other push forward their understanding, thus creating a collaborative ZPD. This extends the definition of the level of potential development to the level of understanding a learner can obtain through guidance or collaboration with a peer.

The ZPD is not a physical space, but a metaphorical space created through social interaction which transforms knowledge from inter- to intrapsychological. As such, the ZPD is a theoretical construct which can be used as a framework for analyzing learning. Goos (2004) interprets the ZPD as taking three forms:

1. The ZPD as scaffolding: This manifestation of the ZPD aligns with Vygotsky’s original definition of the ZPD as the distance between the learners’ actual development when working alone and the potential development created by interacting with a more advanced partner (teacher or peer). To construct this type of ZPD the teacher structures tasks so that students are pushed beyond their current developmental stage. Goos emphasizes that for the task to be effective at creating a
ZPD the learner must actively co-construct the ZPD, otherwise the interaction is not transformative.

(2) The ZPD as collaboration: This manifestation of the ZPD follows from the realization that the ZPD can occur through collaboration between students of equal expertise. Collaborating with peers gives learners the chance to construct and test new understandings and contribute to each other’s knowledge. However, not all peer interactions result in the construction of a ZPD.

(3) The ZPD as interweaving: Unlike the other two forms of the ZPD which result from social interaction, this manifestation of the ZPD emerges from the interaction between concepts. Vygotsky (1986) categorized concepts as either everyday/spontaneous, arising from intuition, and scientific/formal, arising from social conventions. Nelson (1996) elaborates on the relationship between spontaneous and scientific concepts. She believes students create spontaneous concepts based on their experiences with their environment. It is the students’ interactions with society which leads their conceptual systems to shift from spontaneous concepts to formal concepts. Students’ formal concepts are not static; changes in formal concepts can provide feedback, leading to new spontaneous concepts, creating a cycle of continuous conceptual shifts. Goos (2004) suggests the “ZPD is conceptualized here in terms of the distance between learners’ intuitive notions and the formalized concepts, or cultural tools, of a particular academic community. Mature knowledge is achieved with the merging of everyday and scientific concepts—not by replacing the former with the latter as in a transmission model of teaching, but by interweaving the two conceptual forms” (p. 263).
Theoretical Framework – Vygotsky Space

Speech is not only the medium through which thought is conveyed, but also the vehicle through which thought is realized. To serve these different purposes, people use different types of speech. For social interaction, learners use external speech, which serves the purpose of conveying thought to the outside world (Vygotsky, 1978; 1986). The complexity of external speech grows as people develop, but its function and purpose remain the same. The difficulty with external speech lies in the interaction with other people. For external speech to be successful, the speaker and listener must have the same meaning for the external speech. In other words, to have communication the individuals involved must have a shared apperception. All too often it is failure to create shared meaning which leads to a breakdown in communication.

Once interpersonal exchange has occurred the child must have an intrapersonal exchange to internalize the process. For this internal conversation, the child uses inner speech which serves the purpose of shaping thought. Unlike external speech, inner speech is dynamic - both shaping thought and being shaped by thought. Although not obvious, as with external speech, social interaction is connected to and necessary for the construction of inner speech and therefore thought. Since social interaction, which occurs through external speech, is crucial for the development of inner speech, these two forms of speech are intertwined. Furthermore, both types of speech are imperative for the development of thought and the construction of meaning.

For a researcher or teacher, a difficulty comes from trying to observe the interplay between external and inner speech for the construction of thought. Harré (1983) suggests that thought occurs not just as a manifestation of speech as Vygotsky proposes, but also as a social location (collective or individual). As a manifestation of speech, thought occurs publicly as external speech, or privately as inner speech, but Harré views thought as also arising through
social interaction either collectively or individually. Harré rejects the idea that cognitive processes can be described as a duality external/collective or internal/individual. As a result of this rejection of a dichotomous model, Harré presents a framework, labeled by Gavelek and Raphael (1996) as the Vygotsky Space, for characterizing the construction of knowledge through the relationship between collective and individual knowledge, and external and internal social interaction.

The Harré model of Vygotsky Space (see Figure 1) is the Cartesian product of two dimensions, manifestation (external or internal) and social location (collective or individual), resulting in four quadrants.

*Figure 1. The Vygotsky Space. (Adapted from Harré, 1983 by Gavelek & Raphael, 1996)*

Each quadrant represents a different space in knowledge construction:

- Quadrant 1 (Q1), External and Collective: This space is where an individual first experiences new concepts. Individuals use external speech to interact with other individuals to create a shared, collective understanding or to think about a text. In a
school setting, students could be positioned in this space through social interactions with a teacher, peers, or a text.

- Quadrant 2 (Q2), Internal and Collective: This space is where the individual thinks personally using external speech, meaning the individual talks to herself using the same language as she had publicly or in the language of a text. This may occur when a student writes in a math journal or copies a definition from a text.

- Quadrant 3 (Q3), Internal and Individual: This space is where the individual internalizes knowledge. This is where collective knowledge has been transformed into individual knowledge using inner speech. A student may internalize knowledge using a number of tools including individual practice, discourse, writing, or reflection.

- Quadrant 4 (Q4), External and Individual: This space is where the individual uses external speech, either through discourse or writing, to share their internalized knowledge. By sharing her individual knowledge with society, this individual knowledge becomes part of the collective knowledge and moves the individual back to Quadrant 1.

Individual development is presented as a cyclical process through the quadrants. Harré (1983) positions the process as starting in Q1 because according to Vygotsky learning is positioned first interpsychologically which is an external-collective space and then internalized. Interactions between the quadrants are a four-phase iterative process (Harré, 1983; McVee, Dunsmore, & Gavelek, 2005):
• Appropriation: The transition from Q1 to Q2 occurs when an individual adopts the ways of thinking and practices she encounters through social interactions without internalization.

• Transformation: The transition from Q2 to Q3 occurs when an individual internalizes collective knowledge into personal knowledge. This transition involves creating personal meaning and practice in terms of one’s own experiences and context. The construction of a ZPD (scaffolding, collaboration, or interweaving) in Q1 can help a student to transform knowledge positioning them in Q3.

• Publication: The transition from Q3 to Q4 occurs when an individual shares or makes public their personal meanings.

• Conventionalization: the transition from Q4 to Q1 occurs when the individual’s published new learning is adopted into practice by the individual and/or integrated into the community’s shared knowledge and practice.

Although the learning process is cyclical, the Vygotsky Space does not suggest that the learning process is sequential. Students may be functioning in any of the quadrants at any time. One limitation to the Vygotsky Space model puts forward the notion that students will iterate through each quadrant in sequence; but in reality it is possible a student may start in Q1, transition to Q2, and then transition back to Q1, never iterating through each quadrant and never transforming or internalizing knowledge. Another issue with the model is that only Q1 and Q4 can easily be observed because they are in a public space – although Gavelek and Raphael (1996) found instances of student writing which serve as evidence of being positioned in Q2. Furthermore, the only way for a student to be positioned in Q4 is if the student transformed knowledge and transitioned first into Q3 before transitioning into Q4 (Harré, 1983; Gavelek & Raphael, 1996).
Gavelek and Raphael (1996) demonstrate that although it is not possible to observe Q3, by examining classroom discourse and students’ publications it is possible to infer that a student had transformed knowledge and passed through Q3.

Another drawback is that the model does not allow for a temporal component. To allow for the model to reflect the nature of learning more accurately, Gavelek and Raphael (1996) add a third dimension to the model: time (see Figure 2). The addition of the time dimension means there is now a way to distinguish the Q1 space before the student has conventionalized their knowledge and after. The first Q1 exists in a different time than the later Q1 which also means that the later Q1 exists in a new context as the student’s experiences and learning has changed since their last position in a Q1 space. This re-conceptualization of the Vygotsky Space allows for a framework that not only characterizes the construction of knowledge through collective/individual knowledge and private/public social interaction, but also locates individuals in their sociocultural context. In other words, the model through Vygotsky Space provides a way to describe an individual’s construction of knowledge over time.

Figure 2. 3-dimensional Vygotsky Space. (Adapted from Gavelek & Raphael, 1996; McVee, Dunsmore, & Gavelek, 2005)
In order to examine the role of discourse in the development of mathematical understanding, the present study will focus on students’ conceptions of limits. I will begin by defining a limit using the definition given in the textbook used by observed Calculus class.

Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly at \( a \) itself. Then we say that the \textbf{limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \)}, and we write \( \lim_{{x \to a}} f(x) = L \) if for every \( \varepsilon > 0 \) there is a number \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \varepsilon \) (Stewart, 2015).

More informally, the “limit is a number that the \( y \)-values of a function can be made arbitrarily close to by restricting \( x \)-values” (Williams, 1991). The limit of a function was selected because it is the first new concept covered in the calculus curriculum and the limit of a function is a topic students struggle to understand conceptually (Monaghan, 1991; Williams, 1991; Tall, 1993; Bezuidenhout, 2001; White & Mitchelmore, 1996). The limit concept “creates a number of cognitive difficulties” (Tall, 1993, p. 2). Studies looking at student conceptions of limits have found students’ understanding of limits are largely procedural (Tall, 1993; White & Mitchelmore, 1996; Bezuidenhout, 2001), specifically seeing limits as a process to be carried out on functions (Williams, 1991; White and Mitchelmore, 1996). Understanding students’ conceptions of limits can help shed light on how students construct knowledge, as well as how students apply their notions of a limit to procedures, new problems, and new topics.

Cornu (1981), looking at students’ initial conceptions of mathematical terms related to limits in French found that students’ everyday meanings persist well into their study of calculus, which can be problematic when those every day meanings are at odds with the mathematical meanings for the same terms. Following Cornu’s work, Monaghan (1991) surveyed students
about their notions of terms related to limits in English. Monaghan found, for students, the word “limit” conjured images of a speed limit, physical limit, or mental limit which does not align with the mathematical definition. Monaghan did not explore how or if these images could be useful for thinking about limits. The majority of students thought of “approach” as meaning drawing nearer. In this use of approaches, such as “the train approaches the station,” the train will eventually reach the station. Mathematically speaking if a train approaches a station, it may not reach its destination. Thinking of “approaches” as drawing near could be useful but would need more precision. Other definitions of “approaches” included a method, route, or resembling and “tends to” meant a general trend which do not align with the mathematical meaning.

Students’ everyday notions of limits and terms related to limits can influence their conception of these terms mathematically. Tall and Vinner (1981) posit that students create a concept image, “the total cognitive structure that is associated with the concept, which includes all mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152) which may or may not be based on the concept definition (the formal mathematical conception). Tall and Vinner argue that the concept image affects a student’s understanding. When students experience a new formal concept, if it does not conflict with the student’s concept image, then the new concept is connected with the existing concept image. “The difficulty of forming an appropriate concept image, and the coercive effects of an inappropriate one having potential conflicts, can seriously hinder the development of the formal theory in the mind of the individual student” (Tall & Vinner, 1981, p. 167). When students are faced with contradictions in their concept image they have two choices:
Reconcile the old and the new by re-constructing a new coherent knowledge structure [or] keep the conflicting elements in separate compartments and never let them be brought simultaneously to the conscious mind (Tall, 1993, p. 3).

Students who reconcile conflicts are able to construct deep connected understanding, but students who compartmentalize develop disjoint and at times contradictory understanding. The research related to concept images and concept definitions looks at the connection between students’ informal conceptions and the formal \( \varepsilon - \delta \) definition of a limit of a function (Tall & Vinner, 1981; Cornu, 1983; Monaghan, 1991; Williams, 1991; Bezuidenhout, 2001).

Williams (1991) surveyed 341 students enrolled in a second-semester calculus class about their views of limits. Based on previous research on student conceptions or misconceptions of limits, Williams provided students with six mental models of the limit of a function: a formal model, four informal models, and one procedural model. All of these conceptions are accurate, but only the formal definition is mathematically rigorous.

1. dynamic-theoretical (motion) – a limit describes how a function moves as \( x \) moves toward a certain point,
2. acting as a boundary – a limit is a number or point past which a function cannot go,
3. formal – a limit is a number that the \( y \)-values of a function can be made arbitrarily close to by restricting \( x \)-values,
4. unreachable – a limit is a number or point the function gets close to but never reaches,
5. acting as an approximation – a limit is an approximation that can be made as accurate as you wish,
(6) dynamic-practical (procedural) – a limit is determined by plugging in numbers closer and closer to a given number until the limit is reached (Williams, 1991, p. 221).

The classification most frequently viewed as the best description of a limit was a motion definition, followed by the limit as unreachable and then the formal definition. Moreover, students tended to describe their conception of limits using two or more of the limit classifications, highlighting the complexity of the students’ informal ideas but also the difficulty in categorizing notions of limits. Although Williams never acknowledges it, one reason students selected multiple responses may have been because Williams used preselected statements which may not have captured completely the students’ notions of limits; as a result the students may have chosen more than one definition to encompass all of the features of their notion of a limit especially if the students thought the informal definitions were incomplete.

In the second phase of the study, Williams (1991) selected 10 students for a series of five interviews to glean more insight into their understanding of limits. The first interview had students create a definition of limits, followed by a series of interviews where students discussed opposing descriptions of a limit, solved limits problems, and amended their definition of a limit if the student thought it was necessary. Even though during the interviews the students were given problems which were meant to contradict or highlight gaps in their definitions, the students held onto their incomplete idea and/or contradicting conceptions of limits rather than revise their definition to encompass the new example. Williams concludes that students value practicality and simplicity over mathematical accuracy of their limit conception. As a result, their procedural and conceptual knowledge remained separate. It is important to realize that the students Williams surveyed and interviewed had already created a definition of limit in their first semester of
calculus. He did not explore how these conceptions formed or how students applied these models in class. This may be one reason why students were reluctant to change their conception of limits, since to this point their definition may have been sufficient for mathematical success.

Bezuidenhout (2001) studied first-year college students’ responses to a series of questions about the limit of a function and continuity at a point, focusing on misconceptions. Like Williams (1991), Bezuidenhout found students held a dynamic viewpoint of limits. He also found that students’ responses at times contradicted each other, for example signifying a function is continuous at a point but not indicating the function is defined at this point. This indicates that students may hold contradictory notions of limits, continuity, and differentiability. Bezuidenhout (2001) interviewed students about contradictions in their responses. When faced with a contradiction, a student was willing to abandon conceptions, even if mathematically correct, so that the statements would align with her previous interpretation instead of revising her original interpretation. The students demonstrated a reliance on algebraic (i.e., analytical or procedural) approaches to limits. Bezuidenhout concludes students’ understanding of limits rests on unconnected facts and procedures, which prohibits the development of conceptual understanding and makes it more likely to hold contradictory notions.

Szydlik (2000) interviewed 27 college calculus students about their source of conviction and understanding of limits. Students provided three types of definition of limits: (1) intuitive, meaning the limit of a function is \( L \) if whenever \( x \) is close to the limiting value \( c \), the function is close to \( L \), (2) motion, the limit is \( L \) if the function is getting closer and closer to \( L \) as \( x \) approaches \( c \), or (3) incoherent. The intuitive and motion definitions were mathematically accurate but not rigorous, while the incoherent definitions were mathematically inaccurate. Szydlik found all of the students who provided an incoherent definition of limit held external
sources of conviction, looked to authorities as their source for mathematical truth, while all of
the students who provided an intuitive static definition of a limit held internal sources of
conviction, utilized empirical evidence and logic to determine mathematical truth for themselves.
Students who held motion definitions were a mix of external and internal sources of conviction.
Although students only provided three types of definitions of limits, in students’ explanations of
how they solved problems involving limits some alternative conceptions arose, namely the limit
as unreachable, a bound, or the formal definition of a limit. In addition, all students used a
motion description of limits at some point even if their original definition of a limit did not
include the idea of motion. The fact that students used multiple descriptions of a limit shows they
have flexible and complex ways of thinking about limits.

As Williams (2001) points out, the majority of studies on students’ conceptions of limits
view students’ intuitive models of limits in terms of the mathematical definition of limits. This
means the studies compare students’ understanding to this mathematical definition or the
likelihood the student model will lead to this formal mathematical definition. “The mathematical
approach to limit and the cognitive approach to limit are quite different … much as we cannot
understand young children’s thinking about addition and subtraction solely in terms of binary
operations and commutativity we cannot understand calculus students’ reasoning in terms of
inequalities and universal qualification” (Williams, 2001, p. 342). The problem is these studies
do not try to investigate how students’ conceptions are constructed or how students apply the
conceptions to problems involving limits.

Lakoff and Núñez (2000) suggest a theoretical model of how students come to
understand the limit of a function. They suggest that students first understand the limit as
“approaching a limit” in geometric terms and in arithmetic terms through the Basic Metaphor of
Infinity (BMI). The BMI is a metaphor created outside mathematics which people use as the basis for understanding of infinity in all mathematics. Lakoff and Núñez claim the problem is that students are taught using these intuitive ideas of a limit and then told the mathematical definition conveys the same idea, even though the epsilon-delta and the arbitrarily close definition do not involve the notion of the approaching motion. The researchers also criticize the idea that an intuitive model cannot be precise and outline how the approaching a limit intuitive model can be extended to create a rigorous mathematical definition of a limit.

In a follow-up study, Williams (2001) studied two students’ informal conceptions of limits and how these conceptions develop. Williams found that the students’ conception of limits were similar to Lakoff and Núñez (2000) description of approaching a limit. Students saw limits as a method of choosing values to get closer and closer to the limit. The researcher’s intervention did not change either student’s view of the limit as a dynamic process. Both students struggled with the idea of reaching a limit. However, Williams (2001) attempted to change the student’s conception instead of trying to use the intuitive student conception of the limit as dynamic process and build onto this notion to construct a rigorous definition. Williams believes coming to terms with the notion of reaching a limit requires a cognitive leap from the intuitive to the abstract, which may explain why he attempted to change student’s conceptions instead of trying to extend student’s conceptions.

Oehrtman (2009) surveyed 120 students’ thinking about limit concepts (limit of a function, derivative definition, \(0.9 = 1\), l’Hôpital’s rule, limit comparison test, Taylor series of sine, sequence of sets, multivariable continuity). Oehrtman noticed five salient metaphors across students and limits topics:

1. Collapse metaphor – see limits as a collapse of dimension
2. Approximation metaphor – involves unknown quantity and approximations close to this unknown
3. Proximity metaphor – distance between point-locations, used language such as close, wrapping around, hugging, and clustering
4. Infinity as a number – treated infinity as a really big number or representing infinity as a point
5. Physical limitation – small-scale object and measuring against said object

Of these metaphors, when describing the limit of a function the approximation and proximity metaphors were used most frequently with the approximation metaphor being the most common. This finding aligns with Williams’ (2001) finding that students view limits as an approach of choosing values of \( x \) so that the function gets closer and closer to the limit. The other metaphors for limits had not been observed in previous research. Oehrtman (2009) suggests that “students’ reasoning about limit concepts appear to be influenced by metaphorical application of experiential conceptual domains” (p. 420). The student generated metaphors could be powerful for developing students’ conceptions of limits, but Oehrtman (2009) found much of what the students said when applying these metaphors was mathematically inaccurate.

Other imagery which came up but was not as far-reaching included:

1. Motion imagery – seen in students’ choices of words such as “approaches”, but did not indicate using moving objects in their images/understanding
2. Zooming imagery – instructor used this imagery but the students did not adopt this image when describing limits
3. Interpretations of “arbitrarily” and “sufficiently” – students used both words regularly as did the instructor and text, but the students did not have a rigorous definition of these terms (i.e. used ambiguously).

Williams (1991) found that a dynamic or motion definition was most prevalent among students, which contradicts Oehrtman’s (2009) finding that motion imagery was not frequent and when students did evoke motion images it was through words like “approaches,” not a physical moving of an object as Williams described. Students did not adopt the imagery of the instructor, but instead developed their own spontaneous metaphors for limits (Oehrtman, 2009). When students did adopt the language of the instructor and text, specifically “arbitrarily” and “sufficiently,” the terms did not have a rigorous meaning for the students.

Instead, students hold conceptions of limits based on everyday language and informal meanings (Cornu, 1981; Tall, 1993). Research has classified students’ conceptions of a limit (Williams, 1991; Szydlik, 2000) and the metaphors students find salient for conceptualizing limits (Oehrtman, 2009). Students often do not base their conceptions on the imagery and language of the instructor (Oehrtman 2009) or mathematical definitions (Tall, 1993). Even when students adopt the language or imagery, students use the language, such as “approaches,” “tends to,” “arbitrarily,” and “sufficiently,” without constructing mathematical meaning for these terms (Monaghan, 1991; Oehrtman, 2009). Moreover, students prefer conceptions that are simple or follow previous experience instead of attending to mathematical accuracy (Williams, 1991) and can simultaneously hold contradictory notions about limits because they rely on facts and procedures to understand limits (Bezuidenhout, 2001). However, research has not explored how students construct their conceptions of limits.
Summary

The reviewed literature suggests that discourse supports the development of mathematical knowledge; specifically through types of learning opportunities that result in discourse which in turn helps construct understanding. A sociocultural perspective is based on the key tenets that to understand mental processes researchers need to understand how and where mental processes occur, social interaction is necessary for the development of mental functions, and mental processes are mediated through language. After establishing the importance of the relationship between discourse and development of mathematical knowledge, I looked at how mathematical knowledge is constructed through discourse by constructing a ZPD, either through scaffolding, collaboration, or interweaving. These constructs were introduced as the foundation for the Vygotsky Space model (Harré, 1985), which was introduced to provide a framework for understanding the construction of knowledge through collective/individual knowledge and internal/external social interaction. In addition, the Vygotsky Space framework was extended to include a temporal dimension (Gavelek & Raphael, 1996) to permit the model to describe individuals not in a single moment but as they develop over time.

This study focuses on the development of mathematical knowledge about the limit of a function. An in-depth review of students’ conception of limits was provided to understand students’ intuitive ideas of limits. The research shows students hold complex notions of limits, at times using multiple and conflicting models. Student-constructed models are very powerful for developing understanding, but these student conceptions are not necessarily mathematically-based and may lead to students possessing contradictory notions of limits. Most research on limits identifies student conceptions of limits, not on how they construct and apply these intuitive models. This study will use the Vygotsky Space framework to investigate how students construct
a conception of limits. It will also investigate the discourse which supports the development of these conceptions of limits and the relationship between collective and individual students’ knowledge of limits.

**Chapter 3 – Methodology**

The primary objective of this study was to understand the nature of discourse which supports the development of conceptual knowledge of limits, how individual students construct knowledge of limits through discourse, and the relationship between collective and individual students’ knowledge of limits. To address this objective, a longitudinal qualitative research methodology was employed (Yin, 1994; Saldaña, 2003). This approach was selected because qualitative research allowed for a complex and detailed understanding of an issue, that also helps “to understand the contexts or settings in which participants in a study access a problem or issue” (Creswell, 2007, p.40). The setting of the study was the first semester of a year-long college calculus course. Although quantitative measures could have been used to assess knowledge of limits, a quantitative approach would have overlooked the unique experiences and understanding of each individual student. The choice of a qualitative approach allowed for a detailed exploration of the nature of discourse which supports the development of mathematical knowledge. Moreover, a qualitative study provided rich description of the whole class and individual students’ experiences as well as insight into individual students’ knowledge located within the context of the collective knowledge of the whole class.

Furthermore, a qualitative approach aligns with the sociocultural perspective (Vygotsky, 1978; 1986) adopted for this study. In qualitative research the researcher is the primary instrument (Creswell, 2007). This means that my understanding of the research questions was mediated through my interaction with students both collectively and individually. Since the
researcher is the primary instrument, it is important to describe the role of the researcher as part of the research design. In the study, I assumed the role of participant-observer. According to Merriam (1988), the observer as participant role means my observation for the purpose of research is known to the students and I participated in social interactions with the students, mainly answering questions about the mathematics, but was only partially involved due to my role as an information gatherer.

Research Design

To reminder the reader, the research questions are as follows:

(1) How do individual students construct mathematical knowledge through discourse?

(2) What is the nature of discourse which supports the development of mathematical knowledge?

(3) What is the relationship between the collective and individual students’ mathematical knowledge?

Based on the research questions and the sociocultural perspective adopted for this study, a case study design was selected because it affords an in-depth study of the “how” and “why” of a current phenomenon within a real-life context, particularly when it is impossible to separate the phenomenon from its context as is the case with formal education. However, a holistic case study of the class would be insufficient to explore the complexity of the phenomenon of interest here, namely the relationship between individual and collective knowledge. The development of knowledge cannot be divorced from the social interaction (Vygotsky, 1978; 1986). For this reason, an embedded case study design was selected to allow for investigation at two levels (class and student), and the relationship between these levels.
A single case study of a calculus class provided a holistic picture of the context for social interaction and collective meaning making. According to Yin (1995) a single case study is appropriate when a case represents a unique case. The second level or the embedded cases were six students enrolled in the class selected by stratified purposeful sampling (Patton, 2002); sampling criteria are described in detail later on. All of the students were enrolled in the same class which means the students had a shared context. These embedded case studies offer a picture of each individual student’s construction of conceptual knowledge of limits. Embedding the students in the classroom allowed for examination of this shared context, specifically the relationship between conceptual knowledge at the student and class level as well as the role social interaction in the classroom had on individual development.

**Setting and Participants**

**Setting.**

This study took place in an introductory differential calculus course (Calculus I) at a public university in New York in 2013. The specific Calculus I course observed was designed to support students enrolled in the Equal Opportunity Program (EOP), a program whose mission is to provide “access, academic support, and supplemental financial assistance to make higher education possible for students who the potential to succeed despite poor preparation and limited financial resources” (Equal Opportunity Program, 2016). The calculus course selected for this study had a few very important features. First, description of instructional practices given by the instructor prior to the start of research indicated that she provided students opportunities for peer discussion. This meant the class would be a setting where I would be able to observe the students engaging in discourse about limits. Second, most students in the EOP program have not experienced calculus in high school. This means most concepts and ideas covered in the course
were new to students, and students have not internalized any of these ideas before entering the course. Third, the class was a two-semester Calculus I course which met for 55 minutes, five days per week as compared to the university’s standard course, meeting three days per week for one semester. The curriculum for the first semester focused on limits, continuity, and derivative rules with an algebra review, while the second semester covered applications of derivatives, optimization, related rates, and an introduction to integrals. This slower pace allowed more time for students to make meaning of the content. The class was observed every day for seven weeks during the unit on limits, including the limit definition of continuity and derivatives.

**Participants.**

The calculus class observed consisted of about 25 students, predominantly freshmen, all of whom were enrolled in the EOP program. Participants were 17 students, aged 18 and older, enrolled in the selected Calculus I course who consented to participate in the study and the instructor of the course.

**Instructor.**

I had met the instructor, Profesor Morgan\(^3\), prior to the study and based on our informal conversations about teaching calculus with discourse felt she would be a good individual to recruit for this study. Prof. Morgan had over 15 years of teaching experience and was in her fourth year as faculty for the EOP Program. She was invited to participate in the study through a recruitment letter (Appendix A) emailed in the summer of 2013. Based on her response to questions regarding the characteristics of the calculus course, specifically the frequency students engaged in discourse and the frequency of class meetings, Prof. Morgan and her calculus class was selected for participation in this study.

\(^3\) All names are pseudonyms to protect the identities of the participants
Prior to the beginning of the semester but after agreeing to participate in the study, I met with Prof. Morgan to inform her of the study details, including the purpose of the study, observation, collection of in-class work, interviewing, and student participation, and received informed consent (Appendix B). Prof. Morgan agreed for me to observe her class throughout the fall semester, as well as video and audio record her during these observations. She was not asked to make any changes in instruction.

Prof. Morgan also consented to participate in two interviews, conducted at the beginning and end of the study (Appendix C). The first interview was at the beginning of the semester to learn about Prof. Morgan’s course design and how she defined and assessed student understanding. The second interview occurred at the end of the semester and involved Prof. Morgan watching video of students during small-group activities and discussing student learning observed in the video.

This research is predicated on the belief that an individual’s learning is developed through social interaction and experience within a broader social context (Vygotsky, 1978; 1986; Wertsch, 1985). To understand how students constructed understanding of limits, it is necessary to first understand the classroom environment. However, the research questions focus on student’s construction of knowledge and the interactions which support this development. Since Prof. Morgan is not the focus of study, my observations of her during class and the results of her interviews will be discussed to provide insight into the learning environment.

**Learning Environment.**

Prof. Morgan’s goal for students was to build their confidence with mathematics and help them to succeed in mathematics. During the first instructor interview (September, 2013), Prof. Morgan outlined this goal for her students when she said:
Probably my biggest goal is that they leave class not being scared of math. A lot of students coming into [this course] are here because they don’t have strong math backgrounds. They are already coming in with a negative attitude towards math and the vast majority are freshmen. Most of which haven’t really decided what they want to do with their lives. I’m hoping that at the end of class, if they decide not to do a math major or a major that requires a lot of math, it’s because that’s what they want not because they don’t think they can do math…I want them to be able to accomplish whatever goals they want to accomplish in their future.

Prof. Morgan’s statement about student’s negative attitudes towards math was supported by classroom observations over the semester and interviews with students. Moreover, this goal aligned with the goals of the EOP program to provide opportunities for students to succeed in college. This demonstrated that she understood the needs of her students and worked to support those needs.

In terms of mathematical content, Prof. Morgan worked with her students to help them build what she called an intuitive understanding of calculus. Her goal for the unit on limits was for students “to know what a limit is in general… my ultimate goal when they are done with the unit is that they would be able to, in any situation, understand not just how to find a limit but what it represents” (First Instructor Interview, September, 2013). This means she distinguished between procedural knowledge and a deeper understanding of limits. Prof. Morgan placed a great deal of emphasis on helping students to construct a deeper understanding of limits and derivatives, not just a procedural understanding. Throughout the semester, Prof. Morgan never provided a rigorous mathematical definition of a limit, but instead helped students to build an
intuition about limits and construct their own definitions of the limit of a function which grew from the students’ initial conceptions of limits.

To reach these goals, Prof. Morgan created a classroom which supported student collaboration and student discovery. She set the expectations for learning the first day when she stated in her syllabus (Appendix D):

*I believe that anyone, with the proper motivation, can learn anything. I know that a lot of people don’t like math and think they can’t learn it but I think with some patience, persistence and assistance you can. Everyone learns at different rates and some of you will catch on to topics sooner than others. We all have different strengths and weaknesses but if you put them together we can help each other out. So, if you are ready and willing to learn, we’ll work together as a class to be sure everyone is able to learn.*

The course began with an algebra review to refresh students’ memories and ensure they had the prerequisite skills needed to be successful in calculus. The unit on limits began with understanding limits graphically, so that students could construct understanding for the concept of a limit without being bogged down with algebraic calculations. Prof. Morgan (First Instructor Interview, September, 2013) tried “to establish an environment where they are not afraid to make a mistake, where they realize hearing what other people think, whether it is right or wrong, can further their own understanding.”

The course was designed with minimal lecturing so students could work in groups to explore the mathematics and learn independently. Prof. Morgan utilized a variety of instructional techniques which allowed opportunities for discourse, both whole-class or small-group. Most classes were structured with students working in small groups of 3-4 people to define key concepts and the apply these concepts to a problem. Prof. Morgan selected new groups every 5-
10 days so that students would have the opportunity to work with everyone in the class over the semester. Students then presented their group solutions to their classmates at the board. Prof. Morgan rarely lectured, instead she helped students to formalize their understanding while working in their groups. She encouraged students to become independent learners, giving assignments in which students had to find and vet online mathematical resources. She also continually asked students to push their thinking forward by posing questions or problems about concepts they had not yet discussed, but for which students possessed the requisite knowledge to figure out. For example, on the first limits quiz students were given a bonus question asking them to find a limit at infinity graphically even though limits at infinity had not been discussed in class. This question served as the starting point for a discussion on limits at infinity. Additionally, students were never given answers, and were instead expected to find solutions through peer collaboration.

Prof. Morgan embraced her role as a facilitator, and positioned students to build understanding with each other. She expected all students to participate and be responsible for not only their own learning but also supporting their classmates’ learning. She encouraged students to be independent learners, supporting them to learn from each other and find reliable sources online to answer their own questions. She also continually asked students to apply their understanding to new situations. Each assignment, homework or in-class, included a question on a topic that students had not explicitly discussed in class but for which they have all the requisite knowledge. This question was used to prime students to new topics and help them to learn to apply their knowledge, not just expect to be handed the new concepts. As will be described in the individual case studies, some students were resistant to learning in this type of environment but by the end of the unit, all of the students acknowledged the efficacy of the learning experience.
**Students.**

During the first week of classes, I attended class to recruit student participants. Students were informed of the study including the purpose of the study, observation, questionnaires, collection of in-class work, and interviewing. Students provided informed consent (Appendix E) to participate and indicating the level of participation (recorded during class observations, classwork and questionnaires collected, and interview). Of the 25 students in the class, 18 consented to participate in this study in the form of permission to record during classroom observations and collection of questionnaires, class work and assessments. Of the 18 students, 17 were included in the study, as one student withdrew from the course and was therefore omitted. Also during the first week of classes, student participants completed a 5-question background questionnaire about their year in school and previous math coursework (Appendix F), and a 5-factor mathematics self-concept questionnaire (Skaalvik & Rankin, 1995) (Appendix G), gauging their self-perceptions of their mathematical ability, aptitude, effort, intrinsic motivation, and anxiety.

Seven research participants consented to individual interviews (Appendix H). These students were interviewed at the end of the unit to explore the their understanding of limits and their experience learning in a classroom which utilized discourse. Of these seven, six students were selected for case studies. The students were selected based on the results of the math self-concept questionnaire, group dynamics, understanding of material as evidenced by classroom observation, classwork (i.e. homework, quizzes, etc) and interviews. According to the math self-concept, and supported by interviews, three students had low math self-concept and all three were selected as case studies (Kyle⁴, Fern⁴, and Mariah⁴). Four students demonstrated a high

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⁴ All names are pseudonyms to protect the identities of the participants
math self-concept, of which three were selected for case studies. Two of these participants worked extensively together (Javier and Daniella), so they were both included to allow for exploration into each of their individual development as a result of their prolonged collaboration. The third student (Yara) was selected because she was the only student to connect her conceptual and procedural understanding of limits and construct a generalized definition of a limit that was mathematically accurate. She also worked extensively with another student.

**Data Collection**

A good case study uses multiple sources of data (Yin, 1994), including observations, interviews, and physical artifacts (Merriam, 1988). All three data sources were collected to provide data that represented both collective and individual student’s understanding of the mathematical concepts and social interaction which supported this development. In the following sections I will describe in detail collection of each data source.

**Table 1. Data collection timeline.**

<table>
<thead>
<tr>
<th>Week and Date</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before Semester</td>
<td>Informed consent from instructor</td>
</tr>
<tr>
<td>Week 1 (8/28)</td>
<td>Informed consent from students</td>
</tr>
<tr>
<td>Week 1 (8/28)</td>
<td>Administered Background and Math Self-Concept Questionnaire</td>
</tr>
<tr>
<td>Week 1 (8/26 – 8/30)</td>
<td>Informal observation of class</td>
</tr>
<tr>
<td>Week 4 (9/21)</td>
<td>Conducted First Instructor Interview</td>
</tr>
<tr>
<td>Week 4 – 11 (9/16 – 11/8)</td>
<td>Observed class daily during unit on limits and collected student artifacts</td>
</tr>
<tr>
<td>Week 11 (11/4 – 11/9)</td>
<td>Conducted student interviews</td>
</tr>
<tr>
<td>Week 12 (11/13)</td>
<td>Conducted second instructor interview</td>
</tr>
</tbody>
</table>
Instructor interview.

The instructor was instrumental in students’ conceptual development of math. To understand the instructor’s instructional decisions and how she assessed understanding the instructor was interviewed twice during the semester in the instructor’s office. The first interview was conducted the fourth week of the semester; the same week that the unit on limits began. The 40 minute interview (Appendix C) was a semi-structured interview (Lindlof & Taylor, 2002) which focused on the instructor’s design and instructional goals at the beginning of the course. Probing questions were asked to provide a clear picture of how the instructor defined and assessed student understanding. An additional interview was conducted during Week 13 of the semester, after the completion of the unit on limits to have the instructor reflect on the students’ learning in class. For the second interview the instructor watched video of the students during small-group activities to discuss the goal of the activity, what understanding the instructor saw the students building, and what the students took away from the activity. Probing questions were asked to elicit reflection on student learning. Field notes were taken during both interviews, along with audio recordings that were transcribed.

Questionnaires.

Two paper questionnaires were administered in class, the first week of the semester, at the beginning of the data collection prior to the start of the unit on limits. The first questionnaire asked students background questions about the number and types of mathematics courses the student took in high school and college (Appendix F). The second questionnaire (Appendix G) was mathematics self-concept questionnaire (Skaalvik & Rankin, 1995). The survey was a 37 question instrument designed to interpret students’ self-concepts about mathematics along five factors: self-perceived aptitude, self-perceived ability, intrinsic motivation, math anxiety, and
effort. Using Cronbach’s alpha Skaalvik and Rankin (1995) found the factors have reliabilities of .90 (aptitude), .83 (ability), .97 (motivation), .90 (math anxiety), and .84 (effort). This survey was selected for two reasons. The first was that the survey was short enough to only take about 10 minutes to complete which helped ensure students completed the whole survey. The second was that this self-concept questionnaire has been shown to be a good indicator of general mathematical self-concept and academic achievement as indicated by grades (Marsh & Hau, 2004; Möller, Pohlman, Köller, & Marsh, 2009). Both questionnaires provided background context for each student and information needed for selection of case studies.

The mathematics self-concept questionnaire contained 37 five-point Likert scale items grouped into five factors (self-perceived aptitude, self-perceived ability, intrinsic motivation, math anxiety, and effort). Items were written as positive (e.g. “I feel calm in mathematics lessons”) and negative (e.g. “I am tense in mathematics lessons”) statements about mathematics. In order to create a composite score for each factor, the negative statements were rescaled to reflect a positive value (meaning a score of “5” to the question “I am tense in mathematics lessons” was rescored as a “1”) which is consistent with the methodology used by Skaalvik and Rankin (1995). Questions in the same factor were averaged to provide a composite score for that factor. The aptitude and ability factors each only had 3 items so all six questions were averaged to provide a composite aptitude and ability score. Some students did not answer all questions on the questionnaire; missing questions were removed from analysis. Students who were in the top or bottom quartile were labeled as having high or low, respectively, levels of anxiety, effort, intrinsic motivation, or belief in ability/aptitude as compared to their peers.
**Classroom observations.**

Classroom observations were the primary source of data used to gain insight into the nature of discourse which influenced the development of mathematical knowledge. The first week of the semester, I observed the class informally to observe the class and get a better feel for the course design and instructional approach before interviewing the instructor. No recordings or field notes were collected at this time. During the seven weeks when limits were the subject of instruction, every class meeting (5 days a week for an hour) was observed. Each classroom observation was video and audio recorded. One stationary video camera was focused on the white board and projection screen during lectures and the whole class during discussions and activities. The goal of the camera was to record all classroom discussion and also to capture any board work or other visuals projected during class which was referred to during classroom discussion. Each group was given an audio recorder which recorded discussions during small group activities. In addition to video and audio recording, I recorded field notes during and immediately after each classroom observation.

**Student artifacts.**

During the semester students produced a number of artifacts in class during individual and small group activities, and also outside of class. For all students participating in the study, a copy of work relating to limits was collected from the instructor after each class. All work was collected before the instructor had graded it; this ensured that no grades from assessments, class work, or course grades were collected. Student artifacts collected included:

1. Define/Explain – Assignments where students were asked to “tell me,” “describe,” or explain a concept or term. For example, describe how to find a limit from a graph (Appendix I).
2. Problems – Assignments where students were given problems and asked to apply the new concepts they have learned to solve the problem (Appendix I).

3. Quizzes – Students completed quizzes both individually and in groups. For group quizzes, one response was submitted per group. These typically took 20-30 minutes and were a combination of define/explain questions and problems (Appendix I).

4. Tests – There were two tests during the unit on limits and a final exam. The exams took the full class time, completed individually, and were comprised of define/explain questions and problems.

5. Concept Map – Twice students were asked to construct concept maps about limits, including all terms and concepts they associated with limits and showing the connection between concepts, once as a group and once individually (Appendix I).

6. Board Work – All instructor and student work on the white board was copied in my field notes plus video recorded or digital picture taken.

7. Homework – Students completed problems from the textbook or handouts created by the instructor.

8. Project – As a group project students were given a topic on limits, such as rationalization or finding limits graphically, and asked to create a YouTube video to explain how to find a limit using their assigned technique (Appendix I).

Products from in-class activities were collected in tandem with audio recordings. The two together provided insight into the group’s thinking during in-class discussions. Homework, course assessments, and end-of-unit summaries were used to understand the whole class’ understanding of course material, as well as the individual understanding of students selected for case studies.
Student interviews.

The six students selected for case studies were interviewed at the end of the unit on limits. Typically, a pre- and post-interview would be used to compare the student’s change and growth in understanding, but because the students selected had not taken calculus previously and limits are not a standard topic in the high school curriculum outside of calculus, the students had little previous mathematical understanding of the topic. For this reason, an interview was not conducted before the unit began. The interviews were semi-structured (Lindlof & Taylor, 2002) (Appendix H) and took 45-70 minutes. They were conducted in a conference room on the university’s campus which was quiet and had little distractions. An audio and video recorder were used to document each interview, and all written artifacts generated by students during the interview were collected. The first portion of the interview focused on the student’s experience learning during the semester and what they believe helped them the most to learn new concepts in this course. I used probing questions during this portion of the interview to provide more insight into what the student thought about using discourse to learn mathematics. In the second portion of the interview, students were asked seven questions about limits to gauge their understanding of the content. The questions were ordered in this manner so that students could get comfortable with being interviewed and to reduce any anxiety the students may have had about answering mathematics problems in an interview setting. The first two questions about limits were selected to elicit the student’s own definition of a limit and to see how their definition aligned with previous categorizations of student’s models of limits (Williams, 1991). Students were also asked to define a limit in their own words. The remaining five questions were a combination of visual or analytical, and procedural or conceptual questions about limits to provide more detail about the student’s approach to and understanding of limits. I used probing
questions during this portion of the interview to provide more insight into the student’s thinking and understanding of limits.

**Analysis**

A qualitative longitudinal analysis was chosen to focus analysis on mathematical development in terms of time, process, and change. According to Saldaña (2003), the three foundational principles of qualitative longitudinal research are duration, time, and change. Moreover, he emphasizes that time and change are products of their context, “since time is and our social actions and circumstances within it are contextual, change is contextual” (Saldaña, 2003, p. 9). Although this study is not a longitudinal study in the traditional sense, the research questions are concerned with understanding the process of development within the context of a classroom which has a temporal component. In addition, the data involved intensive tracking of the process of learning over the course of the unit on limits. The nature of the research questions and the structure of the data make qualitative longitudinal analysis an appropriate approach for data analysis.

Longitudinal analysis consists of separate analysis of each case through the course of the study, “following individual trajectories and identifying critical moments and change” (Holland, Thomson, & Henderson, 2006, p. 35). Observations and interview data were analyzed in two dimensions: (1) cross-sectionally, analyzing each classroom session and interview, to identify individual mathematical understanding at that moment, and (2) longitudinally, analyzing individual mathematical development across time. In addition, comparative analysis was used to compare cases both cross-sectionally and longitudinally to look for convergence and divergence of cases through time (Thomson & Holland, 2003). Table 2 provides a summary of how the data source and analysis tools align with the research questions.
Table 2. Alignment of research questions with data collection and data analysis methods.

<table>
<thead>
<tr>
<th>Research Question</th>
<th>Data Collection Method</th>
<th>Data Analysis Method</th>
</tr>
</thead>
</table>
| How do individual students construct knowledge of limits through discourse? | Student interview  
Classroom Observation  
Student Artifacts | Longitudinal Analysis |
| What is the nature of discourse which supports the development of mathematical knowledge, specifically the limit of a function? | Classroom Observations  
Student Artifacts  
Teacher Interview | Cross-Sectional Analysis  
- Thematic Analysis  
- Typological Analysis |
| What is the relationship between the collective and individual students’ mathematical knowledge? | Classroom Observations  
Student Artifacts | Comparative Analysis |

Cross-Sectional analysis.

Since most qualitative analysis is cross-sectional, there are a variety of techniques available (Saldaña, 2003). Thematic (Aronson, 1994) and typological (Hatch, 2002) analyses were used for cross-sectional analysis. Video and audio data from classroom observations and interviews were the primary source of data for analysis. The first phase of coding involved looking through field notes to identify all observations where either students or the instructor engaged in discourse about limits. Over the 35 days observed there were 11 days of small group discussion, 9 days of instructor lectures, 3 days of individual work, 3 days of students taking turns explaining how to solve problems on the white board at the front of the whole class, 3 days that students worked on their projects in lieu of the class meeting, 3 days to view the video projects, and 5 days for exams or quizzes. I checked recordings from classroom observations to confirm they contained instances of discourse and to familiarize myself with the data. Of the 11 days students worked in groups, 10 were transcribed because they contained discussions about limits. The day not transcribed was a review day where students remembered how to add and
subtract rational functions. I transcribed video and audio recordings for all class sessions which were identified as including discourse about limits. This meant 10 videos were transcribed to catch the whole class. For the audio recording for the 10 days, only the groups containing case study participants were transcribed. This resulted in a total of 38 group discussions, each 30-50 minutes long, that were transcribed. In addition, I transcribed 6 of the instructor lessons because they pertained to limits and all six student interviews. In addition, all student artifacts for each case study participant (a total of 17 in-class worksheets, 1 video project, 2 concept maps, 6 quizzes or exams, and 25 homework assignments for each student) that contained work related to limits were included as documents for analysis.

I constructed an excel spreadsheet for each case study participant. A tab was created labeled for the date and day of the observation. In the spreadsheet was all data collected that day for the student (e.g. transcribed class discussions, in-class student work, or tests) with a separate tab for homework assignments that were placed after the class it was assigned and before the class it was collected. The spreadsheet also had a tab for the student interview. I also constructed an excel spreadsheet for the whole class which had the same format as the case study spreadsheets but containing transcribed instructor lessons, instructor handouts, and instructor interviews. This resulted in the construction of a timeline of work for each of the 6 case study participants, and for the whole class.

The second phase of coding consisted of typological coding of observations, interviews, and student artifacts. For observations and interview data, each transcript was coded using turn-by-turn coding. I defined a turn as continuous dialogue by the same individual; a break or change in idea or speaker indicated the end of a turn. Not all typologies were applicable for all data, but the following list of typologies were used:
• Direction (e.g. teacher to student, student to teacher, student to student, teacher to group, or student to group),
• Topic (e.g. definition of limits, continuity, function, etc.),
• Purpose of Statement (e.g. ask a question, answer a question, share an idea, explain, justify, etc.),
• Model of a limit (e.g. unreachable, boundary, proximity, etc.),
• Nature of Discourse (e.g. ZPD – scaffolding, collaboration, or interweaving),
• Position in the Vygotsky Space (e.g. Q1, Q2, Q3, or Q4).

The model of a limit was coded using the descriptions of student models of limits outlined in Williams (1991), Szydlik (2000), and Oerhtman (2009). The student conception did not have to have identical wording to the models of limits described in research, instead it needed to have the important characteristics of the model. For example, when a student described “a limit is when a y-value comes close to, but never touches the x-values” this was coded as a model of the limit as unreachable because it aligned with the idea that “a limit is a number or point the function gets close to but never reaches” (Williams, 1991) even though the wording was not identical. Coding for this typology also allowed for open coding in case a student presented a model not described in the literature. The nature of discourse was coded for instances of discourse which resulted in a ZPD (either scaffolding, collaboration, or interweaving) as defined by Goos (2004). For example, scaffolding was coded when it was observed one student pushed another student beyond their current mathematical understanding, and collaboration when it was observed students worked together to co-construct mathematical understanding. Coding for this typology also allowed for open coding so that instances of discourse which
supported the development of student knowledge that did not fit into these categories could be coded.

Position in Vygotsky Space was coded based on repetition of wording and precision of language. I looked at students’ description of limits for wording that was identical to the instructor or their group members to give indication the student was using collective language. Students reveal they have constructed meaning for a concept when they can put it in their own words (Lemke, 1990) so change in wording was used to indicate students were using individual language. Since students used mathematical terminology when putting concepts into their own words, the statements were also examined for precision. Precision looked at the student’s use of terminology, such as “approaches,” to see if the student demonstrated the term was being used vaguely without having a clear definition as described by Oerhtman (2009) when discussing students use of “sufficiently.” When students’ descriptions of limits repeated collective language without precision they were coded as being positioned in Q1 or Q2, depending the location (internal or external) of the description. When students put concepts in their own words or used collective language with precision they were positioned in Q3 or Q4, again depending the location of the description. Students were coded as being positioned in Q1 having completed a cycle through Vygotsky Space when either the group changed their notion of a limit, used the same wording as the student who publicized their thinking, or the group confirmed the notion verbally. It is important to note that mathematical correctness was not a factor in determining position in Vygotsky Space. Completing a cycle in Vygotsky Space indicates that students have internalized a concept and confirmed their individual knowledge with the collective, not that they have constructed correct or accurate knowledge. This means in theory a student can complete a
cycle through Vygotsky Space with incorrect knowledge which would lead to the construction of a misconception.

An example of the typological phase of coding is provided in Table 3. In this discussion, the students were providing a definition for the word *limit* as a verb prior to any instruction on the concept of a limit. The direction of all utterances in this discussion was student-to-student, and the topic stayed focused on providing a definition for the term *limit* as a verb. The students seemed unsure of how to define the term, so they started to express partial thoughts which served the purpose of thinking aloud to stimulate an idea from themselves or one of their group members. One member provided an initial definition through sharing an idea, and another group member expanded on this definition. The third group member asked a question to try and look at the problem in a new way. Two group members answered and the third group member built-on one of his group member’s answers. The final utterance was a partial thought which was thought aloud, possibly to convince herself of the suggested definition. In summary, the students clearly provided a model of the limit as a boundary. The students were using external speech to interact with each other to create a shared definition, so most of the discussion occurred in Q1. However, there were a few instances where the students took their individual knowledge and shared with the group, placing the individual who shared in Q4, but the ideas were then conventionalized by the group moving the students back to Q1 (see Figure 1).

*Table 3.* Example coding for cross-sectional analysis.

<table>
<thead>
<tr>
<th>Discussion</th>
<th>Direction</th>
<th>Topic</th>
<th>Purpose</th>
<th>Model</th>
<th>Position</th>
<th>Nature of Discourse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Javier: um…limit</td>
<td>Student-to-Student</td>
<td>State problem</td>
<td>Limit as a</td>
<td>Q1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The third phase of coding was thematic coding. This consisted of comparing codes across groups and class sessions to look for emerging themes. This phase focused on the nature of discourse to understand when and how types of discourses were used to construct understanding of limits. Codes were also compared for purpose of statements, mathematical topic, and models of limits to understand how these categories related to the nature of discourse.

**Longitudinal analysis.**

Longitudinal analysis was concerned with following each individual student, selected for a case study, as they constructed conceptual knowledge of limits throughout the unit. Due to the
volume of data, I decided to focus on conceptual knowledge of limits because this is the part of limits that research has shown students struggle with the most (Monaghan, 1991; Williams, 1991; Tall, 1993; Bezuidenhout, 2001; White & Mitchelmore, 1996). During the cross-sectional analysis, each student’s conceptual knowledge of a limit was identified and coded for position in Vygotsky Space. The first phase of analysis involved going through the timeline of student data and identifying every instance (from classroom observations, artifacts, and interview) where the student revealed their conceptual thinking about limits. These instances were listed sequentially by day of occurrence, with the code for the position in Vygotsky Space, to construct the timeline of their conceptual understanding of limits (see Table 4 for example of this type of timeline).

The second phase of analysis involved constructing a trajectory through Vygotsky Space plotted on the Harré model (1983) for each student. The student’s position in Vygotsky Space for each instance they revealed their conceptual knowledge of limits was plotted using Geogebra on a Cartesian coordinate system (the x-axis indicated collective/individual and the y-axis indicated external/individual) with each quadrant, starting in the upper right and moving clockwise, representing a position in Vygotsky Space (Q1-Q4). Arrows were added to show progression through time to provide the temporal revision of the Harré model (Gavelek & Raphael, 1996; McVee, Dunsmore, & Gavelek, 2005). The axes represent a dichotomy and not a continuum so position in the quadrant did not have meaning; points were placed so that passage of time would be easier to follow. Figure 3 provides an example of this phase of analysis and the resulting Harré model through Vygotsky Space. A trajectory through Vygotsky Space was constructed for each case study participant.
Figure 3. Example: Javier’s trajectory through Vygotsky Space.

It is important to note that the position in Vygotsky Space does not indicate that all knowledge on that day was in the indicated quadrant in Vygotsky Space. Instead the position in Vygotsky Space on a specific day indicates the position of conceptual knowledge of limits on that day. If a student was positioned in Vygotsky Space in multiple quadrants in terms of their conceptual knowledge of limits then all positions in that same day would be indicated in the figure. For example, in Figure 4 Javier is positioned in Q1 on Day 6. This indicates that on Day 6 the only position in Vygotsky Space observed for conceptual knowledge of limits was in Q1. Furthermore, the trajectory through Vygotsky Space only represents the observed positions in Vygotsky Space. Looking at Figure 4, Javier jumped from Q1 on Day 6 to Q3 on Day 9. This
jump does not mean Javier never positioned in Q2; it simply indicates when the positioning of conceptual knowledge of limits was observed. Most likely Javier transitioned to Q2 and then to Q3 sometime between Day 6 and Day 9, but the data does not confirm when these transitions occurred. Gaps in time in the trajectory also mean it is possible that there was a regression and progression, or vice versa, through Vygotsky Space that was not observed. Even with the possibility that not all transitions in Vygotsky Space were observed, Gavelek and Raphael (1996) point out that with the difficulty of observing internal procedures this analysis allows the researcher to infer transitions in Vygotsky Space even when not observed. Even with the possibility that not all transitions in Vygotsky Space were observed, students were observed every day during the unit on limits and their discourse and work during that period was analyzed. This means that there was regular observation of students’ construction of knowledge, both procedural and conceptual. The trajectory through Vygotsky Space provides a map, although imperfect, of an internal process and affords insight into student’s construction of conceptual knowledge of limits.

Once student’s individual trajectories were constructed, the third phase of analysis focused on identifying impetus for change in student’s conceptual knowledge. The timeline of student’s thinking about limits, constructed during cross-sectional analysis, was examined to identify what led to changes in a student’s conceptual knowledge of a limit. This analysis focused on the relationship between individual and collective student’s knowledge and how they affected each other. Student’s conception of limits was compared with the collective knowledge during that class session, previous class discussions, instructor lessons, and student work during that time to pinpoint what led to changes in the student’s conception of limit and position in Vygotsky Space. This analysis also provided additional proof to support the positioning of
student’s conceptual knowledge of limits in Vygotsky Space. The student interviews were used to verify the identified impetus for change and final conceptual knowledge of limits. This analysis was repeated for each case study participant.

Table 4 provides an example of this phase of analysis. In this example, three instances are provided of Javier’s conceptual knowledge of a limit with the time of conception, position in Vygotsky Space, and model of the limit. The first instance, “you set a boundary,” was constructed during the first discussion on limits so there is no prior lectures, discourse, or student work on this topic. This means students were drawing on their individual experiences with this concept. Javier and his group constructed a definition of a limit as a boundary (see Table 3 for full discussion). The definition was co-constructed by the group placing the knowledge as collective and the term boundary came from a group member so the language was external. This positions Javier’s conception of a limit in Q1 for Javier.

In the second instance, Javier provided the definition, “the limit of a function is a value that the function approaches but never touches.” that was presented in lecture the previous day without putting it in his own words or adding any of his own interpretation. In this instance, Javier has appropriated the teacher’s definition without transforming the definition which positions him in Q2 because he is using Prof. Morgan’s (external) language for his written quiz (internal). The impetus for change from the first conception of a limit to the second was Prof. Morgan’s lecture. Comparing Javier’s definition to the language Prof. Morgan used in her lesson it was almost identical. In addition to Prof. Morgan’s lecture, the students worked individually on a worksheet where they were given the graph of a function with a vertical line at given points. The students were asked to find the limit at the $x$-values which had vertical lines going through them. This required the students to apply a procedure which reinforced the notion of a limit as
unreachable since there was no point for the function to reach, having been blacked out by the vertical line. Javier’s definition does not reflect any of the language from the discussion on Day 1, meaning that the definition presented on Day 4 indicates a change in Javier’s conceptual understanding of a limit from a boundary to unreachable. The impetus for change was the instructor’s lesson, the new collective language constructed as a result of the lesson, and individual practice with a procedure which reinforced this conception.

The final instance, “a y-value that the function approaches but never touches the function gets close to the y-value in question but it is not concerned with what’s happening at that point,” came from an individual quiz that Javier completed in-class. Like the previous instance on Day 4, this was written work so it is located internal to Javier. Javier continued to use terminology and concepts he heard in lecture with little change to the previous conception other than the addition of mathematical terminology as specified in the assignment. The difference in his Day 4 and 5 conceptions came from the group discussion prior to the quiz coupled with the nature of the assignment. On Day 5, the groups were given another group’s quiz and asked to provide feedback on the quiz. One of Javier’s group members suggested that the definition on the quiz could be strengthened by clarifying that it is the limit as “x approaches a y-value.” Javier agrees that the limit is the y-value. This means the connection between the limit and the function was highlighted for Javier. In the quiz students were asked to include x- and y-value in their definition so Javier incorporated the new idea discussed into his existing conception of a limit. This means he was positioned in Q2 since it is internal discourse using collective language, Javier repeated his definition from the day before and appropriated the y-value terminology from his group along with Prof. Morgan’s language describing the limit as “not concerned with what’s happening at that point.”
Table 4. Sample of Javier’s conception of limits over time.

<table>
<thead>
<tr>
<th>Day</th>
<th>Student’s Conception of Limit</th>
<th>Position</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>“You set a boundary.”</td>
<td>Q1</td>
<td>Boundary</td>
</tr>
<tr>
<td>4</td>
<td>“The limit of a function is a value that the function approaches but never touches.”</td>
<td>Q2</td>
<td>Unreachable</td>
</tr>
<tr>
<td>5</td>
<td>“A y-value that the function approaches but never touches the function gets close to the y-value in question but it is not concerned with what’s happening at that point.”</td>
<td>Q2</td>
<td>Unreachable</td>
</tr>
</tbody>
</table>

This example highlights how this phase of analysis focused on change in student’s conceptual knowledge of limits using class discussions, instructor’s lectures, and student work to elucidate the student’s construction of knowledge over time. Analysis focused on comparing individual and collective knowledge, events such as arguments, collaborations, and new concepts which may lead to change in conception. Student’s computations and discussion of procedural knowledge were also analyzed to see how the assignments and student work tied in to inform or reinforce construction of knowledge.

The fourth phase of analysis was comparative analysis. This consisted of comparing students’ pathways through Vygotsky Space and impetus for change to identify patterns, relationships, and themes in the construction of knowledge across cases. This phase focused on students’ conceptions of limits, their trajectory for construction of knowledge through Vygotsky Space, and the relationship between individual and collective knowledge of limits.

This study employed multiple types of data collection and data analysis methods to gain insight into the nature of discourse which supports the development of conceptual knowledge of limits. In addition, the choice of a qualitative longitudinal approach allowed for analysis of change in the context of the classroom, specifically the relationship between collective and individual knowledge constructed through classroom discourse. The next chapter will describe the results of this analysis.
Chapter 4 - Results

The three research questions for this study examines (1) how individual students construct mathematical knowledge through discourse, (2) the nature of discourse that supports the development of mathematical knowledge, and (3) the relationship between the collective and individual students’ mathematical knowledge. To address these research question, this study begins with a case study of how the whole class constructed their knowledge of limits. This illustrates the environment in which students constructed their understanding and considers the first week of the unit on limits including a reconstruction of the teacher-directed lesson which provided an initial definition of a limit. This description is pivotal for exploring all three research questions because it provides context for understanding the collective knowledge on limits. Next, to address the first two research questions, this study will explore six students’ construction of knowledge for the concept of a limit using Vygotsky Space as a lens. Within each student’s case, a background of the student is provided along with a discussion of their construction of knowledge of limits focusing on position in Vygotsky Space and impetus for change. Additionally, the nature of discourse and how it manifested for each student is discussed in terms of their construction of knowledge. Finally, to address the third research question, this study will investigate common themes and trends for all participants.

Professor Morgan’s Class

Prof. Morgan’s class had 17 student participants; all who had taken at least 3 years of high-school math, including Algebra I and II, and Geometry, and 13 had completed a fourth year of high-school math, either Trigonometry, Pre-calculus, or Statistics. The average scores of the student participants on the math self-concept questionnaire in each of the 4 factors were: math anxiety ($M = 3.56, SD = 1.13$), effort ($M = 3.82, SD = 0.46$), intrinsic motivation ($M = 3.31, SD = 0.46$), and
0.68) and belief in mathematical ability/aptitude ($M = 3.31, SD = 0.81$). These were used as benchmarks to compare individual students to the whole class.

A timeline of topics covered over the unit of limits is provided in Table 5 to provide an idea of the topics and pace of the unit. I describe the first week of the unit on limits to portray the most formative portion of the course for developing students’ understanding of limits.

*Table 5. Timeline of limit topics.*

<table>
<thead>
<tr>
<th>Week in Semester</th>
<th>Day in Unit</th>
<th>Topics Covered</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 4</td>
<td>Days 1 - 4</td>
<td>Definition of the Limit of a Function at a Given Point; Finding Limits Graphically</td>
</tr>
<tr>
<td>Week 5</td>
<td>Day 5 - 9</td>
<td>Finding Limits Graphically for Infinite Limits; Sketching Graph of Function Given the Limit</td>
</tr>
<tr>
<td>Week 6</td>
<td>Day 10 - 13</td>
<td>Finding Limits Numerically and Analytically</td>
</tr>
<tr>
<td>Week 7</td>
<td>Day 14 - 17</td>
<td>Limits of Piecewise Functions</td>
</tr>
<tr>
<td>Week 8</td>
<td>Day 18 - 20</td>
<td>Limits at Infinity</td>
</tr>
<tr>
<td>Week 9</td>
<td>Day 21 - 24</td>
<td>Presented Limits Projects</td>
</tr>
<tr>
<td>Week 10</td>
<td>Day 25 - 30</td>
<td>Continuity; Slope of the Tangent Line</td>
</tr>
<tr>
<td>Week 11</td>
<td>Day 31 - 34</td>
<td>Limit Definition of Derivative</td>
</tr>
</tbody>
</table>

On Day 1 of the unit on limits, students were given a list of words: get close to, near (v.), approach (v.), limit (v.), limit (n.), $x$-value, $y$-value, function value, asymptote, undefined, and piecewise function. Students were broken into small groups and asked to define these terms in their own words. Prof. Morgan stressed to students that these definitions need not be mathematical in nature; they should represent what the terms meant to the students. After defining each term, the groups shared their definitions with the whole class constructing a master list of students’ initial meaning for these words. It is from this exercise Prof. Morgan began to build upon students’ initial conceptions of a limit.

On Day 2, Prof. Morgan began class by reminding students of the definitions students created for the list of terms from last class. She focused students’ attention on “get close to,”
“near,” and “approach” then provided an intuitive definition of the limit of a function. She had students focus on the door to the classroom. Walking to the door she stood first outside the classroom facing the doorway and then stepped inside the room facing the doorway. While moving to either side of the doorway she asked students “will I see the same thing from both sides?” She tied this image to limits explaining “the view from one side [of a function] may not be the same as the view from the other side, so we have to look at both sides to see the whole picture.” Prof. Morgan emphasized that this is “their way” of thinking, that students cannot use this language and have individuals outside the class understand, but she explained it was a useful metaphor for the class to understand limits. Here Prof. Morgan established a collective language unique to this class. During this lecture, Prof. Morgan constructed collective language for the class. Next she illustrated finding limits graphically using the graph of a piecewise function and the door metaphor. She drew a box (see Figure 4) around the $x$-value where a jump discontinuity occurred, then showed the “door” getting smaller and smaller by drawing thinner and thinner boxes around the indicated $x$-value as she explained “narrowing down the focus on that spot, who had the word infinitesimally? …getting so close it’s infinitesimally small. What’s happening as it gets very, very small?”

![Figure 4](image.png)

*Figure 4. Illustration of finding limit of a piecewise function using “door” metaphor.*
One of the students asked, “Does it touch?” to which Prof. Morgan replied “without touching, getting as close as you possibly can without touching.” Returning to the door image, she emphasized “she doesn’t care what’s happening under the door,” only what happens when she “approaches the door.” The door metaphor and getting closer and closer without touching were not the only imagery Prof. Morgan employed. Prof. Morgan gave students another way to think of limits by tying the concept of a limit to the students’ use of the word “boundary” when defining limit as a noun the previous class. She had students brainstorm a list of boundaries they encounter in their everyday lives. Selecting examples from the student-generated list of boundaries, such as fences, borders, doors, and laws, Prof. Morgan explained how the notion of a boundary ties to a limit. Finally, Prof. Morgan defined a “limit in math – gets close to [the] value in question but not concerned with what’s happening at that value.” The conception of a limit as a point that did not need to be reachable (unreachable) or a boundary became salient in students’ personal definitions of a limit.

For the rest of class, Prof. Morgan demonstrated finding the limit of a function and then had students practice finding limits given the graph of a function. Day 3 of class followed similarly with a video to refresh students’ memory on limits and then students practiced finding limits graphically. On Day 4 of limits, students were given a group quiz and then individual quizzes. In the group quiz students were asked to define the limit of a function from “a mathematical perspective.” This is the first time they were asked to formalize their understanding. In addition, the groups also needed to explain the difference between a function at a given point and the limit of the function approaching the given point, the difference between a left-hand, right-hand and general limit, and to describe how to find a limit from a graph. The quiz also gave students the graph of a piecewise function and indicated values at which to evaluate the
limit of the function. The individual quiz followed a similar format, with three piecewise functions and indicated limits to evaluate for each graph, plus a question about the difference between a function value and the limit of the function. On Day 5, small groups were handed the quiz of another group, with the names removed, and asked to provide feedback to their classmates to improve their responses. On Day 6, groups received their quizzes back with feedback from their classmates. The groups were asked to revise their quizzes based on their feedback and resubmit.

**Javier**

Javier was a first semester freshman who identified himself as being good at math. When asked about learning calculus over the semester, during the one-on-one interview at the end of the unit on limits, Javier indicated “I mean all of what we’ve learned in Calc[ulus], in class has been easy for me.” He took four years of high school math including Algebra I and II, Geometry, and Pre-Calculus/Trigonometry. His math self-concept questionnaire revealed he average level of belief in his mathematical ability/aptitude ($M = 4.33$), lack of math anxiety ($M = 4.25$), effort ($M = 3.88$), and intrinsic motivation ($M = 3.67$) compared to his peers. Javier attended class regularly (32 of the 35 days on limits or 91%) and always actively participated in class discussions, both small group and whole class. Moreover, Javier displayed comfort working in groups. He often led group discussions, sharing his ideas, and asking his classmates for confirmation. Table 6 provides a timeline of Javier’s group members throughout the unit on limits. When Javier was unsure of how to solve a problem, he listened to his classmates’ explanations, asked questions, and made sure they understood the mathematics.

*Table 6. Timeline of Javier’s group members.*

<table>
<thead>
<tr>
<th>Day 1-5</th>
<th>Day 6-12</th>
<th>Day 15-17</th>
<th>Day 18</th>
<th>Day 19-26</th>
<th>Day 27-34</th>
</tr>
</thead>
<tbody>
<tr>
<td>Javier</td>
<td>Javier</td>
<td>Javier</td>
<td>Javier</td>
<td>Javier</td>
<td>Javier</td>
</tr>
</tbody>
</table>

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Construction of knowledge.

On Day 1 of the unit on limits, Javier’s group constructed a definition of limit as “you set a boundary.” Although Javier had been active in his group’s discussion of defining the list of terms, he never presented his own definition of a limit. This positioned the definition in Q1 – external to Javier and using the collective language.

*Figure 5. Javier’s path through Vygotsky Space.*
During group discussion on Day 4 of the unit on limits, Javier presented the definition of a limit as “The limit of a function is a value, is a value that it approaches but never touches.” This is the first observation of Javier appropriating the collective narrative about limits. Javier was positioned in Q1 (see Figure 5) as he attempted to utilize Prof. Morgan’s definition of limits, focusing on the idea that a limit is unreachable. During the same class, after working in groups, students took an individual quiz. In this quiz, Javier described the limit as “The limit of a function is where the x is approaching but never touches.” Javier was positioned in Q2 (see Figure 5) – internal to Javier but using the collective language. This definition was given in the same class period as the first definition and Javier continued to use the language from group discussion, specifically “approaches but never touches.” However, Javier was beginning to transform the definition as he elaborated on the value being approached specifying it is an x-value. This definition is in stark contrast to the definition of function value as “the value of the closed point on the graph” which Javier also defined on the quiz. In this definition, we can see Javier has constructed personal meaning for the concept of a function value. He provided his individual language to define the concept. The notion of “closed/open point” was very important in Javier’s development in understanding limits.

On Day 6, students were asked to revise their definition of a limit based on feedback from classmates and whole class discussion the previous day. Javier’s group revised their definition from “The limit of a function is a value, is a value that it approaches but never touches” to “A y-value that the function approaches but never touches the function gets close to the y-value in question but it is not concerned with what’s happening at that y-value [emphasis added].” It is clear the students believed they needed to refine their definition but there is no transformation in their definition. Javier, in particular, emphasized the need to include
“mathematical terms in here, like x-value, y-value.” This suggestion led to the addition of y-value in the definition, but the group also referred to their notes and wrote verbatim what they interpreted to be Prof. Morgan’s rigorous definition of a limit, so they were positioned in Q1 (see Figure 5). During the group discussion Javier repeatedly described the limit as “a point that approaches but never touches.” This seems to be the key idea of a limit that Javier has internalized and forms the basis for how he thinks of limits.

Three days later, Day 9, the students were asked once again to individually define a limit. In Javier’s definition, “the limit of a graph is the y-value of a function that a x point approaches but never touches” shows that Javier was positioned in Q3 (see Figure 5) – internal to Javier and using individual language. He has been synthesizing classroom terminology such as y-value, which he previously identified as being an important aspect of a mathematical definition and the y-value specifically for the definition of a limit, and combined it with his personal notions of a limit. A week later students were asked to define left- and right-hand limits. Javier offered a procedural definition, “trace x as it approaches c from the left side.” He also stresses, “If the limit from the left and right have the same y-value then that’s the limit.” Javier transformed and publicized his personal understanding of how to find a limit given the graph of a function. Javier’s use of tracing imagery was unique to his understanding, as it had not previously been discussed in class either in group discussions or as a whole class. By sharing his understanding of how to find a limit, Javier positioned himself in Q4 (see Figure 5) – external to Javier and using his personal language. The definition he shares underscores how important the y-value of the function was in his understanding of a limit.

On Day 15, students were asked to construct in groups a concept map for a limit and include a definition. When his group was ready to define a limit, he responded “I’m going to tell
you right now. Definition of a limit: It is the y-value.” This definition is a regression in Javier’s understanding of limits. Javier voiced frustration at having to explain ideas to his classmates; his simplification of the definition of a limit was to make it more understandable to his group. His new definition focused on the “y-value” which Javier had previously identified as a key feature of a limit. By equating the limit of a function with the function value, he had missed the essential features of a limit. Javier’s group questioned this definition of the limit as the y-value.

Javier: Hold on, hold on. Erase that. I’m going to tell you right now. Definition of a limit. It is the y-value.
Anna: Gets close to the y-value but never touches it.
Javier: It’s always a y-value.
Anna: I put, it did this a long time ago. I don’t know why I put this, but it says “gets close to value in question, but is not concerned with what is happening at that value.”
Javier: What do you have, Mariah?
Mariah: Never close to but approaching a y-value, an x-value close to y-value.
Javier: Okay.
Anna: Okay, so what should we put?
Javier: Gets close to a y-value.
Mariah: This one looks weird though.
Javier: Gets close to a y-value.
Mariah: I know that it is an x-value that comes close y-value.
Javier: That’s right, so an x-value that gets close to a y-value but never touches.

Based on the discussion with his group, Javier revises his definition. Javier is positioned in Q1 (see Figure 5) again as he readopts the collective language “close but never touches.”

On day 32, Javier was interviewed about his understanding of limits. During this interview, Javier was asked one more time to define a limit: “a limit is the value of the function
that approaches a point but will never touch it.” What is striking is how little Javier’s definition had changed over the course of the unit. This definition did not reflect Javier’s previous need to include mathematical terminology in the definition; instead, it was almost identical to the definition he gave the first week of class. This suggests Javier transformed his initial conception of a limit but never transformed it. The addition of terms such as y-value seem to be included when Javier believed he needed a mathematical definition. In his concept map of a limit, completed on the final day of limits, Javier included “never touches,” “approaches,” “y-value,” and “graphing – open circle/closed circle,” which illustrates he still associated these words with a limit even though they were not included in his definition. The definition and concept map show that by the end of the unit on limits Javier is positioned in Q4 (see Figure 5).

*Table 7. Javier’s construction of knowledge of limits.*

<table>
<thead>
<tr>
<th>Day</th>
<th>Student Conception of Limit</th>
<th>Origin</th>
<th>Codes</th>
<th>Quadrant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>“You set a boundary.”</td>
<td>Group members</td>
<td>Boundary</td>
<td>Q1</td>
</tr>
<tr>
<td>4</td>
<td>“The limit of a function is a value, is a value that it approaches but never touches.”</td>
<td>Javier, group discussion</td>
<td>Unreachable</td>
<td>Q1</td>
</tr>
<tr>
<td></td>
<td>“The limit of a function is where the x is approaching but never touches.”</td>
<td>Javier, individual work</td>
<td>Unreachable</td>
<td>Q2</td>
</tr>
<tr>
<td>6</td>
<td>“A y-value that the function approaches but never touches the function gets close to the y-value in question but it is not concerned with what’s happening at that y-value.”</td>
<td>y-value from Javier; rest of addition from group members</td>
<td>Unreachable</td>
<td>Q1</td>
</tr>
<tr>
<td>9</td>
<td>“The limit of a graph is the y-value of a function that a x point approaches but never touches.”</td>
<td>Javier, individual work</td>
<td>Unreachable</td>
<td>Q3</td>
</tr>
<tr>
<td>15</td>
<td>“If the limit from the left and right have the same y-value then that’s the limit.”</td>
<td>Javier, group discussion</td>
<td>Left &amp; Right Hand Limits are Equal</td>
<td>Q4</td>
</tr>
<tr>
<td>18</td>
<td>“Definition of a limit. It is the y-value.”</td>
<td>Javier, group discussion</td>
<td>Limit is Function Value</td>
<td>Q1</td>
</tr>
</tbody>
</table>
By the end of the unit, Javier revealed he thought of a limit visually when defining it. When describing a limit, he emphasized coming from the left side of the graph and from the right side to a given point. Javier explained that a function can be undefined and the limit will still exist because you are approaching the point. When asked what would make the limit not exist, Javier drew a piecewise function with a jump discontinuity and explained the limit does not exist because the left side and the right side of the graph were not approaching the same value. This demonstrates that Javier has developed an understanding of limits and how they differ from a function value. However, when given the equation for two functions, equivalent except for a removable discontinuity, Javier indicated that not only were the limits of the functions equal but the functions themselves were equal. This shows that Javier has not yet connected his conception of limits and function values to his procedural knowledge.

To summarize Javier’s construction of conceptual knowledge of limits, he began the unit by adopting a notion of a limit that is based on Prof. Morgan’s lecture on limits. Javier viewed the teacher as an authority and viewed her role in the classroom to be to tell them how to do math. Javier’s pathway in constructing knowledge of limits predominantly involved sharing his interpretation of Prof. Morgan’s definition and using discourse with his group to reinforce his thinking about limits. However, the interactions with his group were particularly important after Javier was positioned in Q4 for the first time. It seemed that Javier felt he had become an authority on limits in his group, but when presented with the opportunity to conventionalize his
understanding he oversimplified his definition of a limit. Throughout the unit, Javier’s group had affirmed his thinking, but on this occasion, they pushed back, questioning his definition. The discourse with his group not only helped Javier to highlight a flaw in his thinking, but also identified what he viewed as the key features of a limit. As a result of this interaction, Javier cemented his personal meaning of a limit and constructed a generalized definition of a limit. Although he did not complete a cycle in Vygotsky Space, he was repositioned back in Q4.

Nature of discourse.

Working in groups, Javier engaged in three different types of interactions which supported his understanding of limits: Independent Group Work, ZPD as Collaboration, and ZPD as Scaffolding.

Independent group work.

Independent group work took the form of Javier working on problems alone and checking his answers with his group members. This type of independent group work provided Javier with opportunities to reflect on his understanding of limits. When the group questioned Javier’s answers, he evaluated his work to determine if there was a problem with his process or a mistake in his computations. Once he pinpointed his mistake, he revised his work and then rechecked his answer. This type of discourse aligns with Javier’s idea of how group work should be structured. During his interview, Javier indicated he thought groups were most effective when “I try to do them [problems], but I don’t feel like I’m doing them correctly… I’ll see how another person did it and how I did it and I’ll compare the differences and similarities that we have.”
Javier seemed to enjoy collaborating with his peers. He appeared receptive to sharing his ideas and reflecting on feedback from classmates. Collaboration transpired early in the semester when Javier’s group was trying to conceptualize a limit. These exchanges positioned Javier in Q1 where the group was able to build a collective understanding. This served as a jumping off point for Javier’s path through Vygotsky Space. Later in the unit when working on a problem that students had never experienced before, these collaborations were fruitful in supporting Javier’s construction of knowledge of limits because the group established an approach for working with limits in a new way. This type of discourse was particularly important for Javier when he provided a definition for a limit which was overly simplistic, as previously described. The collaboration with his group was instrumental for Javier to identify shortcomings in his definition and revise his thinking. Because of the revision, Javier had a robust conception of limit which distinguished between the limit and the function value and positioned him in Q4.

Javier’s group discussions involved a great deal of scaffolding. No one overtly scaffolded his learning, but he or another group member often scaffolded his groupmates’ understanding. Javier revealed mixed feelings about working in a group. He expressed frustration with working in groups particularly because he felt he regularly had to explain material to his group members. “Just focusing on one question because the person doesn’t get it. Yeah, that’s the most frustrating.” When asked to explain why this was frustrating, Javier explained “I’ve been in a group where I was the only one who knew how to do something so when we’re in like a group of 4, the other 3 people don’t know how to do it and that also gets a little annoying.” He also voiced frustration that scaffolding his peers “will hold us back.” Javier felt that he was not as well
equipped to help his classmates as the teacher, saying “it seems easier for someone to grasp it from the teacher then from a student.” Javier seemed to feel uncomfortable with the role of teaching his group. Moreover, this type of discourse did not meaningfully support Javier’s understanding of limits. The scaffolding was usually needed to help classmates understand the process for finding limits numerically and analytically which Javier was already confident in his understanding of this aspect of limits.

However, part-way through the semester Javier ended up in a group of three with Daniella (case study described below), who like Javier is a high-achiever. Javier found this group to be very conducive to learning and continued to work with this group for the remainder of the semester. Javier and Daniella both indicated in their interviews that they supported each other and found it helpful to explain concepts to their third group member Anna. Working with Daniella allowed Javier to check his solutions when he was working independently, but it also created opportunities for Javier and Daniella to scaffold Anna for conceptual understanding of limits.

Daniella: Wait, okay. That one would be like
Javier: If both of these are the same value that’s what it is, if they’re different values then
Daniella: It does not exist.
Anna: I remember that.
Javier: Yeah.
Daniella: If the answer you have from right there (pointing to right-hand limit) and the answer for that (points to left-hand limit) are the same then that’s the answer.
Javier: That’s your answer.
Anna: I’m trying to think of a good way to say it.
Daniella: Like literally write, if the limit of that (points to left-hand limit) is equal to the limit of that (points to right-hand limit)

Anna: If the limit

Daniella: Then the limit is…I don’t know how to say it, then the limit is that value.

Anna: I got it! The, if the left-hand limit. What is it? If the left and right hand limits are the same then that limit is accurate. If the left-side and the right-side limits are different it does not exist.

Javier: If the limit from the left and right have the same y-value then that’s the limit.

Daniella: When you trace, yeah that’s what you get.

Javier and Daniella’s collaborative scaffolding was the only instance of scaffolding for conceptual understanding among all students. Helping another student to develop conceptual knowledge with the support of Daniella helped to reinforce Javier’s conception of a limit. It was through engaging in this collaborative scaffolding that Javier was first positioned in Q4.

Daniella

Daniella was a first semester freshman who viewed herself as a good student for whom math comes easily, but did not identify herself has strong in math. When asked what she struggled with over the semester, she replied “I don’t think nothing was really hard except for drawing, like going backwards, like drawing a graph given a limit.” She took 4 years of high-school math including Algebra I and II, Geometry, and Pre-Calculus/Trigonometry. Her math self-concept questionnaire revealed she was in the top quartile for all factors, meaning she had strong belief in her mathematical ability/aptitude ($M = 5.00$), lack of math anxiety ($M = 4.63$), high level of effort ($M = 4.38$), and high intrinsic motivation ($M = 4.40$) compared to her peers. Daniella attended class regularly (33 out of 35 classes on limits or 94%) and actively participated in class discussions, both small group and whole class. She worked well in all groups she
participated in over the unit. Table 8 provides a timeline of Daniella’s group members throughout the unit on limits. Daniella never shied from sharing her ideas and questioning her classmates; at times this led to her dominating the group discussions.

Table 8. Timeline of Daniella’s group members.

<table>
<thead>
<tr>
<th>Day 1-5</th>
<th>Day 6-12</th>
<th>Day 15-26</th>
<th>Day 27-34</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yara</td>
<td>Yara</td>
<td>Javier</td>
<td>Javier</td>
</tr>
<tr>
<td>Daniella</td>
<td>Daniella</td>
<td>Daniella</td>
<td>Daniella</td>
</tr>
<tr>
<td>Mariah</td>
<td>Juanita</td>
<td>Anna</td>
<td>Anna</td>
</tr>
<tr>
<td>Noel</td>
<td>John</td>
<td>John</td>
<td>John</td>
</tr>
</tbody>
</table>

**Construction of knowledge.**

On Day 1 of limits, Daniella constructed two definitions of a limit with her group. The first “A restriction or a set boundary within a range” was given to define a limit as a verb. This definition positioned Daniella in Q4 (see Figure 6) when she publicized her personal meaning of limits to her group. This definition also served as the basis for her construction of knowledge of limits. To define a limit as a noun Daniella suggests “Isn’t a limit like the point where a graph or a function no longer exists or something like that. Like if the limit is 2 it won’t ever pass 2, it will like approach up to 2.” This shows Daniella had previous exposure to limits in one of her math classes. She tried to recollect the definition she has previously learned, which positioned her in Q1 (see Figure 6). She parroted a definition she had been exposed to, but it has no personal meaning to her. This is further supported by the fact that when Daniella wrote the definition for her group she slightly changed it to be “The point or value where a function approaches. If the limit is 2, the function will not ever equal 2 but can equal 1.99.” This change in the definition also changed the meaning of the definition from describing the limit as a boundary to an approximation process. It also positioned Daniella in Q2 (see Figure 6) since she copied a definition from their discussion without internalizing the definition.
On Day 4, during group discussion, Daniella gave “The value a graph approaches but never crosses” as the definition of a limit. Daniella was positioned in Q1 (see Figure 6) because she had abandoned the collective understanding from her previous exposure to limits and adopted the shared language of the class without transforming the definition. Daniella’s group agrees with this definition but when she wrote the definition on the quiz for her group she modified the definition based on her group’s feedback to be “the limit of a function is the value at which the graph of the function approaches, but never touches or crosses.” This positioned Daniella in Q2 (see Figure 6) since she used the language of her group without internalizing the
definition. The students were then asked to individually define a limit. Daniella used the ideas discussed in her group as the basis of her definition, but also included additional concepts that she identified as important to her understanding of a limit to construct the definition:

The limit of a function shows/explains the behavior of a graph as it approaches a number. The limit of a graph is a value. The graph approaches and gets near too, but does not cross. The function value is the value of the function as a specific number. The function value is \( f(a) \), where \( a \) is any \( x \)-value. Whatever the function equals only when “\( a \)” is entered into the function is the function value at that specific number. The limit is showed as \( \lim_{x \to a} f(x) \), and the function value is \( f(a) \). The limit cannot exist, where the function value can be undefined. The function value is the range, and the \( y \)-values. For example for #3, above; the limit of -2 is 2, but the function value is 1, because -2 is undefined on the graph. The limit can approach an undefined point. The function value is either a number or it is undefined.

Daniella transformed the definition from the group discussion, inserting her understanding that a function can be undefined and the limit will exist. In this definition, Daniella constructed a more comprehensive definition than the group definition which positioned her in Q3 (see Figure 6). Although she developed her own meaning for a limit, she was not ready to publicize this understanding. Two days later, Day 6, Daniella had an opportunity to share this new definition during the group discussion, but choose not to publicize her meaning. Daniella described this group as a group where she felt she had to give them the answers. It seems that she felt she was cast in the authority role in her group and did not want to be embarrassed by explaining her thinking in a way that was wrong or her group members would not understand.
On Day 9 students once again were asked to individually define a limit. Daniella defined “the limit of a graph is the y-value at which the function approaches for the given x-value.” Daniella moved backwards in her development of a limit, positioning herself in Q2 (see Figure 6). This definition reverted back to the approaches imagery of the class and does not include the features Daniella identified as key in her previous definition. This reversion in her thinking seems to be due in part to the fact that students were asked to integrate mathematical terminology, specifically to discuss the x- and y-value of the function. In an effort to integrate this terminology, Daniella removed her personal meaning from her definition.

On Day 15 Daniella and Javier worked as a group with Anna. Daniella indicated this grouping was a good collaboration, “I feel like in the class I feel like me and Javier are like on the same level but I wouldn’t want to work with just a bunch of Javier’s. I want to work with like someone else. I like how Anna when she doesn’t understand something she like, we communicate well with her.” On this day, Daniella and Javier worked together to explain limits to Anna. Daniella explained to Anna that “If both of these [the left and right hand limit] are the same value that’s what it [the limit] is; if they’re different values then it does not exist.” This moment is important because Daniella finally publicized a salient part of her personal definition to her group, positioning her in Q4 (see Figure 6).

During her interview on Day 32, Daniella defined “the limit of a function is the y-value at which that function approaches but never crosses for the given x-value.” This definition reflected the wording she had used throughout the unit which also aligned with Prof. Morgan’s imagery for limits. In her final concept map, on Day 34, Daniella echoed this definition when she wrote “approaching y-value for given x-value,” but her concept map also emphasized that the limit “exist if left-hand and right-hand limit are the same, limit has that number.” This revealed that
although the left- and right-hand limits being equal was an important piece in her understanding of limits, Daniella did not view this as being part of the definition of a limit. She also included terms such as $x$-value and $y$-value, which had become viewed as part of a rigorous mathematical definition by the class. Daniella’s individual knowledge became part of the collective knowledge, as seen when Anna adopted Daniella’s understanding of a limit, and positioned her back in Q1 (see Figure 6) thus completing a full cycle in Vygotsky Space.

Table 9. Daniella’s construction of knowledge of limits.

<table>
<thead>
<tr>
<th>Day</th>
<th>Student Conception of Limit</th>
<th>Origin</th>
<th>Codes</th>
<th>Quadrant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>“A restriction or a set boundary within a range.”</td>
<td>Daniella, group discussion</td>
<td>Boundary</td>
<td>Q4</td>
</tr>
<tr>
<td></td>
<td>“Isn’t a limit like the point where a graph or a function no longer exists or something like that. Like if the limit is 2 it won’t ever pass 2, it will like approach up to 2.”</td>
<td>Daniella, group discussion</td>
<td>Boundary</td>
<td>Q1</td>
</tr>
<tr>
<td></td>
<td>“The point or value where a function approaches. If the limit is 2, the function will not ever equal 2 but can equal 1.99.”</td>
<td>Daniella, written submission for group</td>
<td>Approximation</td>
<td>Q2</td>
</tr>
<tr>
<td>4</td>
<td>“The value a graph approaches but never crosses.”</td>
<td>Daniella, group discussion</td>
<td>Boundary</td>
<td>Q1</td>
</tr>
<tr>
<td></td>
<td>“The limit of a function is the value at which the graph of the function approaches, but never touches or crosses.”</td>
<td>Daniella, written submission for group</td>
<td>Boundary</td>
<td>Q2</td>
</tr>
<tr>
<td></td>
<td>“The limit of a function shows/explains the behavior of a graph as it approaches a number. The limit of a graph is a value. The graph approaches and gets near too, but does not cross. The function value is the value of the function as a specific number. The function value is $f(a)$, where $a$ is any $x$-value. Whatever the function equals only when “$a$” is entered into the function is the function value at that specific number. The limit is showed as $\lim_{x \to a} f(x)$, and the function value is $f(a)$. The limit cannot exist, where the function value can be</td>
<td>Daniella, individual work</td>
<td>Boundary</td>
<td>Q3</td>
</tr>
</tbody>
</table>
undefined. The function value is the range, and the y-values. For example for #3, above; the limit of -2 is 2, but the function value is 1, because -2 is undefined on the graph. The limit can approach an undefined point. The function value is either a number or it is undefined.”

| 9 | “The limit of a graph is the y-value at which the function approaches for the given x-value.” | Daniella, individual work | Unreachable | Q2 |
| 15 | “If both of these are the same value that’s what it is; if they’re different values then it does not exist.” | Daniella, in group discussion | Left & Right Hand Limits are Equal | Q4 |
| 32 | “The limit of a function is the y value at which that function approaches but never crosses for the given x-value.” | Daniella, individual work | Boundary | Q1 |

Daniella viewed limits at the end of the unit as a series of characteristics: the definition, the left- and right-hand limits needing equality, and finding limits graphically and analytically. These were distinct features in Daniella’s mind: “when you describe a limit like there are a lot of different factors so they all, so a bunch of them put together. Like if you say the definition someone won’t really understand like based off of one sentence. You have to give different factors.” When asked to explain these “factors” she described finding a limit graphically and analytically. Daniella makes a strong distinction between the limit and the function value, which, coupled with her emphasis on thinking about limits graphically, has provided her with a strong intuition about limits. However, this intuition has not been connected with her thinking of limits analytically. When given the equation of two functions, she acknowledged one function has a hole which would mean the two functions are not equal, but still claimed the functions were equal. She did not know how to come to terms with the fact that the limits were the same, but the functions were not equal. Daniella had a complex definition for limits which at times contradicted itself. Because she perceived the limit as a series of characteristics, whenever she
came across a new idea about limits she revised her conception to include this new idea without connecting these characteristics or considering whether they contradicted each other. Since she never identified any inherent contradictions in the factors, she never resolved the contradictions to create a more generalized definition that would apply to all situations involving limits.

To summarize Daniella’s construction of conceptual knowledge of limits, she actively engaged in group discussions throughout the semester. She used discourse with her group to help her identify her initial notions about limits. This understanding became the foundation for building her understanding of limits. Daniella integrated Prof. Morgan’s discourse about limits, as well as her group members’ notions. Her openness to discourse facilitated the construction of knowledge because it allowed her to confirm and revise her knowledge about limits. Daniella was able to progress through each quadrant, transforming and internalizing her knowledge of limits; however, during discourse she was never questioned or contradicted. As a result, Daniella never identified any conflicts in her notion of a limit and constructed a definition of a limit which was a disjoint series of characteristics. Never resolving inherent contradictions in her thinking is problematic because Daniella was never able to create a more generalized and connected definition.

**Nature of discourse.**

Daniella expressed frustration with her groups during her interview, “There was one group I had I felt like I was just giving the answers just basically. Another group me and someone else were just arguing the whole time so like that’s not helping me learn because we’re just arguing over the question but the last group she put us in, it was a coincident [sic] that I actually liked the group… because it was like a balance.” The dynamic of the group had a large impact on the type of discourse Daniella engaged in during group discussions. For instance,
Daniella engaged in independent group work, often while working with Javier. But unlike Javier, Daniella was not able to use this type of discourse to construct her knowledge of limits. It merely served as a way to check her work. The only type of discourse which supported Daniella’s construction of knowledge was collaboration. Discourse took two forms for Daniella: ZPD as Collaboration and ZPD as Scaffolding.

**ZPD as collaboration.**

Collaboration as a ZPD occurred when Daniella worked with her groupmates to construct a collective understanding of limits. This happened in the start of the unit, positioning Daniella in Q1 and provided an initial notion of a limit. These early collaborations also expanded Daniella’s thinking about limits. These collaborations helped Daniella to identify characteristics of a limit and led to her revising her definition to include key characteristics, but never resulted in her combining these characteristics into one coherent definition.

**ZPD as scaffolding.**

The second form of collaboration was collaborative scaffolding: the collaboration with Javier to scaffold Anna’s understanding of limits. This collaboration was the impetus for Daniella being positioned in Q4. The scaffolding provided the opportunity for Daniella to share her individual knowledge. Moreover, the collaboration with Javier confirmed Daniella’s personal meaning conventionalizing it as part of the collective understanding. Scaffolding alone was not sufficient for Daniella to complete a cycle through Vygotsky Space. Scaffolding provided Daniella the opportunity to be positioned in Q4 and it was the addition of the collaboration with Javier that positioned her in Q1 since he validated her thinking confirming it as part of the collective knowledge.
Yara

Yara was a first semester freshman who identified herself as strong in math. When asked about learning math, Yara replied “I think it’s easier to do math [than other subjects].” She took 4 years of high-school math including Algebra I and II, Geometry, and Trigonometry. Her math self-concept questionnaire revealed an average belief in her mathematical ability/aptitude (M = 3.83), lack of math anxiety (M = 4.00), and level of effort (M = 3.71), and high intrinsic motivation (M = 4.00) compared to her peers. Yara attended class regularly (35 out of 35 classes on limits or 100%) but was hesitant to participate in group discussions. Table 10 provides a timeline of Yara’s group members throughout the unit on limits.

Table 10. Timeline of Yara’s group members.

<table>
<thead>
<tr>
<th>Day 1-5</th>
<th>Day 6-12</th>
<th>Day 15-17</th>
<th>Day 18-24</th>
<th>Day 25-26</th>
<th>Day 27-34</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yara</td>
<td>Yara</td>
<td>Fern</td>
<td>Fern</td>
<td>Mariah</td>
<td>Fern</td>
</tr>
<tr>
<td>Daniella</td>
<td>Daniella</td>
<td>Yara</td>
<td>Yara</td>
<td>Yara</td>
<td>Yara</td>
</tr>
<tr>
<td>Mariah</td>
<td>Juanita</td>
<td>DeShawn</td>
<td></td>
<td>Sally</td>
<td>Bethany</td>
</tr>
<tr>
<td>Noel</td>
<td>John</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Construction of knowledge.

On Day 1, Yara offered her initial notions of a limit as something “you can’t cross.” This positioned her in Q4 (see Figure 7) when she publicized her thoughts about limit, but it had not become part of the collective knowledge.
For the first few weeks, Yara worked with Daniella in a group. Yara was very confident in her ability to understand mathematics, but was not comfortable sharing her ideas with her group. As a result, much of the group discussion was dominated by Daniella. For example, on Day 4, Yara defined “the limit is just like get close to, approach a value.” It is clear Yara had constructed this definition from what she learned from Prof. Morgan’s lesson on limits the previous day. This positioned her in Q1 (see Figure 7), since she attempted to use the shared language presented by Prof. Morgan. Daniella had presented a definition for a limit during
discussion for the group quiz, of “the value at which the graph of the function approaches, but
never touches or crosses.” However, Yara did not attempt to adopt the shared language of her
peers.

Noel: What is the limit of a function?
Daniella: Oh, it’s the value like it approaches but doesn’t ever cross. Something
like that.
Noel: When a value approaches
Daniella: Wait, the value a graph approaches but never crosses.
Noel: The what? Never crosses the
Daniella: huh? Nah, I said a value a graph approaches but never crosses.
Noel: Crosses what?
Daniella: The value.
Noel: Oh…but you can’t say the value of a graph that approaches a value.
Yara: Yeah, the limit is just like get close to.
Daniella: Yeah. So what are we going to write?
Noel: The limit of a function is when
Yara: (simultaneously) just get close to a…approach a value.
Noel: The limit of a function is the value of graph approaches.
Yara: touch the value
Daniella: (writes “the value at which the graph of the function approaches, but
never touches or crosses”) good?
Noel: Yeah that’s close enough

In her individual quiz, Yara’s definition did not reflect the ideas or language of this group
definition. Instead she defined it as “the limit of a function mean get close to the value. The
function value means y-value, output, range.” The definition is still vague and does not show an
attempt to make personal meaning. Yara used Prof. Morgan’s language, as compared to her
classmates, which illustrates that Yara was constructing knowledge separate from her group.
Yara began to identify the ideas about a limit that were salient for her understanding, such as the distinction between a function and a limit. This positioned her in Q2 (see Figure 7).

On Day 9, Yara revisited her idea of a limit, now defining “the limit of a function is a graph that shows the y-value when x approach, but never touches the point.” This definition used some of the wording of her previous definition, but also incorporated some of the language used by her group during discussion, specifically “never touches.” Yara was still trying to transform the notion of a limit and different facets of the collective language to find an image of the limit that made sense to her, which positioned her in Q2 (see Figure 7).

During the next week, Yara was moved into a group with Fern. This significantly changed the group dynamics for both students. Yara felt comfortable working with Fern and began sharing her mathematical thinking; becoming an authority for her group. On Day 18, Yara shared with her group her definition of a limit, “A y-value when x approaches, when x approaches but never touches.” At this point, Yara was positioned in Q4 (see Figure 7). She publicized the ideas that she has identified as the most crucial elements of a limit, namely that “x approaches but never touches” the function value.

During her interview on Day 32, Yara presented a similar definition, “the limit of the function equals the y-value when x approaches but never touches the point.” The definition became more precise, while still reflecting the same key ideas which seemed to become a mantra for Yara. Namely, the notion of “approaching but never touching.” This notion became part of the collective language when Fern also adopted this turn of phrase. This positioned Yara in Q1 (see Figure 7), since her personal meaning became part of the collective understanding for both her and Fern, and completed a full cycle in Vygotsky Space.

Table 11. Yara’s construction of knowledge of limits.
At the end of the unit, Yara made a strong delineation between a limit and a function value, as evidenced by her final definition of a limit. In addition, her ability to think abstractly about a limit given a function value and vice versa illustrates that she had a robust definition of continuity and the relationship between a limit and continuity. Although her definition of a limit is simple, it encompasses the ideas that Yara identified as unique to the limit. The definition is robust enough to be applied to multiple problems and connect to related mathematical concepts. Yara also had strong analytical skills which allowed her to evaluate limits easily. She connected her conceptual and analytical understanding of a limit, but this connection was still tenuous at the end of the unit. She demonstrated some trepidation when explaining the difference between the limit and the function analytically, which indicates that this connection between conceptual and procedural knowledge is still tentative in Yara’s mind.

To summarize Yara’s construction of conceptual knowledge of limits, she began by adopting the discourse of Prof. Morgan, but she predominantly constructed knowledge through

<table>
<thead>
<tr>
<th>Day</th>
<th>Student Conception of Limit</th>
<th>Origin</th>
<th>Codes</th>
<th>Quadrant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>“You can’t cross it.”</td>
<td>Yara, group discussion</td>
<td>Boundary</td>
<td>Q4</td>
</tr>
<tr>
<td>4</td>
<td>“The limit is just like get close to, approach a value.”</td>
<td>Yara, group discussion</td>
<td>Unreachable</td>
<td>Q1</td>
</tr>
<tr>
<td>9</td>
<td>“The limit of a function is a graph that shows the y-value when x approach, but never touches the point.”</td>
<td>Yara, individual work</td>
<td>Unreachable</td>
<td>Q2</td>
</tr>
<tr>
<td>18</td>
<td>“A y-value when x approaches, when x approaches but never touches.”</td>
<td>Yara, group discussion</td>
<td>Unreachable</td>
<td>Q4</td>
</tr>
<tr>
<td>32</td>
<td>“The limit of the function equals the y-value when x approaches but never touches the point.”</td>
<td>Yara, individual work</td>
<td>Unreachable</td>
<td>Q4</td>
</tr>
<tr>
<td>35</td>
<td>“Close to the value but never touches.”</td>
<td>Yara, individual work</td>
<td>Unreachable</td>
<td>Q1</td>
</tr>
</tbody>
</table>
personal reflection. Yara was passive in group discussions, rarely inputting her thinking or adopting the discourse of her group. Instead, Yara listened to her group, reflected on their notions of a limit, and used this to inform her own thinking. When the group’s notion conflicted with Yara’s, she seemed to transform the new conception and resolve any contradictions with her own conception when appropriate or reject the new notion. Yara did not engage in discourse until she had constructed a personal meaning of a limit and felt confident enough to publicize her thinking. When she was ready to publicize her personal meaning, her group conventionalized her thinking and then Yara’s discourse became the discourse of her group, completing a cycle in Vygotsky Space.

**Nature of discourse.**

Yara did not engage in much discourse over the unit. On the first day learning about limits, Yara tried to collaborate with her group. However, her conceptions were not integrated into their collective knowledge. In addition, when Yara attempted to collaborate with her group on procedural questions, her group did not seem to understand what Yara was suggesting. This did not support construction of knowledge for Yara or her group members. These experiences help to explain why Yara spent the first half of the unit on limits observing her group without contributing much to the group discussion. Nevertheless, this passive participation seemed to support Yara’s construction of knowledge of limits, along with scaffolding Fern during the latter half of the unit.

**Passive participation.**

Although Yara did not often engage in discourse, she was still active in the group in the sense that Yara listened to the group’s ideas and reflected on the group discussion. It is clear that Yara was not ignoring her group or just working alone because when she felt the group was
missing an important idea, she contributed to the group. It was just that her contributions were few during group discussion.

*Marie:* I think it’s negative infinity. What do you guys think it is?

*Noel:* I think it’s -3 because when you do this, -5, from both sides of -5. So this is here.

*Marie:* But this is at -3 though.

*Noel:* No for -5 you have to go from both sides of -5.

*Daniella:* Yeah.

*Yara:* They didn’t put the things out so it’s just a line so it’s -3.

Her generally passive participation served as a catalyst for reflection which supported Yara’s internalization of limits and her positioning in Q2 and then Q3. Moreover, this reflection facilitated Yara’s development of a definition of a limit which was generalizable. She was able to consider her groupmates’ notions and integrate or revise her personal definition so there were no contradictions in her conception of a limit.

**ZPD as scaffolding.**

Partway through the unit on limits, the dynamics in Yara’s group changed. Instead of being in a group with Daniella, she was working with Fern. This switch was significant because she was no longer working in a group which was constructing ZPD as collaboration; she was now in a group where Yara facilitated ZPD while scaffolding Fern. This change was imperative for Yara’s progression through Vygotsky Space. Scaffolding Fern solidified Yara’s personal meaning and gave her opportunity to publish knowledge and enabled Yara to be positioned in Q4. By sharing her knowledge with Fern, the scaffolding provided an opening for Yara’s understanding to become part of the collective knowledge. When Yara worked with Daniella, her
knowledge was never integrated into the group’s thinking, but Fern adopted Yara’s notion of a limit, positioning Yara in Q1 and completing her cycle through Vygotsky Space.

**Fern**

Fern was a first semester freshman who viewed herself as a good student, but did not identify herself as strong in math. When asked what she struggled with over the semester, she replied “the easiest thing was factoring because I’m really good… with basic algebra. But then when it comes to the other things, I don’t understand.” She took 4 years of high-school math including Algebra I and II, Geometry, and Pre-Calculus/Trigonometry. Her math self-concept questionnaire revealed she has a low belief in her mathematical ability/aptitude ($M = 2.67$), high math anxiety ($M = 2.00$), average level of effort ($M = 3.88$), and low intrinsic motivation ($M = 2.60$) compared to her peers. Fern attended class regularly (33 out of 35 classes on limits or 94%) but was often passive during group work. Table 12 provides a timeline of Fern’s group members throughout the unit on limits. Fern usually refrained from sharing her ideas and questioning her classmates except when working with Yara, who helped Fern to be more active in group discussions.

**Table 12.** Timeline of Fern’s group members.

<table>
<thead>
<tr>
<th>Day 1-5</th>
<th>Day 6-12</th>
<th>Day 15-17</th>
<th>Day 18-24</th>
<th>Day 25-26</th>
<th>Day 27-34</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fern</td>
<td>Fern</td>
<td>Fern</td>
<td>Fern</td>
<td>Fern</td>
<td>Fern</td>
</tr>
<tr>
<td>Juanita</td>
<td>Kyle</td>
<td>Yara</td>
<td>Yara</td>
<td>Nell</td>
<td>Yara</td>
</tr>
<tr>
<td>Alice</td>
<td>Nell</td>
<td>DeShawn</td>
<td>Raul</td>
<td>Lena</td>
<td>Bethany</td>
</tr>
</tbody>
</table>

**Construction of knowledge.**

On Day 1, Fern acted as a scribe for her group, but contributed little to her group discussion. The only suggestion she made was when she said “isn’t a limit like infinity.” This
revealed Fern had some previous exposure to the notion of a limit. Fern was positioned in Q1 (see Figure 8) because the definition did not have personal meaning; it was her attempt to mimic what she remembered from previous exposure to the definition of a limit prior to the start of Prof. Morgan’s class. This definition did not have strong meaning for Fern because she abandoned her suggestion and wrote down a classmate’s definition of “a point that can’t be touched but has a boundary” when selecting what to put for her group’s response, instead of including her own ideas.
Figure 8. Fern’s path through Vygotsky Space.

On Day 4, Fern once again refrained from participating in her group’s discussion. She observed her group construct the definition “an imaginary [sic] line that a function approaches but never actually touches, because it plays as a boundary.” During the day’s activities, the students also individually were asked to provide a definition of a limit. This was the first time Fern disclosed her understanding of a limit as “the limit of a function is approaching but never touching the number. The function value is an open circle or else it does not exist.” Fern’s definition presented an idea of the limit as unreachable. This definition demonstrates that she had not transformed the ideas about limits that her group has been discussing since they had been presenting the limit as a boundary, and instead she had latched onto the imagery that Prof. Morgan presented the class during her lecture on limits. This definition shows Fern positioned in Q2 (see Figure 8) since she used the language established by her teacher. This was in stark contrast to her definition of a function value where she was positioned in Q4 since she publicized her meaning when she uses “open circle” in her definition. Fern’s concept that “the function value is an open circle or else it does not exist,” is important to note because it demonstrates that Fern struggled to understand the function value. This was problematic for her in the sense that without a solid understanding of a function, it became difficult for Fern to make the distinction between a function value and the limit of a function, especially when the function was undefined.

On Day 6, Fern’s group received feedback on their definition of a limit. The group was resistant to feedback and refused to change their work, to the point where her group became combative with the group that gave them feedback and Prof. Morgan had to step in to defuse the situation.

Brad: (reading) “your answer could have been strengthened if you included more information about when and when it’s not existent”
Raul: When and when (Chuckles)

Juanita: When and when it is not existent.

Raul: Oh…

Juanita: Kyle, what is the general rule? We didn’t learn that. Nope it doesn’t count.

Kyle: What? You’re not going to let me explain it?

Juanita: NO we didn’t learn it!

Although her group was very vocal defending their notion of a limit, Fern once again did not participate in the group discussion. Fern showed no sign of ownership of her group’s definition.

On Day 9, Fern was asked again individually to define a limit. Fern’s definition was drastically different from the definition given five days earlier. “The limit of a graph is what the y-intercept is when a is approaching a number.” She defined a limit in terms of proximity, instead of her previous definition which described a limit as unreachable. This definition positioned Fern in Q3 (see Figure 8). She had shed the language of Prof. Morgan which she used in her previous definition while incorporating the y-value terminology specified in the assignment. This definition was problematic from a mathematical standpoint since Fern confused a y-value and y-intercept (a mistake she made frequently throughout the unit). This confusion aligns with her misunderstanding of a function she previously exhibited. Nonetheless, it demonstrated how Fern conceptualized the limit and that she had begun to transform this concept.

The next week, Fern’s group changed and she began working with Yara and another classmate, DeShawn. This is important to note because Fern developed an antagonistic relationship with DeShawn. Unlike her previous experience in a group where a disagreement broke out, Fern actively participated in disagreements with DeShawn.
DeShawn: Alright this is not going to work out attitude girl. It’s a team effort.
Fern: Shut up!
DeShawn: Everybody plays their part.
Yara: I think that both sides are… I don’t know how to.
DeShawn: Coming from both sides. Approaching $c$ coming from the left hand and right side where $c$ is at.
Yara: The $y$-value is the same I think. I don’t know how to explain it.
DeShawn: That’s not even going to do it.
Fern: What makes you think I know what to write?
DeShawn: You know what to do.
Fern: No I don’t. She [Yara] knows what to do. All I know is that she just put in the $c$. The same thing that we’ve been doing it’s just that the $c$
Yara: It’s just that the left side and right side and they have the same value.

The disagreements between Fern and DeShawn helped to cement a bond between Fern and Yara, but also pushed Fern to take ownership of Yara’s notion of a limit. Fern mentioned she viewed Yara as very knowledgeable mathematically and had a great deal of respect for Yara’s opinions and ideas. On Days 15 and 18, Yara presented her ideas of a limit, specifically defining a limit on Day 18 as “the function of $y$-values when $x$ approaches but never touches.” Fern once again acted as scribe, writing down Yara’s ideas, making suggestions on notation, and refusing to include DeShawn’s additions. This positioned Fern back in Q1 (see Figure 8) but not because she completed a cycle in the Vygotsky Space; instead, she was beginning the cycle afresh by adopting Yara’s language without integrating any of her own meaning.

Fern worked with Yara for most of the remainder of the unit with different students as their third group member. When asked about working in groups she said “But then again I guess it depends on the people because I feel like when I was with Bethany [group member] and Yara… We all did it and if we didn’t understand it we would ask each other. And there was a
part where I was like ‘how’d you get that?’ and she [Yara] was like ‘this is what you did’ and she corrected us, you know. So that I don’t mind, that’s like a good group.”

During the one-on-one interview on Day 32, Fern defined “a limit is the x or y-intercept approaching a line but never touching.” This definition positioned Fern in Q1 (see Figure 8). It mimicked Yara’s ideas about limits as evidenced by the description of a limit as “approaching… but never touching.” This choice of phrase was very important to Yara in her conception of a limit. However, this phrase did not have meaning for Fern as shown by the fact that she referred to the “x or y” because she is not sure which variable it should be and again called it an intercept. On the final day working with limits, Day 35, Fern gave a final definition of “Limits (vb./noun) near, close to approach x-y values (function value).” Fern is positioned in Q2 (see Figure 8). Fern tried to connect her notions of a limit together, but the definition was not a fleshed-out conception of a limit and was imprecise. Fern demonstrated she is trying to transform Yara’s notion of a limit and make it her own, but the concept is not fully formed or defined well enough to have meaning for her.

Table 13. Fern’s construction of knowledge of limits.

<table>
<thead>
<tr>
<th>Day</th>
<th>Student Conception of Limit</th>
<th>Origin</th>
<th>Codes</th>
<th>Quadrant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>“A point that can’t be touched but has a boundary.”</td>
<td>Group members</td>
<td>Boundary</td>
<td>Q1</td>
</tr>
<tr>
<td></td>
<td>“Isn’t a limit like infinity?”</td>
<td>Fern, group discussion</td>
<td>Infinity</td>
<td>Q1</td>
</tr>
<tr>
<td>4</td>
<td>“An imaginary line that a function approaches but never actually touches, because it plays as a boundary.”</td>
<td>Group members</td>
<td>Boundary</td>
<td>Q2</td>
</tr>
<tr>
<td></td>
<td>“The limit of a function is approaching but never touching the number. The function value is an open circle or else it does not exist.”</td>
<td>Fern, individual work</td>
<td>Unreachable</td>
<td>Q2</td>
</tr>
<tr>
<td>9</td>
<td>“The limit of a graph is what the y-intercept is when a is approaching a number.”</td>
<td>Fern, individual work</td>
<td>Proximity</td>
<td>Q3</td>
</tr>
</tbody>
</table>
At the end of the unit, Fern did not recognize the difference between the limit of a function and the function value. During her interview, when given the graph of a function with a removable discontinuity, Fern indicated the limit does not exist because “there is an open circle.” This was also supported by the fact that when given a limit and asked to discuss the function, she assumed the function was continuous and gave the function value as the limit. She made the same assumption when given the value of a function and asked to discuss the limit of the function. Not only does this illustrate a fundamental misunderstanding of limits, but highlights a misunderstanding in the relationship between limits, functions, and continuity. When given limits to evaluate analytically, Fern could solve them algebraically but reduced the functions to their limit value. Therefore, she was not able to identify that two functions were not equal, even if their limits were equal. Due to the fact that Fern feels confident in her ability to work algebraically, she sees functions as equations to be manipulated. She struggled to work graphically, think abstractly about limits, and make connections between conceptual and procedural understanding of a function and its limit.

To summarize Fern’s construction of conceptual knowledge of limits, she initially took a passive role during discourse on limits. She adopted Prof. Morgan’s discourse on limits because she questioned her group’s notion of a limit. This distrust of her group caused Fern to reflect on the group’s ideas and to accept or reject their notions as part of her personal meaning. It was
during these group interactions that Fern positioned herself in Q3. However, during the unit the dynamics changed when she switched to a group with Yara. Fern trusted Yara’s discourse; moreover, Yara was more passive than Fern in group discussions causing Fern to adopt a more active role in the group discourse. This was problematic for Fern’s construction of knowledge because she felt she needed to defend Yara’s ideas. In the process, Fern abandoned her own conception of a limit and adopted Yara’s conception, without challenging or reflecting on Yara’s notions. This resulted in Fern deserting the knowledge she had developed about limits and assuming Yara’s discourse without constructing personal meaning.

**Nature of discourse.**

As with many of the students, Fern’s discourse can be broken into two movements. The first movement was the first round of groups where Fern was a passive participant. She was active in her group by serving as the recorder for the group, but she did not contribute much to the group discourse. When asked about working with this group, Fern said “I felt like the people in my group didn’t really care.” In the second part of the unit, there was a very different group dynamic and Fern actively engaged in group discourse. This discourse either involved Yara scaffolding Fern’s understanding of limits or Fern arguing with their group member DeShawn.

**Passive participation.**

Fern’s initial discourse took the form of listening and writing for her group. She read the group’s response to questions and checked to make sure she represented the group’s ideas accurately. In addition, Fern occasionally interjected her thoughts to add to the group discourse, especially when she had questions or thought the group’s answer was unclear. The nature of the discourse provided opportunities for Fern to reflect and make sense of the concept of a limit and eventually construct her own personal meaning of a limit, positioning Fern in Q3.
Moving into a group with Yara influenced the trajectory of Fern’s path through Vygotsky Space. Before working with Yara, Fern had developed a tentative personal meaning for limits. Initially Fern and Yara were working collaboratively, but DeShawn challenged their ideas. Yara was positioned in Q4 and publicizing her personal meaning, while Fern was not ready to position herself in Q4. In an attempt to defend their collaboration, Fern let Yara scaffold her understanding of limits because she did not have confidence that her personal notion of a limit was valid. Instead, she used Yara’s conceptions to argue with her groupmate. This interplay between group members was a watershed for Fern. She forsook her personal meaning and adopted Yara’s understanding. Fern was not able to voice her personal meaning yet so she adopted Yara’s conception so she could argue with DeShawn. Because Fern was arguing this new thinking, she did not have an opportunity to reflect on this new conception or develop her own meaning for the definition. This set Fern back to Q1. The dynamics of the group became so acrimonious that the third group member was removed. Unfortunately for Fern, she had already accepted that Yara’s conception was the way to understand limits. This hindered Fern’s construction of knowledge, since she never reflected on Yara’s definition even when the group changed. This means Yara was unable to construct a personal meaning of limits using Yara’s conception.

Kyle

Kyle was a first semester freshman who took 4 years of high-school math including Algebra I and II, Geometry, and Trigonometry. His math self-concept questionnaire revealed he had a low belief in his mathematical ability/aptitude ($M = 2.50$), high math anxiety ($M = 2.50$), and average level of effort ($M = 3.38$) and intrinsic motivation ($M = 3.25$) compared to his peers.
He attended class regularly (34 out of 35 classes on limits or 97%). Kyle was outgoing and usually contributed to group discussions. Over the course of the semester, Kyle became more interactive and began to project an air of confidence in mathematical discussions, often directing discussion and disagreeing with his group. Table 14 provides a timeline of Kyle’s group members of the unit on limits. When talking about learning mathematics, Kyle explained that “It’s essential for you to understand why it’s used; not just to memorize it. I’ve learned that, like, just recently. Because before I always relied on my memory but if you really understand it and try to connect it with real-life then you can kind of make it out a lot easier.” Kyle’s realization may explain the contrast in his math self-concept at the beginning of the semester and what was observed during the semester. This new confidence was still tentative for Kyle; he admitted that if he did not understand something quickly he would just copy his groups’ work without really understanding what was going on. Kyle indicated that he “likes to keep challenging myself” and tries to push himself to “not memorize them [new mathematical concepts] but more like understand it.”

Table 14. Timeline of Kyle’s group members.

<table>
<thead>
<tr>
<th>Day 1-5</th>
<th>Day 6-12</th>
<th>Day 15-24</th>
<th>Day 25-26</th>
<th>Day 27-34</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kyle</td>
<td>Fern</td>
<td>Kyle</td>
<td>Kyle</td>
<td>Kyle</td>
</tr>
<tr>
<td>Raul</td>
<td>Kyle</td>
<td>Raul</td>
<td>Noel</td>
<td>Raul</td>
</tr>
<tr>
<td>Brad</td>
<td>Nell</td>
<td>Nell</td>
<td>Nell</td>
<td>Brad</td>
</tr>
</tbody>
</table>

Construction of knowledge.

On Day 1, Kyle served as the scribe for his group. Although Kyle gave some suggestions, he refrained from participating actively during the discussion and instead decided which of his group’s ideas would be selected to be turned in for the assignment. During the discussion, his group indicated that they thought of a limit as “a boundary.” The only prior notion Kyle shared
with the group was a definition of an asymptote as something that “gets close to but never touches.” These two ideas, the limit as a boundary and the phrase “gets close to but never touches,” together formed Kyle’s initial conception of a limit. This positioned Kyle in Q1 (see Figure 9), since the idea for the limit came from the group and not from Kyle.

Figure 9. Kyle’s path through Vygotsky Space.

This initial conception was cemented in Kyle’s mind on Day 2 when Prof. Morgan used both the idea of a boundary and unreachable imagery, along with describing a limit as “getting close to, but never touching” an object. These ideas were further solidified on Day 4, when Kyle began by defining a limit as “how close a function can get to the boundary.” His group did not adopt this definition, instead they continued an in-depth conversation about what it means for a limit to exist.

Raul: The limit of a function is a boundary.
Kyle: or a point.
Brad: How far the limit can go.
Kyle: The limit, how far.
Brad: How far a function goes before it is
Kyle: How far a value can go without crossing the boundary or something.
Brad: before it’s cut off by the boundary. Nah.
Kyle: Before it reaches the boundary.
Brad: Well it could be, a function is how far something goes and the limit is like a cut off, like it can’t pass it.
Kyle: Yeah.
Brad: So the limit of a function would be how far a function can go.
Raul: So how far a function can go.
Kyle: To the boundary, how close it can get to the boundary.
Brad: You can make it sound smarter, but that’s like the basic thing. Like how far it travels.
Raul: The limit of the function is the value of the graph.
Brad: Cannot cross.
Kyle: Cannot reach, cannot cross, cannot reach, cannot approach.
Brad: Nah, it approaches it.
Raul: Oh, the value.
Kyle: Cannot surpass.
Raul: In which a line or graph…
Kyle: Cannot surpass (talking over Raul)
Raul: can get extremely close to, but never touch it.

Later during this discussion Kyle explained that the limit is “the value as y approaches x.”

Through the course of the discussion Kyle presented the idea of a limit as a boundary, as well as in terms of proximity. This shows that Kyle is still positioned in Q1 (see Figure 9), since he was trying different language to find a notion of a limit that made sense to him. Kyle’s group did not include any of Kyle’s input and presented as their definition “a point on a graph where the function can get extremely close to but cannot reach or touch.” Kyle did not adopt his group’s definition when he individually defined a limit. Instead Kyle returned to the original idea of a
limit from the first day, defining “the limit of a function relates to how/where the function reaches with passing a certain boundary \((x \rightarrow n)\) whereas a function value focus on a certain point and its exact value \((f(n))\).” This definition positioned Kyle in Q2 (see Figure 9), as he tries to appropriate the notion of the limit as a boundary and construct personal meaning.

On Day 5, as previously discussed, the group critiqued another group’s definition and gave them feedback on how they could refine their definition. Thinking aloud, Kyle told his group that “approaches is the same things as the boundary, boundary is a thing you never touch because it approaches and stops right there.” This illustrates that Kyle was taking his notion of a limit, specifically as a boundary, and using it interchangeable with his classmates’ description of a limit without identifying there was a shift in his conception. This positioned Kyle in Q1 (see Figure 9), since he was using his classmates’ conception of a limit without thinking about its meaning or how that meaning related to his own conception of a limit.

On Day 9, students provided individual definitions of a limit again. Kyle defined the limit as “a set point or value on the y-axis in which x (the x-axis) approaches but, never touches it due to this boundary.” This definition was a more formalized version of the definition from Day 5; like the previous definition, it included Kyle’s notion of the limit as a boundary. This definition also included his group’s notion of approaching which Kyle integrated into his understanding on Day 9 when Prof. Morgan required more formal terminology for the assignment. This positioned Kyle in Q2 (see Figure 9), as he was appropriating his group’s definition without constructing meaning for this new addition. Instead it appeared that Kyle viewed a boundary and “approaching but never touching” as interchangeable.

On Day 18, Kyle began by describing a limit as “something you can’t touch” and later in the discussion he elaborated on this notion adding “a point where y-value can't be touched as the
x-value approaches it.” This definition shows Kyle had abandoned the idea of a limit as a boundary and adopted the image of the limit as unreachable. This may be because his group’s definition aligned with Kyle’s definition of an asymptote on Day 1, which felt more natural since it related to his prior knowledge and the fact that Kyle never realized the difference in his original conception of a limit and that of his group. Kyle was positioned in Q1 (see Figure 9), since he was still changing his definition, adopting that of the group. Kyle acted as the scribe for the group and wrote “a point of value in which x approaches but does not touch.” He wrote the definition for his group without any discussion or input from them. This positioned Kyle in Q2 (see Figure 9) only because the discourse transitioned from external to internal. Kyle still had not transformed the notion of a limit. The definition lost some of the specificity of the definition he had shared with his group which indicates that Kyle had not created personal meaning for this definition. He was mimicking his classmates since he did not realize he had changed the definition.

On Day 32, during the interview, Kyle defined a limit as “a number or a point it doesn’t touch. The way we find the limit, if the limit is 5 you can’t pass that. It’s a barrier because it never reaches it.” Kyle’s definition combined a couple of images: the limit as a barrier and as unreachable. It was not clear if Kyle had identified these as separate concepts and felt they both are important features of a limit, or if he viewed them interchangeable. He also demonstrated a misunderstanding of the relationship between a function and its limit. Kyle was positioned in Q1 (see Figure 9) because he still did not have a personal meaning for a limit and equated the limit and the function as the same thing.

*Table 15. Kyle’s construction of knowledge of limits.*

<table>
<thead>
<tr>
<th>Day</th>
<th>Student Conception of Limit</th>
<th>Origin</th>
<th>Codes</th>
<th>Quadrant</th>
</tr>
</thead>
</table>

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At the end of the unit, Kyle viewed the limit as a barrier that cannot be passed. This definition served Kyle well enough when working with a continuous function; however, when faced with a discontinuous function he was unable to reconcile the difference between a function and the limit. He knew that a function can be undefined, but struggled to connect what this meant for the limit. Kyle used the terminology of functions and limits interchangeable, particularly
“undefined” and “does not exist.” He did not realize that one describes a function and the other a limit or the difference between the two concepts. Moreover, he did not grasp the relationship between the two (i.e. a function can be undefined but its limit exists and vice versa). Kyle illustrated an understanding that limits and continuity are connected, but had not formalized the relationship. He assumed all situations involving limits were continuous functions. For example, Kyle was given a limit and asked to describe the function value. He assumed continuity to make sense of the problem. He was not able to realize that the information did not support this assumption or that the function could be undefined. The same occurred when given a function value and asked to describe the limit. He was unable to draw or describe a function with a limit that did not exist. Although Kyle struggled to think about limits abstractly, he was comfortable evaluating limits analytically. In addition, he struggled to connect his thinking about limits analytically with his thinking of limits graphically. Like many of his classmates, he was not able to distinguish between a function and the limit of a function when given the equation for the function. He wanted to simplify the equation without realizing that the simplification changed the representation of the function into something not equivalent to the original. Kyle thought of limits as a series of techniques, as evidenced by his concept map constructed the last day of limits which only gave methods. He did not include any characteristics of a limit or a definition for a limit. This was in part due to the fact that Kyle never developed his own personal meaning or definition for a limit. Without a definition of a limit, he was unable to connect the concepts he learned about limits, instead boiling limits down to a series of computations.

To summarize Kyle’s construction of conceptual knowledge of limits, he began the unit taking a very passive role in group discussion. Even though he was not actively engaged in the discussions, he did evaluate the group discourse and decided what narrative his group would turn
in to Prof. Morgan for the group’s written work. During this time, Kyle showed he was thinking about his notion of a limit compared to his classmates’ and trying to integrate these notions into one idea. Partway through the class, Kyle repositioned himself as a leader of his group. He actively participated in discourse including sharing his ideas of limits and directing his group on what to do during discussions. Kyle still served as the scribe for his group, but now he wrote his notions verbatim, representing his ideas as the groups’ with no reflection or revision of his ideas based on the group discussion. Kyle’s notion of a limit became stagnant. He established himself as an authority for his group without being receptive to his group’s dialogue. Consequently, Kyle never recognized that his notion of a limit was incomplete and never amended his conception.

**Nature of discourse.**

The discourse in Kyle’s groups never helped him to construct knowledge of limits. Over the course of the unit on limits there were no instances of collaboration in Kyle’s groups. Discourse in Kyle’s group took two forms, but neither supported Kyle’s construction of knowledge. For the first half of the unit, Kyle’s group tended to work individually, and the group then shared their answers with each other but if there was a discrepancy the group argued over the answer. No one scaffolded the other group members; they did not work through the problems together or analyze the process to make sense of the problem. Kyle was the recorder for his group, so he reflected on the group discussion and decided what to turn in for the group. There was so much disorder and disparity in the group, Kyle tried to integrate the group’s notions of limits with Prof. Morgan’s, but the group did not provide opportunity for Kyle to share and reflect on this conception.

Eventually Kyle had a new group. They also used independent group work but this time the group helped each other when there were differing opinions. Kyle was now actively engaged
in discourse and tried to scaffold his groupmates which helped to reinforce his procedural knowledge, but it did not help to develop his conceptual knowledge of limits. The format of independent group work never led to occasions for reflection on the definition of a limit, so Kyle was not able to transform his thinking on limits.

**Mariah**

Mariah was an upper classman who felt she struggles with math, especially Algebra. Mariah described learning in math classes as “they’ll give you a problem but they won’t really explain the definition or they won’t really go in to depth about anything.” She took 3 years of high-school math including Algebra I and II, and Geometry, and a semester of Pre-Calculus as a freshman. Her math self-concept questionnaire revealed she has a low belief in her mathematical ability/aptitude ($M = 2.50$) and intrinsic motivation ($M = 2.13$), average level of effort ($M = 4.25$), and very high math anxiety ($M = 1.25$) compared to her peers. In fact, Mariah had the lowest score of all of her classmates on the math self-concept survey in terms of lack of math anxiety, meaning she had the highest level of math anxiety in the class. Mariah regularly missed class (27 of the 35 classes on limits or 77%) but worked very hard to catch up on the material she missed. Table 16 provides a timeline of Mariah’s group members over the course of the unit on limits. Mariah was very active during group work, often asking questions to clarify concepts, and sought additional tutoring out of class.

*Table 16. Timeline of Mariah’s group members.*

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Yara</td>
<td>Mariah</td>
<td>Mariah</td>
<td>Javier</td>
<td>Mariah</td>
<td>Mariah</td>
<td>Mariah</td>
</tr>
<tr>
<td>Daniella</td>
<td>DeShawn</td>
<td>Brad</td>
<td>Anna</td>
<td>Brad</td>
<td>Yara</td>
<td></td>
</tr>
<tr>
<td>Mariah</td>
<td>Casey</td>
<td>Juanita</td>
<td>Mariah</td>
<td>Juanita</td>
<td>Sally</td>
<td>Juanita</td>
</tr>
<tr>
<td>Mariah</td>
<td>Noel</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Construction of knowledge.

Mariah was absent the first day of class when students worked to define their prior notions of a limit. As a result, the first glimpse into Mariah’s ideas about limits is on Day 4, after Prof. Morgan has introduced the concept of a limit to the class. Mariah began the unit working in a group with Daniella and Yara. The definition given by her group was Daniella’s definition that “the limit of a function is the value at which the graph of the function approaches, but never touches or crosses.” Mariah did not participate in the discussion of the definition of a limit. She waited to engage with her group until they were finding limits graphically. However, in the individual quiz, Mariah offered “the limit of a function is when; for example lim \( x \to 1 \) as \( x \) approaches 1 to any number, but never touches it.” Mariah’s definition reflected the ideas present in the group definition, namely that the limit “never touches.” She also tried to integrate the notation she learned in the previous day’s class, but the definition had not been transformed since she was not able to put the notion in her own words. Mariah was positioned in Q2 (see Figure 10) as she tried to use the group’s definition and the limit notation and combine them, but she had not yet transformed these ideas in her own words.
Mariah next revealed her definition of a limit on Day 9, during an individual quiz, as “the limit of a graph is when the y variable approaches x, and comes close to but never touches.” This definition continued the imagery present in the group definition from the previous week and incorporated the x, y-value terminology requested in the assignment. This terminology had been prevalent in her group’s discussion, but Mariah struggled to integrate these ideas into her definition as evidenced by her switching the x- and y-value in her definition. Since Mariah is still using the group’s ideas about limits without incorporate her own meaning, she was still positioned in Q2 (see Figure 10). Mariah’s hesitation to incorporate her own ideas, in part, was due to the fact that Mariah had little confidence in her own mathematical intuition, so relied on others’ knowledge to guide her thinking.

*Figure 10. Mariah’s path through Vygotsky Space.*
During group discussion on Day 15, Mariah explained to her group that a limit is “a function of $x$ as $x$ approaches, in this case it would be $c$ from the left side.” Mariah demonstrated she understood the limit in terms of proximity which illustrates she was beginning to transform the notion of a limit; however, the definition also showed a great deal of confusion with terminology. She attempted to use more rigorous terminology, but her use of the terms demonstrated that $x$-value had no personal meaning for Mariah. As a result of using the class’s terminology, Mariah was positioned in Q1 (see Figure 10). She had not been able to develop understanding using the collective language.

Three days later Mariah suggested to her group that a limit is “never close to but approaching a $y$-value, an $x$-value close to $y$-value.” This definition showed the relationship between the $x$- and $y$-values in a function, but it was not clear if Mariah has internalized this relationship. Javier had presented a definition of the limit as “the $y$-value” to begin this discussion which may have served to prime Mariah to indicate the correct relationship while defining the limit. Regardless of whether Mariah has a clear understand of the relationship of the $x$- and $y$-value in a function, she demonstrated that this definition was beginning to have personal meaning. She did not defer to Javier’s comprehension, as she had in previous discussions, but instead questioned his definition and inserted her own understanding which Javier then confirmed. This definition positioned Mariah in Q4 (see Figure 10) since she publicized her personal meaning, but it also positioned her in Q1 (see Figure 10) as Javier accepted her understanding and presented it in the group’s definition of a limit. This resulted in Mariah completing a cycle through Vygotsky Space.

On Day 32, in the interview, Mariah defined “a limit is when a $y$-value comes close to, but never touches the $x$-values.” This demonstrates she still had not resolved her issue with the
relationship between the \( x \)- and \( y \)-value in a function, but it also shows that this definition had become engrained in her thinking of limits. This positioned Mariah in Q1 (see Figure 10) because she continued to practice with her conception of limits, but she struggled to transform this definition and to identify and revise flaws in her conception. This confusion was still present in her definition on the last day of the unit of limits, specifically terminology confusion when she used \( y \)-variable instead of \( y \)-value. Mariah defined the “limit of a graph is when the \( y \)-variables approaches \( x \) and comes close to but never touches.” Although this definition is incomprehensible in a mathematical sense, it shows Mariah was explaining the limit in her own words. It also included the notions that were key in Mariah’s thinking of the limit, positioning her in Q2 (see Figure 10).

Table 17. Mariah’s construction of knowledge of limits.

<table>
<thead>
<tr>
<th>Day</th>
<th>Student Conception of Limit</th>
<th>Origin</th>
<th>Codes</th>
<th>Quadrant</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>“The limit of a function is the value at which the graph of the function approaches, but never touches or crosses.”</td>
<td>Daniella, written submission from group</td>
<td>Boundary</td>
<td>Q2</td>
</tr>
<tr>
<td>9</td>
<td>“The limit of a function is when; for example ( \lim_{x \to 1} ) as ( x ) approaches 1 to any number, but never touches it. The function value is touching and it is a certain number on the graph.”</td>
<td>Mariah, individual work</td>
<td>Unreachable</td>
<td>Q2</td>
</tr>
<tr>
<td>15</td>
<td>“The limit of a graph is when the ( y ) variable approaches ( x ), and comes close to but never touches. When it asks for the function of ( x ), the line touches the ( x ) value at ( y ).”</td>
<td>Mariah, individual work</td>
<td>Unreachable</td>
<td>Q2</td>
</tr>
<tr>
<td>18</td>
<td>“A function of ( x ) as ( x ) approaches, in this case it would be ( c ) from the left side.”</td>
<td>Mariah, group discussion</td>
<td>Proximity</td>
<td>Q1</td>
</tr>
<tr>
<td>32</td>
<td>“Never close to but approaching a ( y )-value, an ( x )-value close to ( y )-value.”</td>
<td>Mariah, group discussion</td>
<td>Unreachable</td>
<td>Q1</td>
</tr>
<tr>
<td></td>
<td>“A limit is when a ( y )-value comes close to, but never touches the ( x )-values.”</td>
<td>Mariah, individual work</td>
<td>Unreachable</td>
<td>Q1</td>
</tr>
</tbody>
</table>
Limit of a graph is when the y-variables approaches x and comes close to but never touches. When it asks for the function of x, the line touches the x value at y.

At the end of the unit Mariah was tentative about her mathematical thinking and second guessed her thought process. Mariah often hedged her answers to problems and regularly changed her responses, undermining her mathematical intuition. She viewed mathematics as a process you memorize, “like, the calculus isn’t the hard part because that’s just memorizing the functions, I mean the formulas, but it’s like with signs and multiplying and all that.” Mariah made a clear distinction between the limit of a function and a function value both graphically and analytically. Mariah had not initially shown she had explored thinking abstractly about limits, but with probing she was able to describe a function given a limit, and a limit given a function. She demonstrated a tentative understanding of continuity, although she was not able to formalize her thinking. She could also identify that two functions were not equal when given their equations, even though their limits were equal. Mariah’s definition, although problematic in terms of her understanding of the relationship between the x- and y-value in a function, allowed her to cultivate a generalized definition that can be applied to multiple situations. However, she struggled to evaluate limits analytically due to a weak foundation in algebra.

To summarize Mariah’s construction of conceptual knowledge of limits, Mariah was not able to engage in discourse with the class as much as her peers because she missed class regularly. When Mariah was present, she rarely publicized her understanding of a limit, but she was not passive in group discussions. Instead, she listened to her peers and asked questions about her classmates’ conceptions, especially on topics which were introduced in classes when she was absent. To understand the material she missed, Mariah sought out external authorities to help her.
learn a concept. Specifically, she used online lessons, such as the lessons on Kahn Academy, and worked with Prof. Morgan to understand material. Mariah was able to construct her own notion of a limit through listening in class, questioning peers in group discussions, learning independently, and individual discourse with Prof. Morgan. Although Mariah’s conception of a limit echoed ideas found in her peers’ conceptions, her definition was nuanced and robust enough to be generalizable.

**Nature of discourse.**

Initially, Mariah attempted to collaborate with her group members. However, Mariah’s absences from class quickly affected her ability to engage in discourse. Her group constructed a ZPD through collaboration, but Mariah was not able to take part in the group discussion because she had already fallen behind in developing her understanding of limits. Early on, Mariah asked for help from her group, but they did not respond to her or offer any explanation and she did not ask for help again. Gradually, Mariah gained confidence to ask her group questions which assisted in making sense of the concepts being discussed in her group. It was through these questions Mariah was able to construct a ZPD as scaffolding and develop her notion of a limit.

Mariah’s group often employed independent group work, sharing their answers afterwards. Mariah used these exchanges as a chance to ask her group members questions to help her understand the concept being discussed. These questions would not just be on procedure, but also questions about how and why an approach worked. When her groupmates answered Mariah’s questions, she reflected on their approach, inserted her thinking, and connect them to the ideas she had learned independently. This discourse helped Mariah to position in Q2. Mariah’s absences played a large part in her construction of knowledge because she was not present in class to engage in discourses so could not integrate her peers’ conceptions into her
notion of a limit. It necessitated her seeking out discourse and building personal meaning outside of class. After building personal meaning, Mariah continued to use the same discourse practice in her group. It allowed Mariah to publish her personal meaning and have her group accept her conception of a limit. Mariah was able to self-scaffold to complete a cycle through Vygotsky Space.

**Cross-Case Analysis**

In this section I report the findings from cross-case analysis of all the embedded cases. The analysis focused on the relationship between students’ individual and collective knowledge to address the third research question (What is the relationship between the collective and individual students’ mathematical knowledge?). The analysis revealed three salient themes for considering individual and collective knowledge. The first pertains to students’ conceptions of a limit and how their conceptions related to the collective knowledge established by Prof. Morgan and the collective knowledge constructed during group discourse. The second theme is a comparison of students’ pathways for constructing knowledge and how their interactions with their groups influenced that pathway through Vygotsky Space. The third theme encompasses the nature of discourse involved in the construction of individual and collective knowledge and the interplay between the two.

**Conception of limits.**

During Prof. Morgan’s introduction to limits she described a limit using imagery of the limit as a value on the function that does not have to be reachable (i.e. unreachable), as well as the limit as a boundary for $x$. All six of the students adopted one of Prof. Morgan’s images in their final definition of a limit although none of them adopted the “door” metaphor; four of the students (Javier, Yara, Fern and Mariah) described a limit as unreachable, while two students
(Daniella and Kyle) described it as a boundary. Not only was her imagery used by all the students, but they also echoed her wording. Prof. Morgan had described a limit specifically as “getting as close as you possibly can without touching.” All the students who used the unreachable imagery used the phrase “never touch,” while the students who described it as a boundary used the phrases “never crosses,” and “never reaches it” in their definitions of a limit.

It is clear that the collective knowledge established by Prof. Morgan influenced all of the students in their conception of a limit. It is interesting that none of the students used Prof. Morgan’s final definition of a limit (see Table 18), but instead they latched onto the imagery (unreachable or boundary) and language (close as you can without touching) she used to describe a limit.

Table 18. Students’ final definition of limit.

<table>
<thead>
<tr>
<th>Individual</th>
<th>Final Definition of a Limit</th>
<th>Conception</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prof. Morgan</td>
<td>“limit in math – gets close to value in question but not concerned with what’s happening at that value.”</td>
<td>Unreachable; Boundary</td>
</tr>
<tr>
<td>Javier</td>
<td>“A limit is the value of a function that approaches a point but will never touch it.”</td>
<td>Unreachable</td>
</tr>
<tr>
<td>Daniella</td>
<td>“The limit of a function is the y value at which that function approaches but never crosses for the given x-value.”</td>
<td>Boundary</td>
</tr>
<tr>
<td>Yara</td>
<td>“The limit of the function equals the y-value when x approaches but never touches the point.”</td>
<td>Unreachable</td>
</tr>
<tr>
<td>Fern</td>
<td>“A limit is the x or y-intercept approaching a line but never touching.”</td>
<td>Proximity</td>
</tr>
<tr>
<td>Kyle</td>
<td>“A number or a point it doesn’t touch. The way we find the limit, if the limit is 5 you can’t pass that. It’s a barrier because it never reaches it.”</td>
<td>Boundary</td>
</tr>
</tbody>
</table>
Mariah

“Limit of a graph is when the y-variables approaches x and comes close to but never touches. When it asks for the function of x, the line touches the x value at y.”

Unreachable

It is important to note that some of the students never worked together in a group (e.g., Javier and Fern, Daniella and Kyle). This means that not all of the students were able to share their conceptions with each other nor had a chance to construct a collective knowledge between them. Daniella never worked with Kyle nor did they work with any of the others’ groupmates, so the fact that they both developed a conception of a limit as a boundary did not come from a collective knowledge between them since they never had the opportunity to discuss it with each other. In fact, Daniella worked with Javier, Mariah, and Yara throughout the unit, all of whom described a limit as unreachable. This suggests that Prof. Morgan established a collective knowledge which students drew from while constructing their individual conceptions of a limit. In the case of Daniella and Kyle, each student adopted the collective knowledge established by Prof. Morgan which aligned with their initial conception of a limit. For Daniella, discussions with her group did not hinder the construction or change her personal meaning for a limit. Moreover, she was able to take this personal meaning and share it with her group without changing the groups’ collective meaning for a limit. In contrast, Kyle struggled to come to terms with the notion of a limit as a boundary, which aligned with his initial thinking of limits, and the groups’ collective understanding of a limit as unreachable. Kyle wavered between both notions of a limit and never constructed personal meaning for a limit.

For three of the students (Javier, Fern, and Mariah) the construction of the group’s collective knowledge played a part in developing their conception of a limit. In the case of Mariah and Javier, the group discussions helped both students to identify key elements of a limit.
and incorporate these into their conceptions, specifically that the limit is the $y$-value of the function as its graph approaches an $x$-value. Both students echoed this idea and phrasing which came up in their groups’ discussion, demonstrating that the group collective knowledge informed their conception of a limit. In Fern’s case, she mimicked Yara’s conception of a limit. Prior to working with Yara, Fern had begun to conceptualize of the limit in terms of proximity. This was her own notion of a limit, but Fern did not believe in her conception of a limit, possibly because it was not reflected in the collective knowledge on limits. Yara publicized her conception to the groups’ collective knowledge and Fern adopted this conception for herself. Unlike Javier and Mariah who used the group discussions to solidify their thinking about limits, Fern forsook her personal meaning for the collective knowledge.

Looking beyond the students’ definitions of a limit, four of the students (Javier, Daniella, Yara and Mariah) constructed a conception of a limit which made the distinction between a limit and a function value and identified the relationship between a limit and continuity. Three of the four students (Javier, Daniella, and Yara) developed strong procedural understanding. The combination of conceptual and procedural understanding assisted the students in developing a deeper understanding of limit. However, only Yara was able to make a connection between her conceptual and procedural knowledge allowing her to apply her conceptual thinking to procedure. Both Javier and Daniella had disjoint conceptual and procedural knowledge, which meant that conceptually they could identify differences and similarities between two functions and their limits, but when given the equations of two functions, they reduced the functions and limits to a set of calculations, instead of applying their conceptual understanding to think about the functions. Javier had developed a generalizable definition of a limit that could be applied in this type of situation, he just was not ready to make that connection. Daniella, on the other hand,
had developed a disjoint conception of a limit which prohibited her from being able to apply her conception to her procedural knowledge. Mariah did not have a strong procedural understanding of limits, and as a result she struggled through much of the analytical portion of the unit on limits. However, she was able to connect her conceptual and procedural knowledge. Since Mariah had a generalizable conception of a limit and lacked prior knowledge to have a strong foundation for evaluating limits analytically, she applied her strong conceptual understanding of a limit to help develop her procedural understanding.

Both Fern and Kyle struggled to develop conceptual understanding, but developed a procedural understanding of limits. Kyle and Fern never made a distinction between a function value and the limit of the function at that value. This also obstructed their ability to develop an understanding of continuity, how continuity connects to the notion of a limit, or connect their conceptual and procedural understanding of limits. Both students felt confident in their numerical skills which led them to focus on procedural understanding. Kyle demonstrated that he viewed limits as a series of procedures. Constructing these categories allowed him to find strategies to use, but this same compartmentalization scheme impeded his thinking about limits. Kyle viewed thinking about limits graphically its own procedure for finding limits, not a tool to help develop conceptual understanding. Fern, on the other hand, seemed to struggle with the notion of a function which hindered her ability to understand a limit and the relationship between a function and its limit.

Fern, Kyle and Mariah all struggled over the unit to understand limits. However, Mariah was able to develop a nuanced and robust definition of a limit, while Fern and Kyle were not. This is partially due to the fact that Fern and Kyle had a strong algebraic foundation so were able to evaluate limits analytically with little difficulty. Neither student viewed themselves as strong
math students, so this success with procedure led them to focus on procedural understanding. Mariah struggled algebraically, focusing on conceptual understanding allowed her to develop an understanding without using algebra.

All of the students adopted the collective knowledge of limits presented by Prof. Morgan, but the students constructed individual meaning for this knowledge which aligned with their initial concepts of a limit. This is why there were students who used unreachable imagery and others that used boundary imagery. This internalization explains some of the differences seen across students’ conceptions of a limit. Construction of collective knowledge in small groups influenced students’ notions of limits, as did their prior knowledge and personal beliefs about their mathematical ability. It is the interplay between these collective and individual influences which led to each students’ conception of a limit.

**Construction of knowledge.**

Three of the students (Daniella, Yara and Mariah) were able to complete a cycle through Vygotsky Space (see Figure 11). Javier had progressed to Q4, but did not publicize his understanding. Neither Fern nor Kyle completed a cycle and both ended in Q2, still trying to transform the notion of a limit and construct personal meaning.
From the students’ pathway through Vygotsky Space, Q3 appears to be a tentative position in their path. Of the five students who were positioned in Q3, three of the students travelled back to Q1 or Q2 without completing a cycle (see Figure 10). For Daniella and Javier, this was only temporary as they then positioned back to Q3 and moved on to Q4. For both
students, they had transformed the definition of a limit, but were not ready to publicize that thinking. Javier attempted to publicize his understanding, but oversimplified the definition of a limit and his group rejected Javier’s notion of a limit. Eventually Javier reconceived his earlier conception of a limit, positioning himself back in Q4, but did not publicize his understanding to add it to the collective knowledge. Daniella, conversely, was not able to use her own language with mathematical terminology. Initially Daniella’s conception of a limit was worded using colloquial terminology. She reverted to the collective language about limits until she was able to integrate mathematical language, repositioning herself in Q3 and then sharing her personal meaning for a limit with the group which was conventionalized by the group. In the case of Fern, she transformed the definition of a limit but it was not reflective of the group understanding. Not only was she not ready to share her thinking about a limit, she also mistrusted this understanding. She never became ready to publicize her thinking and instead rejected her personal meaning and adopted the collective understanding, thus positioning her in Q1 and requiring her to restart the cycle to transform this new notion of limits. She was not able to construct her own meaning, but that may have been due to a lack of sufficient time to make meaning for this new conception.

Unlike the other case studies, Kyle never transformed the collective understanding of limits. Kyle struggled between his initial conception of a limit and the collective understanding of a limit because these two notions were not the same. When this occurred for Daniella she ignored the collective understanding and Fern, conversely, transformed and rejected her personal meaning. Kyle did not take either of these options, instead he fluctuated between the two notions, trying them both without adopting either. To successfully construct their own conceptions of a limit students either decided to adopt the collective understanding, as did Javier, Yara, and Mariah, and then transform and construct personal meaning or reject the collective
understanding, as did Daniella, and construct their own understanding. Kyle failed to resolve the struggle between his individual knowledge versus the collective knowledge, placing him in perpetually in limbo between Q1 and Q2.

**Nature of discourse.**

Throughout the unit on limits, there were many instances of students constructing ZPD either through scaffolding or collaboration. This was in large part due to the nature of the tasks Prof. Morgan designed. Students were given problems for which they had all the background understanding needed and were asked to apply their knowledge to figure out a new topic. Although all six students voiced frustration with this design in their interviews, they also cited the structure of the course as one of the things that helped them learn. They also all acknowledged they were always able to figure out how to approach a problem, so the tasks were not outside of their understanding.

Scaffolding was an important type of discourse for the construction of knowledge. All the students experienced this type of discourse in the form of Prof. Morgan’s lecture on limits. She took the student’s initial conceptions and scaffolded their notion of a limit to a mathematical conception. Because most of the class activities involved group work, almost all other instances of scaffolding were between students. In the case of Fern, this type of discourse occurred when she worked with Yara. Yara’s scaffolding led to Fern reinitializing her conception of a limit and once again being positioned in Q1. Scaffolding served as an entry point for individuals into the collective knowledge on limits and positioning in Q1. Most instances of scaffolding arose when students were focusing on procedural understanding. The development of procedural understanding occurred separately from their conceptual understanding and most students were not able to connect these two types of understanding of limits. Being scaffolded by another
person was instrumental in positioning students in Q1 but did not seem to support further construction of conceptual knowledge. The support of procedural knowledge through scaffolds did not help students in their construction of conceptual understanding.

However, scaffolding another student served as an opportunity for students to position themselves in Q4. The nature of scaffolding another person requires an individual to have constructed meaning for a concept which meant they were positioned in Q3. The act of sharing their understanding to help their classmates presented students with the opportunity to transition to Q4. Having a chance to scaffold another student’s thinking was instrumental for Yara and Daniella to position themselves in Q4 because it gave them opportunities to publicize their personal meanings. If that understanding was adopted by the group, then the student’s personal meaning became part of the collective understanding and they completed a cycle through Vygotsky Space. Scaffolding another person only offered the opportunity to complete a cycle through Vygotsky Space; as seen with Javier, it did not guarantee that students would push their construction of knowledge forward or transition to the next quadrant in Vygotsky Space. There was a reciprocal relationship occurring between scaffolding and the individual and collective knowledge. The scaffold served to introduce the individual to the collective knowledge and when the student was ready scaffolding another served as a way for them to enter their individual knowledge into the collective understanding.

Collaboration, unlike scaffolding, tended to happen when students were discussing conceptual understanding. This type of discourse served as a chance for students to make sense of new concepts or refine their current conceptions. Collaboration helped students construct knowledge through each quadrant of Vygotsky Space. For example, Javier collaborated with his group the first day of the unit on limits to establish their initial conceptions of a limit. This
interaction positioned students in Q1 and primed them for Prof. Morgan’s scaffolding the next day. Javier’s collaboration with Mariah to define a limit midway through the unit was instrumental in solidifying his personal definition of a limit and helped him to position in Q4. Unlike scaffolding, during collaboration both students were given opportunities to internalize meaning, publicize understanding, and experience new concepts. Collaboration involved students using their individual knowledge to construct a collective knowledge and then in turn, individually internalizing the constructed knowledge.

A key factor which led to discourse facilitating construction of knowledge was opportunities for reflection. Javier, Mariah and Yara all engaged in reflection and were able to construct a generalizable definition for a limit. In the case of Javier and Mariah, they both were able to reflect on their individual knowledge during discourse with each other. It was Mariah’s questioning of Javier’s definition which led to reflection. Javier needed to reflect on the definition; thus, Javier identified problems with his simplified definition of a limit and revised the definition he had publicized. The discussion on how to revise Javier’s definition led both Javier and Mariah to think about what needed to be included in the definition and the essential features of a limit. Mariah used the reflection to validate her personal meaning for a limit and to be ready to publicize her understanding. For Yara, reflection did not come through direct interaction. Instead, she used her passive participation to listen and reflect on her group’s collective knowledge. When one group member became combative during these discussions, this served as a catalyst for her to reflect on her own individual knowledge and solidify her personal meaning for a limit. The act of reflecting and revising their definitions helped all three students construct a generalized definition of a limit.
Fern, Kyle, and Daniella never reflected on the collective or their individual knowledge. For Fern and Kyle, this failure to reflect was detrimental to their construction of personal meaning. Fern had begun to construct a personal meaning, but she never identified that she had her own conception and that this conception was valid. Because she never self-validated her own personal meaning she was willing to discard that meaning. She also adopted Yara’s conception without any reflection because she viewed Yara as an authority. As previously described, Kyle struggled to come to terms with the conception of a limit as unreachable versus as a boundary. He never reflected on the two conceptions so he never came to a resolution with this conflict which hindered his adoption of either notion of a limit. Daniella was successful in constructing a conception of a limit, but her definition was derived by adding elements to her definition without reflecting on whether there were contradictions or redundancies in her conception. Her view of a limit could become problematic in the future because she never constructed a generalized definition of a limit. Although scaffolding, collaboration and debating were used by students to reflect on their individual and the collective knowledge, these interactions were not sufficient to guarantee reflection. Daniella regularly engaged in scaffolding and collaboration and Kyle was active in debating his groupmates and yet neither of the students reflected on their knowledge.

**Summary**

To explore how students’ construct mathematical knowledge through discourse, case studies for six students were discussed, tracking their construction of knowledge of limits through Vygotsky Space. Javier, Daniella, Yara, and Mariah constructed personal meaning for a limit and completed a cycle through Vygotsky Space. However, only Javier, Yara, and Mariah constructed a mathematically generalizable conception of a limit. Javier, Yara, and Mariah all reflected on their thinking about limits. On the other hand, Daniella continually added to her
definition; never refining or reflecting on her conception but rather holding on to lots of different pieces that weren’t necessarily compatible. Both Fern and Kyle were unable to complete a cycle, but for very different reasons. Fern constructed personal meaning, positioned in Q3, but discarded her conception to adopt the collective language resetting her journey through Vygotsky Space to Q1. Kyle was never able to construct personal meaning, therefore never moving beyond Q2. From the student’s experience constructing knowledge, not only is Q1 where students first experience a new concept, but when their initial conception aligned with the collective knowledge shared in Q1 students were able to construct personal meaning. However, students’ positioning in Q3 is tentative. Many of the students regressed in Vygotsky Space before moving forward in their pathway.

Looking at the discourse which supported the development of mathematical knowledge a few trends were discovered. Being scaffolded by another person was instrumental in positioning students in Q1 but did not seem to support further construction of conceptual knowledge. The support of procedural knowledge through scaffolds did not help students in their construction of conceptual understanding. Collaboration, unlike scaffolding, tended to happen when students were discussing conceptual understanding. This type of discourse served as a chance for students to make sense of new concepts or refine their current conceptions. Collaboration helped students construct knowledge through each quadrant of Vygotsky Space. This type of discourse was very productive for the students who engaged in it. Reflection was a key factor which led to discourse facilitating construction of knowledge. Although students used scaffolding, collaboration, and debating to reflect on their individual and the collective knowledge, these interactions were not sufficient to guarantee reflection.
Finally, cross-case analysis was used to explore the relationship between the collective and individual students’ mathematical knowledge. Prof. Morgan designed instruction so that students identified pre-conceived notions and used those as a foundation for constructing knowledge of limits. Students adopted the language and imagery of the instructor when it aligned with their initial conceptions. Students’ construction of knowledge was influenced by their perceptions of each other’s understanding as well as their math self-concept. When tension arose between collective (Prof. Morgan’s and/or the groupmates) and individual knowledge, it acted as a barrier for the construction of personal meaning. In the next section, these findings will be discussed along with implications for instruction and further research.

Chapter 5 - Discussion

The goal of this study was to answer three questions relating to the interplay between students’ individual and collective knowledge constructed through discourse. This section will synthesize the results from the previous chapter to address each research question: how students construct mathematical knowledge through discourse, the nature of discourse which supports the development of mathematical knowledge, and the relationship between the collective and individual students’ mathematical knowledge. Incorporated into this discussion is related research and instructional implications to emphasize how this research ties into the current educational knowledge. The section closes by outlining limitations of the research, implications of the study, and future research.

Individual Students’ Construction of Knowledge through Discourse

The Harré model of Vygotsky Space (Harré, 1983) provided a framework for exploring students’ construction of knowledge. In observing students’ pathways through Vygotsky Space there were two positions in Vygotsky Space that were instrumental for students’ construction of
knowledge: Q1 (public/social) and Q3 (private/individual). Q1 is where students first experience a concept. Students’ encounter with the concept in this space serves as the foundation for the trajectory of their paths to construct knowledge. Q3 not only occurs when students transform a concept and construct personal meaning, but is also a tentative space as students attempt to transition to Q4.

**Quadrant 1: Initial conceptions are sticky.**

Students’ construction of mathematical knowledge through discourse began in Q1. All of the students began by publicizing their prior experiences and notions with each other, using collective language to construct a space that was external to the students. This is where students first experienced a new collective concept. The curriculum was designed so that students would work together to construct a collective understanding of limits. The instructor validated students’ initial conceptions by using these conceptions as the foundation to introduce a definition for a limit. In addition, Prof. Morgan’s definition provided students with access to the conventional language of the mathematical community. This resulted in the construction of a collective language on limits built from the students’ and instructor’s knowledge. From Q1 all the students transitioned to Q2, the collective and individual space. This space is where students used the collective language constructed in class while working individually.

Although most students progressed past Q2, Kyle was never able to transition to Q3. This failure to transform the collective understanding and make personal meaning of this knowledge stems from Kyle’s inability to resolve the conflict between his initial conception of a limit with his group’s collective understanding of a limit. To progress to Q3, some students stuck to their initial conceptions and used them to make sense of a limit mathematically while others rejected their individual understanding and worked to make meaning of the collective conception. Kyle
was not able to internally resolve this dissonance. Unlike other students who had an external force that led them to adopt the collective knowledge to construct meaning, Kyle had no external force and did not relinquish his initial conception.

This suggests that initial conceptions are sticky. Most students clung to their initial conceptions of limits. When a student’s initial conception aligned with the collective language the student was able to develop personal meaning. Javier, Yara, Mariah, and Danielle constructed personal meaning by using their initial conceptions as the basis for their mathematical definition of a limit. Development of personal meaning did not guarantee the students would develop a mathematically rigorous definition, as described with Daniella, but all of the students constructed a definition which allowed them to make sense of limits in the context of Calculus I. However, Fern and Kyle’s initial conception did not align with the collective knowledge of their classmates. They were unable to translate their initial conception of a limit into a mathematical notion consequently failing to make meaning of the collective conception. Thus, their intuitions about limits and inability to reshape their initial conceptions served as a barrier for their learning.

Previous research (Tall, 1993; Cornu, 1981; Williams, 1991; Oehrtman, 2009) has supported the finding that students cling to their initial conceptions of a limit. Students rely on what Cornu (1981) calls spontaneous conceptions; however, this name suggests that the conception occurs without apparent external cause or from impulse. Students’ conceptions are not spontaneous, even though they may appear that way from a mathematical standpoint. Mathematics borrows terminology from the English language; even though students mathematically speaking may have no prior conceptions of a limit, students already have ideas, intuitions, and images of limits which come from their experience and use of the word in the
non-mathematical sense of the word. These students’ conceptions stem from students’ prolonged use and experience with the word limit, hence initial conception. Because students’ initial conceptions come from deeply ingrained thoughts about a limit, it makes sense that these initial conceptions are sticky. Research finds that students do not change their initial conceptions of a limit even after experiencing instruction on the formal mathematical definitions (Tall, 1993; Cornu, 1981; Williams, 1991; Williams, 2001; Oehrtman, 2009; Glüçer, 2013). Even when students encounter examples of limits which contradict their conception of a limit (Bezoidenhout, 2001; Williams, 1991) or experience direct instruction which highlights misconceptions in student conceptions (Williams, 2001) they may not revise their conception of a limit. This is a contrast to what was observed in this study, where students. Students used the imagery and language of their instructor to grow their initial conception of a limit.

**Quadrant 3: Transitioning from Q3 to Q4 is tentative.**

The process of transformation is essential for a student’s individual construction of knowledge (Vygotsky, 1978; 1986). For students to create meaning, they need to move beyond parroting back the collective language and transform it by putting a concept in their own words (Lemke, 1990). It is through transformation that students construct their own knowledge and make sense of a concept; without transformation, students do not truly learn something for themselves. Previous research illustrates the importance of Q3, since it is in this space the individual’s ideas and practices become unique (Gallucci et al., 2010).

All but one student positioned themselves in Q3, constructing personal meaning for a limit. Previous research with the Harré model of Vygotsky Space (Harré, 1983) focused on the impact individual transformation had on the collective knowledge when shared in Q4 (Gavelek & Raphael, 1996; Gallucci, 2010). This study reveals Q3 is a tentative space in students’
construction of knowledge. Of the five students who transitioned to Q3, two of them subsequently transitioned backwards in Vygotsky Space. In the case of Javier, he had a clear personal meaning but oversimplified his definition to make it understandable to his group. However, he lacked the mathematical sophistication to understand what features of a limit are essential. While Fern transitioned to Q3 and then regressed in her thinking, unlike Javier, she was never able to transition back to Q3. Fern’s personal meaning was a fledgling idea which had not fully formed and did not match the collective notion of a limit. She lacked confidence in her conception and when her individual understanding did not match that of her classmates, she rejected her own meaning and adopted the collective understanding positioning her back in Q1.

Yara and Daniella also demonstrated tentativeness to publicize their personal meanings. For Yara, this hesitation stemmed from her lack of language to share her personal meaning. Once she felt confident she could express her thinking, she publicized her conception. Daniella felt hampered by the belief that she was the authority for the group. She did not want to publicize her meaning if it was not correct, so waited to publicize her thinking. While she waited, she was asked to integrate mathematical language which led to a regression in Vygotsky Space while she figured out how to include collective language, specifically mathematical terms, into her personal meaning. Once she integrated the two, she felt confident to publicize her conception.

Javier, Fern, Yara, and Daniella illustrate the tentativeness of students’ positions in Q3. They had the tools to construct personal meaning, but it was not a small step for them to publicize this meaning to their peers. One implication is that students can easily be discouraged at this point when they have a fledgling understanding. Classmates often do not have the expertise to see nuance in personal meaning and may quash emerging conceptions if they deviate from the collective language. It may be helpful for students at this point to write their thinking.
and discuss with a instructor who can help students to develop the language they need or the confidence needed to share their thinking with the class. Prof. Morgan built an environment where students were comfortable with discourse and sharing ideas, but even with this type of environment Fern and Yara struggled to leave Q3. Yara had confidence in her mathematical thinking and eventually developed the language she needed to share her conception, but Fern was very unsure of herself mathematically and quickly discarded her understanding because it did not match the collective knowledge. If Fern’s conception of a limit had been nurtured, she may have been able to form it in to a rigorous definition but Fern needed more support to be able to make this transition. Abandoning her ideas and adopting the collective understanding made Fern feel like she had support through her classmates to make sense of the mathematics. Fern did not have the time needed to transition back to Q3 with the new conception of a limit.

This finding highlights the need for evaluation which provides students the opportunity to publicize their thinking so they can receive feedback and suggestions in a supportive context. Gavelek and Raphael (1996) describe this as the “fragility of the evaluative context.” As they explain, the form and willingness of students to publicize their thinking is influenced by the nature of feedback they receive in the classroom. Prof. Morgan’s class had an environment which supported discourse but students were still nervous about the feedback they would receive from classmates. The students would have benefited from opportunities to receive feedback from their instructor in a low stakes environment, not just from their peers. It is also important to provide students an environment where it is safe for them to make mistakes while publicize their thinking, as Prof. Morgan did, along with chances for the instructor to provide suggestions to help students improve, develop, or revise their personal conceptions.
The Nature of Discourse that Supports the Construction of Knowledge

Three types of discourse that supported the construction of knowledge were observed: independent group work, passive participation, and construction of ZPD. Independent group work occurred when students worked independently and then used group discourse to validate their ideas. This type of discourse was marked by long stretches of independent work and discussion of solutions. Engaging in independent group work did not guarantee construction of knowledge, but could be used by students as a way to reflect on their thinking. Javier engaged in this type of discourse to support his construction of knowledge about limits. He would often begin discussions by sharing his ideas looking for his group to either confirm or question his understanding. When needed, he would revise his ideas and then confirm this new conception with the group. However, most students did not use this type of discourse to construct meaning for the mathematics.

The next type, passive participation, occurred when students observed group discourse providing minimal input to group discussion but actively listened to the group dialogue. Both Yara and Fern were able to use this structure to listen and evaluate the group’s collective understanding, and then integrate or revise ideas from the collective understanding with their individual knowledge. Students who passively participated were critical of the group dialogue, rejecting ideas that did not align with their individual conceptions, and occasionally interjecting comments to focus the group or add to the discussion. As with independent group work, not all students used their observation of group discourse to construct understanding.

The final type of discourse was that which supported the construction of ZPD either as collaboration or scaffolding. Discourse which resulted in ZPD as collaboration tended to occur early in the unit when all students were positioned in Q1 or when students were working on
conceptual problems. Discourse which resulted in ZPD as scaffolding occurred when one student had transitioned to Q3 and was publicizing his or her knowledge. Discussion of this discourse is explored in more depth in the next section when looking at the relationship between students’ individual and collective knowledge.

During this study a new category of ZPD was observed with one group, which I call collaborative scaffolding. This type of ZPD involved students working collaboratively to scaffold a peer. It is the intersection of the ZPD as scaffolding and collaboration (Goos, 2004). Something that was unique to collaborative scaffolding was that students engaging in this type of discourse all identified it as a productive and helpful partnership whether they were scaffolding or being scaffolded. The students who viewed themselves as being the authority for their group felt burdened by this role when they were solely responsible for scaffolding their groupmates (i.e. when engaging in scaffolding that was not collaborative). They generally felt they had no support and it slowed down their learning. However, Daniella and Javier both viewed the act of collaborating to scaffold Anna as helpful for their own construction of knowledge. It reduced the burden of being the authority while allowing them to collaborate with each other. Students of comparable knowledge can help each other push forward their understanding thus creating a collaborative ZPD (Forman & McPhail, 1993; Goos 2004). This finding has implications for instruction. There is a large body of research on ability grouping which supports the value of cooperative learning (e.g. Slavin, 1990; Lou et al., 1996; Zakaria, Chin, & Daud, 2010). The present study adds to this body of knowledge by illustrating how cooperative learning groups comprised of more than one high-ability student with low-ability students promotes ZPD as collaborative scaffolding. This provides opportunities for both collaboration and scaffolding and supports students who adopt an authority role, relieving the burden of being the sole authority.
Regardless of which type of discourse students engaged in, reflection was central to not just the construction of personal meaning but also to the development of a generalizable definition of a limit. Yara, Javier, and Mariah all used class discourse as a catalyst for reflection on their understanding of a limit. Yara reflected on her understanding as a passive participant, meaning the impetus to reflect on her thinking came from Yara. Javier and Mariah had a spontaneous moment during group discussion when Mariah questioned Javier’s conception of a limit. Both students used this moment to reflect on the definition of a limit and deepened their understanding of the concept. This finding is supported by Cobb et al. (1997) who found participation in reflective discourse, the practice of jointly taking what was previously done and making it an object of reflection, creates opportunities for conceptual development.

Some students never reflected on their understanding of a limit. Daniella’s group provided opportunities for reflection when they asked questions about her conception of a limit but Daniella never took advantage of them. She never reflected on her understanding; instead she added exceptions to her initial conception so that she had a series of ideas that together allowed her to think about and evaluate limits. Although Prof. Morgan provided opportunities for reflection, especially when she had students provide feedback on their classmates’ definitions of a limit and then revise their definition based on this feedback. Students were resistant to revising their definitions or to giving credence to their classmates’ feedback. They mistrusted their classmates’ ability to provide constructive and mathematically-accurate feedback. Students’ perceptions of each other were very important for their construction of knowledge (discussed further in the next section). This illustrates the importance of moments for reflection and revision of ideas, but also how difficult it can be to create these opportunities; in this study moments of reflection as a result of dialogic discourse only occurred spontaneously.
The Relationship between Collective and Individual Students’ Knowledge

There is a complex relationship between individual and collective knowledge. The two interweave to inform each other (Vygotsky, 1978; 1986; Goos, 2004; Harré, 1983) but this relationship is not reciprocally experienced or equivalently internalized by each student. It was observed that collective knowledge was constructed from students’ individual initial conceptions and then revised by Prof. Morgan. Her additions to the collective knowledge served two purposes: (1) introducing students to the conventions of the mathematical community (in terms of language and notation) and (2) provide repetition with variation (Lemke, 1990). By providing repetition with variation in her lecture, Prof. Morgan exposed students to different ways to express a concept to ensure students do not become reliant on fixed wording. From here students began to make sense of the collective knowledge, transformed the conception, and then publicized their notions thus informing the collective knowledge. The construction of knowledge followed the cyclical feedback loop that the Vygotsky Space describes (Harré, 1983; Raphael & Gavelek, 1996; McVee et al, 2005; Gallucci et al, 2010).

There is an interesting pattern in the language that formed as a result of this interweaving between collective and individual knowledge. Lemke (1990) describes personal meaning as being constructed when students can put concepts in their own words instead of just parroting the collective language. Many students put the concept of a limit in their own words and publicized their personal meaning; it would be expected the students’ discourse would expand on the repetition with variation, so that the collective knowledge would expand and include more variations since each student would add their own interpretation and wording to the collective knowledge. However, once students expressed a concept in their own words they did not vary their wording of the concept from then on. This meant early in group discussions there were
different variations as students tried to make sense of a concept, but once they began constructing meaning there was less variation since students would not deviate from their personal wording.

This indicates two things. First, students did not have the expertise to identify different variations in wording and determine if they meant the same thing, so they clung to the wording of their conception that they knew as correct. Second, it led to a stagnation in the collective discourse. When students put a concept in their own words, they stopped using variation in their expression of the concept, thus reducing the variation in the collective language. This meant classmates who had yet to construct meaning heard less repetition with variation. The students who had not constructed meaning began to parrot their classmates because they only experienced a fixed wording. This reduced variation in the collective language until there were only two narratives, that of the limit as a barrier or unreachable. Both were notions that Prof. Morgan had presented at the beginning of the unit. Moreover, as the collective language narrowed to a fixed wording, students who has a low self-concept questioned their conception of a limit, even if they had put the concept in their own words, and attempted to adopt the collective knowledge over their individual conceptions. There were many factors which affected the narrative that dominated the group’s collective knowledge, in particular the imagery presented by the instructor, how it aligned with student’s initial conception, and students’ perceptions of their classmates and themselves.

**Role of the instructor in individual student’s knowledge.**

It is important to consider the intended and unintended language and imagery used during instruction and its influence on students’ conceptions. However, careful selection of imagery, language, or definitions does not necessarily result in students adopting the language and
imagery of their instructor. Oehrtman (2009) found that although some students adopted imagery and metaphors their professor used, none of the students adopted the zooming imagery the professor regularly employed to describe limits. Moreover, much of what the students said when applying these metaphors was mathematically inaccurate showing the imagery did not have meaning for the students or had reinforced a misconception. Though students did adopt the language of the instructor and text, specifically “arbitrarily” and “sufficiently,” the terms had no rigorous meaning for the students. Even further, Glücer (2013) noticed that students used different metaphors to describe limits than had been utilized by the instructor and adopted word use that was not reflected in the instructor’s practices. It is not sufficient to carefully select the language, imagery, and mathematical discourse employed during instruction because students may not be able to construct meaning from the language and tools presented in class (Williams, 1991; Oerhtman, 2000; Glücer, 2013). As this study illustrates, when students’ intuition was at odds with the collective understanding of limits, as in the case of Fern and Kyle, students struggled to use the collective imagery to make meaning of the mathematical definition of a limit.

This study provides insight into selecting language and imagery which is salient to students and can be used to construct personal meaning that align with mathematical notions. The students in this study adopted Prof. Morgan’s imagery, either the limit as a boundary or barrier, and none of the students adopted her “door” metaphor. The boundary/barrier metaphor came up in student’s discussion of their initial conceptions where the “door” metaphor came from Prof. Morgan. The students only used the imagery which aligned with their initial conceptions. In addition, four of the students were able to use this imagery to construct a definition of a limit which was meaningful to them and mathematically accurate. Prof. Morgan
had students make their initial conceptions of a limit transparent by publicizing their thinking and using their notions to develop a collective conception of a limit. She used this student-generated collective conception as the foundation for building a mathematical definition for limits. This allowed her to expand on the students’ initial conception using imagery and language that resonated with students to provide a mathematical conception of a limit which aligned with students’ thinking about limits. Lakoff and Núñez (2000) posit that it is possible to use students’ intuitive models to construct rigorous mathematical definitions. This study illustrates this assertion and provides an instructional model for taking intuitive models and extending them to construct a mathematical definition of a limit. However, the students were not able to progress to the point of mathematical rigor in their definition of a limit. Students were not presented a rigorous definition (epsilon-delta or arbitrarily close) of a limit, so did not have the appropriate metaphors to help them characterize their intuitive notions into precise terms (Lakoff & Núñez, 2000).

**Role of students’ perception of each other on individual students’ knowledge.**

Students’ knowledge is constructed through interactions with their world. In Prof. Morgan’s class, most interactions were between peers; that is, student’s individual knowledge was shaped through interactions with their peers and their peers’ collective knowledge. A key factor observed to have influenced students’ individual knowledge was students’ perceptions of each other’s math skills. Studies have demonstrated that perception of learning environments is related to student achievement and emotional experiences (Fraser, 1994; Frenzel, Pekrun & Goetz, 2007). It was observed in this study that students who were viewed as more knowledgeable by their peers influenced the knowledge of their groupmates, especially those students struggling to transition to Q3. Prof. Morgan required students to explore and make sense
of the mathematics they were learning. She positioned her students to be responsible for constructing a collective knowledge, ensuring that each group member understood the concepts, and validating each other’s ideas. This created an authority vacuum since Prof. Morgan removed herself from the role of authority. Depending on the how students perceived each other’s mathematical ability, there were three group dynamics that emerged.

At times one person stepped up to fill in as the authority for the group. As research on ZPD suggests, a more knowledgeable student can scaffold a less knowledgeable peer (Vygotsky, 1978, 1986; Goos, 2004). The results reveal that scaffolding only occurred if the student viewed the peer as an authority and viewed themselves as less knowledgeable. For instance, Fern viewed Yara as a mathematical authority and held a low belief in her own mathematical ability. Fern described Yara as the person who confirmed and guided her thinking, meaning Fern allowed Yara to scaffold her construction of knowledge and adopted the knowledge collective knowledge constructed by Prof. Morgan and her group, over her individual knowledge. In other words, Yara was viewed as an authority and Fern valued Yara’s thinking over her own. On the other hand, Daniella viewed herself as the only authority when she worked in a group with Yara. Yara never scaffolded Daniella because Daniella had a high belief in her mathematical ability and did not hold Yara in the same esteem. This happened despite Yara’s construction of a generalized definition of a limit and Daniella’s inability to synthesize her notions of a limit to create a general definition.

Another dynamic that emerged was when students viewed each other as having similar knowledge so no one acted as authority. When this happened, students worked cooperatively, combining their individual understanding to create a collective understanding of the mathematics. This type of interaction often resulted in a ZPD as collaboration. This occurred for
Javier when working with his first group. During these collaborations, there was a reciprocal flow of knowledge between group members. Each group member added to the collective knowledge and then individually generated understanding based on the group interactions.

The third dynamic was one where none of the students felt there was an authority in the group. In this case, students did not feel that they could act as the authority for the group nor did they have confidence in their classmates’ mathematical ability. When this happened, each person was concerned with his or her own individual understanding and did not use each other’s ideas to construct a collective knowledge. The group members’ inability to trust each other’s ideas led them to talk past each other; they were unable to construct a ZPD either as collaboration or scaffolding to forward their understanding. This situation occurred for Kyle in his first group. The group interactions were not only tenuous, but a barrier for Kyle’s construction of knowledge. Kyle identified that his initial conception of a limit did not align with his group members’ ideas, but he did not trust his group enough to adopt their collective language about limits. Instead, he stayed in limbo between his initial concept and the group’s conception.

This suggests peer esteem has an impact on student achievement. When students hold each other in high esteem it results in a group dynamic which supports the construction of knowledge, but when students do not respect each other’s knowledge it acts as a barrier for learning. This finding is supported by Frenzel, Pekrun, and Goetz (2007) who found peer esteem had a significant impact on students’ enjoyment of the mathematics classroom and led to less math anxiety. In addition, differences in class perception of the classroom environment, including peer esteem, influenced students’ experience in the classroom. Students’ perception of
each other’s mathematical abilities affects the construction of knowledge of both collective and individual knowledge. The present study takes a qualitative look to understand this relationship.

**Role of students’ perceptions of themselves on individual knowledge.**

Not only did students’ perceptions of each other greatly influence individual construction of knowledge, but students’ self-perception of their mathematical ability contributed to the relationship between individual and collective knowledge. Previous research has shown that students’ mathematical self-concept has an impact in their academic success (Bandalos, Yates & Thorndike-Christ, 1995; Skaalvik & Rankin, 1995; Skaalvik, 1997). This study found that students with high mathematical self-concept trusted their initial conceptions and their ability to take the collective knowledge and build personal meaning. On the other hand, students with low belief in their mathematical ability did not trust their initial conceptions and struggled to use the collective knowledge to build personal meaning. However, students with low self-concept but high effort were able to seek out resources to support their construction of knowledge.

All of the students with a strong belief in their mathematical ability (Javier, Daniella, and Yara) transitioned to Q4 and constructed a personal meaning for a limit. In addition, none of them allowed themselves to be scaffold by their peers. This refusal to be let their peers dictate thinking did not mean they were closed to feedback from their peers. It was feedback from Mariah which led to Javier’s moment of reflection and growth in his thinking about limits. Instead, students with high self-concept used the interactions with their groupmates in ways that were productive to their construction of knowledge. They engaged in collaboration to construct knowledge, scaffolding to publicize their personal notions, and reflected on the interactions with their groupmates to refine and revise their understanding of limits.
Students with low mathematics self-concept did not trust their initial conceptions. They were not able to use group interactions in a way that would best support their individual construction of knowledge, as. As demonstrated by Fern who allowed herself to be scaffolded by Yara, even when she had constructed personal meaning. This reset her cycle through Vygotsky Space. Although not specifically a barrier to her learning, it restarted the learning process. She may have been able to use Yara’s notion and construct personal meaning, but the unit ended before she had time. This finding is supported by Bezuidenhout (2001) who observed students will abandon their conceptions, even if mathematically correct, if they doubt their understanding. This study adds to this observation by considering the factors which influenced and resulted in the rejection of personal meaning, even when that meaning is mathematically correct. Kyle’s low belief in his mathematical ability served as a barrier since he neither had self-confidence in his intuition nor confidence in his group. Without either he did not trust his individual knowledge or the collective knowledge and was not able to use either to construct personal meaning. This aligns with findings that students’ self-perception affects achievement and correlates to a self-defeating orientation (Skaalvik, 1997).

Mariah had low belief in her mathematical ability but high effort. Unlike her peers with low math self-concept, she constructed personal meaning and completed a cycle through Vygotsky Space. Mariah’s experience is very different from her other classmates. She took ownership of her learning and put forth effort to enhance her learning when she identified a weakness in her understanding. Research on self-enhancing orientation (Skaalvik, 1997) and self-efficacy (Bandalos, Yates & Thorndike-Christ, 1995) found these factors were positively related to student achievement, so that if a student believes they can be successful in mathematics, it can counter-balance the student’s negative self-concept.
Limitations

Despite careful design and implementation of this study, there are several limitations to this research. The first is this study occurred in an atypical calculus class. Differences from a typical calculus course include the structure, instruction, and composition of the class. This is a two-semester-long calculus course which met daily, so there was significant time to dedicate to developing students’ conception of a limit. In terms of instruction, the course was predominantly student exploration and discourse. Finally, the course was designed to support at-risk students, so the demographics and mathematical background of students vary greatly. The uniqueness of the setting may limit the generalizability of the results. However, the exceptionalness of the class allowed for longitudinal analysis of students’ construction of knowledge of limits through discourse which would not have been possible in a typical calculus class.

Another limitation in the design of the study was the inability to follow the students into the second semester to continue to chart their construction of knowledge and the persistence of the knowledge they had developed in the first semester. This additional time could have revealed if students’ conceptions became more rigorous or problematic and led to revisions. However, the level of time and depth during the semester on limits provided insight into the construction of knowledge. Future research should engage in a longitudinal study with a longer timeframe to understand the full progression from initial conception to a rigorous mathematical conception. I suspect it would require a multi-year study to explore construction of knowledge that surpasses proficiency and develops to expertise.

After completing data collection, during data analysis, students’ perceptions of each other emerged as a salient theme. Although students discussed their views of their classmates during interviews, it would have been helpful to have asked more specific questions about perception
during interviews and administer a perception survey, as was done with math self-concept. Future studies should consider the impact of students’ perception of their peers.

**Implications and Future Research**

The focus of this research was to understand students’ development of mathematical knowledge and how discourse supports the construction of collective and individual student’s mathematical knowledge. Due to the context of the research being set in a calculus course, it also provides insight into the conceptual understanding of calculus teaching and learning. Thus, implications from this study as well as suggestions for future research fall into two main categories, mathematics learning and calculus education.

This research explored the interplay between individual and collective students’ knowledge resulting from student discourse. Previous research has illustrated the benefit of discourse on students’ conceptual development; however, researchers could not infer specific student’s conceptualizations from the collective (Cobb et al., 1997). Use of the Harré model of Vygotsky Space (Harré, 1983) allowed for this type of distinction; moreover, it elucidated the interplay between individual and collective understanding and how this informed student’s construction of knowledge. The study reveals the importance of Q1, Q3, and the transition from Q3 to Q4 in students’ construction of knowledge. This has instructional implications for considering how students experience a new concept in Q1, as well as the need to find ways to kindle notions developed in Q3 so that students do not regress in their understanding.

This research adds to the understanding of students’ conceptions of limits. Prior to this study, there was no research on how student’s conception of a limit develops. This insight into how students construct their understanding of limits has several instructional implications. This research highlights the importance of students’ initial conception and how they can be used as a
foundation to construct mathematical understanding of the concept. It provides proof of Lakoff and Núñez (2000) assertion that it is possible to use students’ intuition about limits to develop a mathematical definition, but more research is needed to understand how to provide precision in this conception. Moreover, it provides an instructional model for this approach, although other instructional models need to be investigated as well. Further research is needed to continue to explore the impact of making students’ initial conceptions transparent and using them to develop conceptual understanding of limits particularly to extend the definition to a rigorous definition.

The study expands on the knowledge of ZPD. It posits that ZPD can emerge during student cooperation that is both collaboration—knowledgeable peers supporting each other to move forward their understanding—and scaffolding—a more knowledgeable peer supporting another to move forward the other student’s thinking. This type of collaborative scaffolding emerged when students worked collaboratively to scaffold a peer. Students engaging in this type of discourse identified it as a productive and helpful partnership which was in stark contrast to the students’ feelings about scaffolding their classmates on their own. This has implications for teaching and learning, particularly for instructors when purposively selecting students for group work.

There is a large body of research on self-concept and anxiety in mathematics (e.g. Fennema & Sherman, 1976; Frost, Hyde, & Fennema, 1994; Skaalvik, 1997; Juter, 2004). For this reason, students’ perceptions of themselves was included to help inform the study. While conducting this research, peer esteem and student perception of ability had an influence on individual construction of knowledge. While investigating this phenomenon, I found very few studies that examined student perception of their peers and its impact on emotion experience and
achievement in mathematics. This study highlights the relationship between student perception of ability and mathematics achievement and can help to inspire further research in this area.

As demonstrated in this study, the Harré model of Vygotsky Space provides a new way to assess student understanding, specifically a qualitative assessment to evaluate growth in conceptual understanding. Standard measures are good at assessing procedural knowledge but conceptual knowledge is more abstract and harder to assess. Using standard calculus assessments, Mariah would have shown minimal understanding because her poor algebraic skills hindered her ability to evaluate limits analytically but this method of analysis illustrated her growth in conceptual understanding; an understanding that was more advanced than some of her peers who would have been identified as having strong knowledge using more traditional assessment measures. With current emphasis on growth models and value-added, future research on using the Harré model of Vygotsky Space (Harré, 1983) as a qualitative assessment tool could provide an alternative way to measure conceptual growth.


Appendix A - Instructor Recruitment Letter

Dear Instructor,

Little research explores how discussions in a college mathematics courses help the whole class and each student individually learn the concepts and ideas of mathematics.

For this reason, this letter is to ask if you would be willing to participate in a research study examining in-class discussions in your classroom. I will not be making any changes in instruction; I just want to collect information to gain insight into how discussion helps students learn mathematics. The study will involve observations during your instruction of the concept of limits.

If you would like to participate, please email back the attached survey with your answers to: progers@albany.edu. Please try and be complete as possible with the survey.

The results of this study will provide valuable information for mathematics and mathematics education courses. The names of teachers and students will not be stated in any published documents and participation is voluntary. All contact information will be kept confidential. Thank you for your time and cooperation.

Sincerely,

Patterson Rogers
Ph.D. student
University at Albany
progers@albany.edu
Dear Instructor,

Below you will be asked to (1) to provide your contact information, and (2) describe generally about your instruction practices in your calculus course.

Contact Information

Name: ______________________________            Date: ____________

Email: ______________________________

1. How often and for how long does your calculus course meet each week?

______________________________________________________________________________

2. How frequently do you engage students in discussion about calculus each week? For what purpose(s)?

______________________________________________________________________________

______________________________________________________________________________

______________________________________________________________________________

______________________________________________________________________________

______________________________________________________________________________
Appendix B - Instructor Consent Form

Dear Professor Morgan,

My name is Patterson Rogers. I am a graduate student in the Department of Educational Theory and Practice at the University at Albany. I am conducting a study for research purposes entitled *Exploring the nature of collective and individual students’ conceptual knowledge developed through mathematical discourse* under the supervision of my advisor Dr. Arthur Applebee. The goal of this study is to find out how having discussions in a college Calculus course help the whole class and each student individually learn the concepts and ideas of calculus, specifically limits. I will not be making any changes in instruction; I just want to collect information to gain insight into how discussion helps students learn mathematics.

Your participation in the study would involve allowing me to observe your classroom during the duration of your unit on limits. Participation will not require anything beyond allowing me to recruit students and go over student consent and the administering of a 20 minute questionnaire during class time. During the observation I will be video recording so I can capture anything written on the board and audio recording, so I can capture what is said during whole and small group discussions. To present the results of my study, I may include examples of what was said in class; no images or sound bites from the recordings will be shared. All of the information you provide will be confidential, this means your identity and the identity of your students in this study will be treated as confidential. The results of the study may be published but I will not give any names (yours or the students) or include any identifiable references. The storage of the data will be kept secured. I will destroy audio and video recordings upon completion of the study.

I will also be collecting samples of students’ classwork (worksheets, course assessments, homework, and products from groupwork) and a questionnaire which asks questions about students’ previous math classes and how they feel about math. In addition to observing your class, I will ask students if they are willing to be interviewed outside of class to talk about the student’s understanding of the course material and their feelings about learning in a class which uses discussions. You will not know which students are participating in the study.

You are free to choose whether or not to participate in this study. You must be at least 18 to participate in this study. You will be provided with any significant new findings developed during the course of this study that may relate to or influence your willingness to continue participation. If at any time you want to stop being in this study, you may stop being in the study by contacting: Patterson Rogers, progers@albany.edu. Your participation in this project is voluntary. Even after you agree to participate in the research or sign the informed consent document, you may decide to leave the study at any time I will retain and analyze the information you have provided up until the point you have left the study unless you request that your data be excluded from any analysis and/or destroyed.

All information obtained in this study is strictly confidential unless disclosure is required by law. In addition, the Institutional Review Board and University or government officials responsible for monitoring this study may inspect these records.
If you have any questions about this study, please contact Patterson Rogers at progers@albany.edu or Dr. Arthur Applebee at aapplebee@albany.edu. You will be offered a copy of this form to keep. Research at the University Albany involving human participants is carried out under the oversight of the Institutional Review Board (IRB). This research has been reviewed and approved by the IRB. If you have any questions concerning your rights as a research subject or if you wish to report any concerns about the study, you may contact University at Albany Office of Regulatory & Research Compliance at 1-866-857-5459 or hsconcerns@albany.edu.

Sincerely,

Patterson Rogers

__________________________________________

I have read, or been informed of, the information about this study. I hereby consent to participate in the study.

Your printed name: ____________________________

Your signature: _______________________________ Date ______________
Appendix C - Instructor Interview Protocol

The following questions will guide the interview, but follow-up questions may emerge during the interview.

First interview.

1. What are your goals for students in your class?
2. How do you organize the material in your class?
3. What instructional strategies do you use to help students develop conceptual understanding? How do you think these strategy help students learn?
4. How do you support discourse in the classroom? What are your goals for class discussions?
5. How do you gauge whether students are understanding the material? What types of understanding are you looking for?
6. How do you think students struggle the most in this class? How do you help them with these struggles?

Second interview.

Teacher will watch video from class and comment on the follow features of what they are observing:

1. What the teacher intended for students to learn during the activity.
2. How well the activity went based on these intended goals.
3. What understanding/misconceptions they see the students building in the video.
4. What the students took away from this activity.
5. What they think is the most important take away from watching the video.
6. Additional thoughts and reflections.
Follow-up questions for both interviews will include questions in the vein of:

- Can you think of an example when you saw this happen…?
- Where did you see students struggle the most? What are some of the things that contributed to their struggle? How do you help them with these struggles?
- Where did you see students demonstrating conceptual understanding? What are some of the things that contributed to this understanding? How do you help them?
- Could you explain why you made the choice we see in the video?
Appendix D - Course Syllabus

Math XXX\textsuperscript{5} – Algebra & Calculus I

Fall 2013

Welcome to Math XXX- Algebra & Calculus 1. My name is Ashley Morgan\textsuperscript{6} and I will be your instructor for this course. I like to keep classes understandable, interesting and fun by using various teaching methods and involving you in class- so be ready! I believe that anyone, with the proper motivation, can learn anything. I know that a lot of people don’t like math and think they can’t learn it but I think with some patience, persistence and assistance you can. Everyone learns at different rates and some of you will catch on to topics sooner than others. We all have different strengths and weaknesses but if you put them together we can help each other out. So, if you are ready and willing to learn, we’ll work together as a class to be sure everyone is able to learn.

How to contact me:

Like you, I am busy. I know how hectic schedules can be so I try to be accessible at various times and in various ways. So, please feel free to contact me at any time with questions. I can be reached…

**By email:** XXX

**By phone or text:** XXX (for voicemail or text)

*(I probably won’t actually answer but leave a message I’ll get back to you ASAP)*

*In person:* My office hours are in XXX:

Mondays and Wednesdays: 1:30 – 3:00

Thursdays: 10:30 – 11:40

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\textsuperscript{5} Identifying information removed from Syllabus

\textsuperscript{6} Pseudonym
Grading:

I grade using a cumulative point system. Everything you do will have a point value and you will be given a grade of actual vs. possible points. For example, if a homework is worth 30 points you might receive a grade of 25/30. What is this in percentages? Hmm? That sounds like a math question to me! Keep track of your grades somewhere (perhaps on the back cover of your notebook) and you’ll always know exactly what your grade is.

There is a grading outline at the end of the syllabus. Fill in your grades as you get them so you will always know exactly what your grade is. Keep track of this yourself. It is not my responsibility to keep track or your grades for you.

Attendance/Class Participation:

If you miss class you are going to miss something important. It’s too easy to fall behind and so hard to catch up when you do so you have to be in class. I will not reteach you something you missed when skipping a class. You are responsible for all material missed for whatever reason.

*Missing a class is not an excuse for missing a test or quiz or for not handing in an assignment.* If you know you will miss class turn it in early or make arrangements with me ahead of time to take the quiz/test. You will still be marked as absent, but I will accept the material if you have made previous arrangements with me. The key here- Be responsible, be prepared.

Requirements:

**Textbook**- Single Variable Calculus by Stewart. 6th or 7th edition. (7th preferred)

**Internet Access**- is necessary to contact me, get materials on line, go to online help chats and do some online summary work. More details will follow.
**Homework:**

Practice makes perfect, right? Homework problems are assigned almost every class. Some of them will be collected and graded. Homework problems will be graded out of 5 points for a total of 20-30 points per unit. Stay on top of your homework- if you can do those problems you can do quiz or tests problems.

**Assignments and Quizzes:**

Assignments and quizzes are graded with no partial credit. These can be corrected and resubmitted. Your grades for the original and corrected assessment will be averaged. Corrections must be turned in before the end of the unit in which they were given.

**Retake Test:**

No extra credit will be given. However, there is one “Retake Test” available to you. But, the following rules apply:

1) You can only retake 1 test over the course of the semester
2) You must have at least an 85% overall attendance percentage
3) The retake must be scheduled before the unit exam of the following unit. **(The last exam can be retaken only if taken before the final exam!)
4) You may retake the test once and you **get the grade you earn on your retake.**
5) If you wish to retake a test you must notify me at least one day in advance to be sure I have a copy of the retake.
6) Retakes are to be done on your own time, not during class time.
7) You may NOT retake the “Final Exam”!
8) You decide which test to retake, if any.

**Late work:**
Sorry, but this is unacceptable. I give you plenty of notice for everything. If you forget something and turn it in the next class it will be graded with a 50% loss of points. After one class, late work will no longer be accepted. Yes, it sounds kind of harsh but that’s life in the “real world” projects and such need to be done on time or you get fired!

Absences are not an excuse for late work or missed quizzes or tests. Assignments are to be handed in on the due date. Watch the schedule, be prepared.

Written summaries:

Together, we’ll be working on writing e-books for subjects in our course! At the end of each topic you’ll turn in a written summary of the material, your understanding, perspectives and description of it in your own words.

Portfolio:

You will be required to keep all of your graded homework, writing assignments, quizzes, projects and exams in a portfolio. Everything you receive a grade on for this course will be put in your portfolio and resubmitted at the end of each unit with a written analysis of your work. More details will follow with each unit. Keep this syllabus! Turn it in with each portfolio check with your updated grade list.

Stuck on a problem? Get Extra Help:

Ask for help when you need it- please don’t be embarrassed. Help is available…

-From me (office hours and contact info above) Sometimes a quick email or text to me will get you the explanation you need. I’m online A LOT! So shoot me a quick note and see if I can help, don’t sit for hours stuck on a problem. Or, stop in my office hours and we can work on it together.
- Tutoring sessions: Seek help from the EOP tutors or Academic Support Services too if you are having difficulty. Talk to your counselor about getting a tutor if needed.

- Online: If you need help and can’t make an office hour we can meet on the BB site for online whiteboard chats. This can be done anytime but you will need to set up an appointment with me. We can do group sessions there too!

Class schedule and due dates:

You’ll get a schedule at the beginning of each unit. We’ll deal with changes as they come up. The schedule will include due dates for assignments, quizzes and exams. So, watch the schedule, keep up, and stay on track. To begin with, the dates of the exams are listed below.

There are no make-up dates for any reason. Be sure to be in class on these dates! 😊

Exam 1: Friday, September 13
Exam 2: Friday, October 11
Exam 3: Friday, November 1
Exam 4: Monday, November 25

Final Exam:

There is a “cumulative final” given at the end of the semester. I will let you know the date of the exam. When possible, it will follow the University final exam schedule. Everyone will take the final.

Under the guidelines of the Rehabilitation Act of 1973 and the Americans with Disabilities Act (ADA), the College is required to provide reasonable accommodations to students with disabilities. If you have a diagnosed disability that might affect your performance in this class, please meet with the instructor as soon as possible. This information will be kept confidential.
<table>
<thead>
<tr>
<th>Unit 1</th>
<th>Quiz 1.1 / 25</th>
<th>Quiz 1.2 / 25</th>
<th>Homework 1 / 20</th>
<th>Unit 1 total:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Exam 1 / 100</td>
<td>/ 235</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Assignment 1.1 / 15</td>
<td>Summary 1 / 25</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Assignment 1.2 / 15</td>
<td>Portfolio 1 / 10</td>
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**Current average:**

<table>
<thead>
<tr>
<th>Unit 2</th>
<th>Quiz 2.1 / 25</th>
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<th>Homework 2 / 30</th>
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<tr>
<td></td>
<td></td>
<td>Quiz 2.3 / 25</td>
<td>Exam 2 / 100</td>
<td>/ 310</td>
</tr>
<tr>
<td></td>
<td>Assignment 2.1 / 15</td>
<td>Summary 2 / 25</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>Assignment 2.2 / 15</td>
<td>Portfolio 2 / 20</td>
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</tr>
<tr>
<td></td>
<td>Assignment 2.3 / 15</td>
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<tr>
<td></td>
<td>Assignment 2.4 / 15</td>
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</tbody>
</table>

**Cumulative total:**

<p>| Quiz 3.1 / 25 | Current average: |</p>
<table>
<thead>
<tr>
<th>Unit 3</th>
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<th>Unit 3 total:</th>
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<tr>
<td></td>
<td>Quiz 3.3</td>
<td>25</td>
<td>Homework 3</td>
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<td></td>
<td></td>
<td>Exam 3</td>
</tr>
<tr>
<td></td>
<td>Assignment 3.1</td>
<td>15</td>
<td>Summary 3</td>
</tr>
<tr>
<td></td>
<td>Assignment 3.2</td>
<td>15</td>
<td>Portfolio 3</td>
</tr>
<tr>
<td></td>
<td>Assignment 3.3</td>
<td>15</td>
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</table>

<table>
<thead>
<tr>
<th>Unit 4</th>
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<th>25</th>
<th>Unit 4 total:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Quiz 4.2</td>
<td>25</td>
<td>Homework 3</td>
</tr>
<tr>
<td></td>
<td>Quiz 4.3</td>
<td>25</td>
<td>Exam 4</td>
</tr>
<tr>
<td></td>
<td>Assignment 4.1</td>
<td>15</td>
<td>Summary 4</td>
</tr>
<tr>
<td></td>
<td>Assignment 4.2</td>
<td>15</td>
<td>Portfolio 4</td>
</tr>
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</table>

Cumulative total: 285
Current average: 830
Cumulative total: 1140
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<tr>
<th>Assignment 4.3 / 15</th>
<th>Current average:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assignment 4.4 / 15</td>
<td></td>
</tr>
<tr>
<td><em>Final Exam 200 points</em></td>
<td>Final total:</td>
</tr>
<tr>
<td></td>
<td>1340 possible points</td>
</tr>
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</table>
Appendix E - Informed Consent Students

Dear Student,

My name is Patterson Rogers. I am a graduate student in the Department of Educational Theory and Practice at the University at Albany. I am conducting a study for research purposes entitled *Exploring the nature of collective and individual students’ conceptual knowledge developed through mathematical discourse* under the supervision of my advisor Dr. Arthur Applebee. The goal of this study is to find out how having discussions in a college Calculus course helps the whole class and each of you individually learn the concepts and ideas of calculus, specifically limits. After talking with your instructor, your class was selected because discussion is an integral part of instruction. I will not be making any changes in instruction; I just want to collect information to gain insight into how discussion helps students learn mathematics.

I will be observing your class during the instruction on limits to understand how discussions are used in your class and how they help the class to understand the material better. During the observation I will be video recording the white/blackboard and audio recording, so I can capture what is said during whole and small group discussions. By participating in this study, you will be allowing me to collect and study your discussions about mathematics in class.

At the beginning of the semester I will administer a questionnaire in class which asks questions about your previous math classes and how you feel about math. Filling out this questionnaire will take about 20 minutes and will be completed in class. I am also asking permission to collect this questionnaire and your classwork, specifically worksheets, quiz/test, homework, and products from groupwork, which are completed during the part of the class which focuses on limits. The purpose of collecting your work and the survey is to get a picture of the whole class’ understanding of limits and the class’ previous experience with math.

In addition to observing your class, I would like to interview some of you to gain deeper insight into individual student’s understanding of the course material and your feelings about learning in a class which uses discussions. There will be two interviews, one near the beginning of the course and one at the end of the semester. Each interview is expected to last about 45 minutes. If you agree to an interview, I may contact you after class to set up an interview. Interview will be selected to allow for discussions with students who have different levels of understanding of limits and amount of class participation. Students who are interviewed may find that the interview helps them to understand the class material more deeply. You may still participate in this study if you are not willing to be interviewed.

To present the results of my study, I may include examples of what was said in class and of student work; no images or sound bites from the recordings will be shared. All of the information you provide will be confidential, this means your identity in this study will be treated as confidential. The results of the study may be published but I will not give your name or include any identifiable references to you. The storage of the data will be kept secured. I will destroy audio and video recordings upon completion of the study. Your instructor will not know which students are participating in the study. Your decision to participate in the study will not impact your grade nor will it be understood in any negative way by the researcher.
You are free to choose whether or not to participate in this study. You must be at least 18 years old to participate in this study. There will be no penalty if you choose not to participate. You will be provided with any significant new findings developed during the course of this study that may relate to or influence your willingness to continue participation. If at any time you want to stop being in this study, you may stop being in the study without penalty or loss of benefits by contacting: Patterson Rogers, progers@albany.edu. Your participation in this project is voluntary. Even after you agree to participate in the research or sign the informed consent document, you may decide to leave the study at any time without penalty or loss of benefits to which you may otherwise have been entitled. I will retain and analyze the information you have provided up until the point you have left the study unless you request that your data be excluded from any analysis and/or destroyed.

All information obtained in this study is strictly confidential unless disclosure is required by law. In addition, the Institutional Review Board and University or government officials responsible for monitoring this study may inspect these records.

If you have any questions about this study, please contact Patterson Rogers at progers@albany.edu or Dr. Arthur Applebee at aapplebee@albany.edu. You will be offered a copy of this form to keep. Research at the University Albany involving human participants is carried out under the oversight of the Institutional Review Board (IRB). This research has been reviewed and approved by the IRB. If you have any questions concerning your rights as a research subject or if you wish to report any concerns about the study, you may contact University at Albany Office of Regulatory & Research Compliance at 1-866-857-5459 or hsconcerns@albany.edu.

Sincerely,

Patterson Rogers

I have read, or been informed of, the information about this study.

I do ____/ do not ____ give my permission for you to record my discussions in class during observation.

I do ____/ do not ____ give my permission for you to collect my classwork (worksheets, quiz/test, homework, and products from groupwork) on limits, and a background/attitudes towards mathematics survey.

I do ____/ do not ____ give my permission to be interviewed

Your printed name: ________________________________

Your signature: ___________________________________ Date _____________
Appendix F – Background Questionnaire

1. Which of the following describes your student status? Please circle the best answer.
   - [ ] Freshman
   - [ ] Sophomore
   - [ ] Junior
   - [ ] Senior
   - [ ] Graduate

2. How many years of math did you take in high school? _____________

3. How many semesters of college math have you taken? If this is your first college math course, please put 0. _____________

4. How many semesters since your last math class? If you took a course last semester, please put 0. _____________

5. Which of the following math courses have you taken? Please circle all courses that apply. Pick the courses which best describe what you have taken.
   - [ ] Math A
   - [ ] Math B
   - [ ] Algebra I
   - [ ] Algebra II
   - [ ] Geometry
   - [ ] Pre-Calculus
   - [ ] Trigonometry
   - [ ] Non-AP Calculus
   - [ ] AP Calculus AB
- AP Calculus BC
- College Algebra
- College Pre-Calculus
- Calculus I
**Appendix G - Mathematics Self-Concept Questionnaires**

Please answer the following questions to the best of your ability by circling your answer. You will provide only one answer for each question. You will choose the answer that most closely represents you and your experiences.

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. I have high aptitude for math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2. I like math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>3. Math classes are boring.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>4. I feel calm in math class.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5. I always do my best in math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>6. Even if I get a difficult math problem, I do not give up.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>7. I have no talent for mathematics.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>8. I look forward to math class.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>9. I am nervous in math class.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>10. I always do my math homework.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>11. Further education with a lot of math does not appeal to me.</td>
<td>1</td>
<td>2</td>
<td>3</td>
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<td>5</td>
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</tr>
<tr>
<td>12. I feel safe in math classes.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>13. I hate math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>14. I am as talented in math as other people in my class.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>15. Working with math is fun.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>16. In the future, I would like to learn more math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>17. I am relaxed in math classes.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>18. I want a job where I do not have to do math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>19. I can learn mathematics if I work hard.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>20. I do assignments in math as quickly as I can.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>21. I want to avoid all math in college.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>22. I am worried in math class.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>23. No matter how much I try, I have problems learning math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>24. I wish I took more math classes.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>25. I work hard in math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>26. I want to get on a track that has as little math as possible.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>27. I wish I did not have to take math classes.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
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<td></td>
</tr>
<tr>
<td>28. I just cannot learn math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>29. I am always nervous when I have to do math in classes.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>30. I give up quickly if I get a difficult math problem.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>31. In my future education, I would like not to have to do math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>32. I am tense in math class.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>33. I do as little as I can get by with when math is involved.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>34. I would like an occupation where I can use math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>35. I always prepared well for tests in math.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>36. I don’t mind a lot of math in my future education.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>37. I am anxious when I have to do math in classes.</td>
<td>1</td>
<td>2</td>
<td>3</td>
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<td>5</td>
</tr>
</tbody>
</table>

From Skaalvik & Rankin (1995)
Appendix H - Student Interview Protocol

Questions about the Course:

These are broad questions that may be asked during the interview; however, follow-up questions may emerge during the interview.

1. What has been the hardest thing to learn understand in this course? The easiest?
2. What in class has helped you the most to learn the material? The least?
3. What are your favorite things about this course? Least favorite?
4. What do you think about talking about math? How have math discussions positively or negatively impacted your understanding of material?
5. What do you think about working in small groups? How has group work positively or negatively impacted your understanding of material?
6. When working in a group to what extent (if any) did you find hearing your partner’s ideas/sharing and getting feedback on your ideas helpful? More difficult?
7. What suggestions would you make for improving this class?
8. How do you study for mathematics outside of class? Do you use any outside resources?
Content Questions:

1. The following are statements about limits, pick the statement(s) that best describe how you think of a limit.
   - limit describes how a function moves toward a certain point
   - A limit is a number or point past which a function cannot go
   - A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values
   - A limit is a number or point the function gets close to but never reaches
   - A limit is an approximation that can be made as accurate as you wish
   - A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached

2. Does the statement you pick match your definition of a limit? If not, could you tell me your definition?

From Williams (1991)

3. Give student a graph of \( f(x) = x^3 + 3 \) and ask them to find the limit as \( x \) approaches 0 and ask how they found the limit.

4. Looking at these two graphs is \( f(x) = g(x) \)? Is \( \lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) \)?
5. Let’s say we know that \( \lim_{x \to 2} f(x) = 3 \), can we find \( f(2) \)? Is \( f(x) \) continuous at \( x = 2 \)?

6. Let’s say we know that \( f(1) = 4 \), does \( \lim_{x \to 1} f(x) \) exist? If yes, what does the limit equal? Is \( f(x) \) continuous at \( x = 1 \)?

7. Are the functions \( f(x) = \frac{x^2 - x - 6}{x - 3} \) and \( g(x) = x + 2 \) equal? Explain. Is \( \lim_{x \to 3} f(x) = \lim_{x \to 3} g(x) \)? Explain.

Follow-up questions:

a. Can you tell me how you solved this problem?

b. What did you do first?

c. How did you know what to do next?

d. Could you write down the steps you took?

e. Could you explain what you did here?

f. Why does your answer make sense?
Appendix I - Examples Artifacts

Explain Assignment

Drawing graphs when given limits

Directions: Work through this worksheet with your group. At first, work out your answers on a scrap piece of paper. When you are in agreement that you have a correct answer, copy that answer onto an answer sheet. Clearly number your answer and be sure your graphs are appropriately labeled. Submit your final answers by the end of class to be graded as a homework.

Things to consider...
- What does a graph look like when the limit at a given value exists?
- What does a graph look like when the limit at a given value does not exist?
- What’s the difference (graphically) between a limit and a function value?
- What do infinite limits look like (when the limit is ±∞ or -∞?)

1. Sketch the graph of a function \( y = r(t) \) for which \( \lim_{{t \to 3}} r(t) = 0 \) but \( r(3) = 2 \).

2. Sketch the graph of a function \( y = f(x) \) for which \( \lim_{{x \to 3}} f(x) = 2 \), \( \lim_{{x \to 3}} f(x) = 5 \) and \( f(3) = 0 \).

3. Sketch the graph of a function \( y = f(x) \) for which \( \lim_{{x \to 3}} f(x) = -\infty \), \( \lim_{{x \to 3}} f(x) = \infty \).

4. Sketch the graph of a function \( y = r(t) \) for which \( \lim_{{t \to 3}} r(t) = -\infty \).

5. Sketch the graph of a function \( y = f(x) \) for which \( \lim_{{x \to -\infty}} f(x) = -\infty \), \( \lim_{{x \to -\infty}} f(x) = -2 \).

6. Sketch the graph of a function \( y = f(x) \) for which \( \lim_{{x \to \infty}} f(x) = DNE \) and \( f(2) = 5 \).
Practice Problems

Calculus 1 Worksheet #5

Limits involving approaching infinity: \( \lim_{x \to \infty} f(x) \)

TO INFINITY AND BEYOND !!!!!

Important theorem: \( \lim_{x \to \infty} \frac{1}{x} = 0 \)

Limits Involving Infinity
(Principle of Dominance)

1. \( \lim_{x \to \infty} \frac{x^a}{x^b}, \) if \( a < b \). Then, limit = 0. (Look for the highest degrees/powers of \( x \))

2. \( \lim_{x \to \infty} \frac{C x^a}{D x^b}, \) if \( a = b \). Then, limit = \( \frac{C}{D} \). (Look for the highest degrees/powers of \( x \))

3. \( \lim_{x \to \infty} \frac{x^a}{x^b}, \) if \( a > b \). Then, limit = \( \infty \) or \( -\infty \). (Look for the highest degrees/powers of \( x \) and check the sign of \( \infty \) by substituting with a large \( x \)-value.)

Problems:

<table>
<thead>
<tr>
<th>( \lim_{x \to \infty} )</th>
<th>( \lim_{x \to \infty} )</th>
<th>( \lim_{x \to \infty} )</th>
<th>( \lim_{x \to \infty} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{7 + \frac{1}{3x} - \frac{2}{x^2}}{x^3} )</td>
<td>( \frac{4x + 8}{5} )</td>
<td>( \frac{3x - 1000}{x + 100} )</td>
<td>( \frac{5x + 5}{7x^2 + 1} )</td>
</tr>
<tr>
<td>( \frac{5x^2 + 2}{4x^3 + 7} )</td>
<td>( \frac{3x^3 + 5}{5x^2 + 1} )</td>
<td>( \frac{2x^2 - 4x}{x + 1} )</td>
<td>( \frac{2x^2 - 4x}{x + 1} )</td>
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<tr>
<td>( \frac{3x^3 + 2}{5x^2 - 1} )</td>
<td>( \frac{3x^2 + 2}{4x^2 - 1} )</td>
<td>( \frac{x^2 + 2}{x - 555} )</td>
<td>( \frac{3 - 2x}{3x^3 - 1} )</td>
</tr>
<tr>
<td>( \frac{3 - 5x}{3x - 1} )</td>
<td>( \frac{3 - 2x^2}{3x - 1} )</td>
<td>( \frac{6x^2 - 2x - 1}{2x^2 + 3x + 2} )</td>
<td>( \frac{3x^3 + 2}{2x^2 - 9x^3 + 7} )</td>
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<tr>
<td>17. ( \lim_{x \to -\infty} \frac{x}{x^2 - 1} )</td>
<td>18. ( \lim_{x \to \infty} \frac{8x^2 + 3x}{2x^2 - 1} )</td>
<td>19. ( \lim_{x \to \infty} 10 - \frac{2}{x^3} )</td>
<td>20. ( \lim_{x \to \infty} 4 + \frac{3}{x} )</td>
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**Answers:**

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</thead>
<tbody>
<tr>
<td>1) 7</td>
<td>2) ( \frac{4}{5} )</td>
<td>3) 3</td>
<td>4) 0</td>
<td>5) ( \frac{5}{4} )</td>
<td>6) (-\infty)</td>
<td>7) ( \infty )</td>
<td>8) (-\infty)</td>
</tr>
</tbody>
</table>
Quiz

1) Define “the limit of a graph”. Be careful to distinguish between which variables (x or y) apply to which concepts and aspects of limits. (2 pts)

2) Find the limits given the graph below: (1 pt. each)

   a) \( \lim_{x \to 0} f(x) \)
   
   b) \( f(0) \)

   c) \( \lim_{x \to \infty} f(x) \)

   d) \( \lim_{x \to 5} f(x) \)

   e) \( f(5) \)

   f) \( \lim_{x \to -\infty} f(x) \)

   g) \( \lim_{x \to -2} f(x) \)

   h) \( \lim_{x \to -5} f(x) \)
Project Guidelines

Limit Project 100 points

Your goal for this project is to work with a group to make a video which demonstrates how to do one of the following:

1- Finding limits from a graph
2- Drawing a graph from given limits
3- Finding limits algebraically through factoring
4- Finding limits algebraically through rationalizing
5- Finding limits algebraically of piecewise functions
6- Finding limits at infinity

Your video should include:

- Appearances by everyone in your group
- An introduction
- Examples and explanations which cover all concepts of the topic
- A conclusion which summarizes any main points, processes and/or steps
- Something fun and/or creative! 😊

by class time on Wednesday, October 16.

Frequently asked questions:

1) How long should our video be?
   a. Your video should be as long as it takes to cover your topic in what you believe is sufficient depth. Sure, you can’t cover every possible little detail in your video but you should cover the main points and variations.

2) Do we each have to talk?
   a. Yes. At some point each person in your group needs to be seen in the video and say something! You can determine how little or how much each person talks.

3) How will we be graded on these?
   a. You will receive four grades of 25 points each: one from the instructors, one from your classmates, one from yourself and one from the other members of your group. More details on how that will work will be discussed in class…
**How you will be evaluated…**

By the Instructors:

<table>
<thead>
<tr>
<th>Topic</th>
<th>Points</th>
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<tbody>
<tr>
<td>Content accuracy</td>
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<tr>
<td>Content depth</td>
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<tr>
<td>Creativity</td>
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Total / 25

By your classmates:

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Total / 25

By yourself:

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Total / 25

By your group members:

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<tr>
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<th>Points</th>
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Total / 25

Name____________________________

Total Grade:      / 100