On combinatorial models for Kirillov-Reshetikhin crystals of type B

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On Combinatorial Models for
Kirillov-Reshetikhin Crystals of Type $B$

by

Carly J. A. Briggs

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Abstract

KR crystals encode the structure of certain finite dimensional representations of affine Lie algebras. There are various combinatorial models for them; type specific ones, such as tableau models in classical types and type independent ones, such as the quantum alcove model. While the type specific models are more explicit, they have less easily accessible information, so it is generally hard to use them in specific computations. As these computations are much simpler in the quantum alcove model [11, 12, 13], an alternative is to translate them to the tableau models, via a crystal isomorphism between the two models. This was achieved in types A and C. The main goal of this thesis is to work towards generalizing these results to type B.
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1 Introduction

Crystals are colored, directed graphs which encode the representations of quantum groups ($q$-deformations of enveloping algebras corresponding to Lie algebras) in the limit of the quantum parameter $q \to 0$. All highest weight representations have associated crystals, for semi-simple Lie algebras and moreover for Kac-Moody Lie algebras. In addition, there is an important category of crystals of affine Lie algebras called Kirillov-Reshetikhin (KR) crystals [5]. They are important because they can be used to construct the corresponding highest weight crystals as a semi-infinite tensor product (via the Kyoto path model). KR crystals are indexed by $r \times s$ rectangles and are denoted $B_{r,s}$. KR crystals are endowed with a grading by the so-called energy function, which originates in the theory of solvable lattice models [2]. Tensor products of KR crystals are constructed via a specific rule (tensor product rule) mimicking the tensor product of the corresponding representations, and they turn out to be connected graphs [3]. There is a unique affine crystal isomorphism between a tensor product of KR crystals and one obtained by permuting its factors, which is called the combinatorial $R$-matrix.

There are type specific models for the highest weight crystals of the classical Lie algebras, in particular, Kashiwara-Nakashima (KN) tableaux [4]. These were extended to the corresponding KR crystals [1]. Also, there are type independent models such as the Lakshmibai-Seshadri (LS) paths model, the Littelmann path model [16], and the alcove model [15]. Some of these were generalized to models for KR crystals, namely quantum LS paths and the quantum alcove model [10, 12, 13]. The objects of the quantum alcove model are sequences of Weyl group elements which correspond to paths (starting at the identity) in the so-called quantum Bruhat graph. The latter originates in the quantum cohomology of flag varieties, and is obtained by adding extra downward edges to the covers of the (strong) Bruhat order on the corresponding Weyl group.

While the type specific models are more explicit, they have less easily accessible information, so it is generally hard to use them in specific computations: of the energy function, the combinatorial $R$-matrix etc. As these computations are much simpler in the quantum alcove model [11, 12, 13], an alternative is to translate them to the tableau models, via a crystal isomorphism between the two models.

In types $A$ and $C$, the so-called filling map from the objects of the quantum alcove model...
corresponding to a tensor product of KR crystals to fillings of a corresponding Young diagram \( \mu \) consisting of Kashiwara-Nakashima (KN) columns [4] was constructed in [9]; these columns are known to index the vertices of a fundamental crystal \( B(\omega_r) \), which in this case is isomorphic to the KR crystal \( B^r,1 \) viewed as a classical one (i.e. the affine edges are removed). In [9, 10] it was shown that the filling map is the desired affine crystal isomorphism.

So the goal is to construct similar affine crystal isomorphisms in types \( B \) and \( D \).

While the map from the quantum alcove model to fillings is essentially a “forgetful map”, the difficulty is in constructing its inverse. One of the complications compared to types \( A \) and \( C \) is that, as a classical crystal, \( B^r,1 \) now splits as a disjoint union of fundamental crystals \( B(\omega_r) \oplus B(\omega_{r-2}) \oplus \ldots \). So the first task we solved is to extend the shorter KN columns to columns of height \( r \), and to combine this procedure with the doubling of each column, via the classical procedure explained for instance in [6]. The second task is to reorder the entries in each column, in order to make the columns compatible with the Weyl group elements (in this case, signed permutations) in the quantum alcove model. The reordering procedure displays additional complexity compared to the one in types \( A \) and \( C \) exhibited in [9]. The third task is to associate with the sequences of reordered columns the corresponding chains of Weyl group elements in the quantum alcove model, which form paths in the quantum Bruhat graph. In types \( A \) and \( C \), the latter is based on a greedy procedure. However, in type \( B \) certain modifications to the greedy procedure are needed. Experiments with Sage were helpful in discovering the needed algorithms. In this thesis the first task is completed. The second and third tasks are completed in cases corresponding to a specific structure of the objects in the quantum alcove model (i.e., sequences of Weyl group elements giving paths in the quantum Bruhat graph). We will address the remaining cases in future work.
2 Background

2.1 Root systems in general

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra, whose rank is $r$. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible root system, $\mathfrak{h}^*_R \subset \mathfrak{h}^*$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. Let $\rho := \frac{1}{2}(\sum_{\alpha \in \Phi^+} \alpha)$. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding simple roots. We denote by $\langle \cdot, \cdot \rangle$ the nondegenerate scalar product on $\mathfrak{h}^*_R$ induced by the Killing form. Given a root $\alpha$, we consider the corresponding coroot $\alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle$ and reflection $s_\alpha$.

Let $W$ be the corresponding Weyl group, whose Coxeter generators are denoted, as usual, by $s_i := s_{\alpha_i}$. The length function on $W$ is denoted by $\ell(\cdot)$. The Bruhat order on $W$ is defined by its covers $w \prec ws_\alpha$, for $\alpha \in \Phi^+$, if $\ell(ws_\alpha) = \ell(w) + 1$. The mentioned covers correspond to the labeled directed edges of the Bruhat graph on $W$:

$$
\begin{align*}
    w & \xrightarrow{\alpha} ws_\alpha & \text{for } w \prec ws_\alpha. 
\end{align*}
$$

The quantum Bruhat graph is defined by adding to the Bruhat graph (1) the following edges, called quantum edges, labeled by positive roots $\alpha$:

$$
\begin{align*}
    w & \xrightarrow{\alpha} ws_\alpha & \text{if } \ell(ws_\alpha) = \ell(w) - 2\langle \rho, \alpha^\vee \rangle + 1. 
\end{align*}
$$

We call the edges from the Bruhat graph “up edges” and the added edges “down edges”.

The weight lattice $\Lambda$ is given by

$$
\begin{align*}
    \Lambda := \{ \lambda \in \mathfrak{h}^*_R : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi \}. 
\end{align*}
$$

The weight lattice $\Lambda$ is generated by the fundamental weights $\omega_1, \ldots, \omega_r$, which form the dual basis to the basis of simple coroots, (i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$). The set $\Lambda^+$ of dominant weights is given by

$$
\begin{align*}
    \Lambda^+ := \{ \lambda \in \Lambda : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+ \}. 
\end{align*}
$$

Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice $\Lambda$, which has a $\mathbb{Z}$-basis of formal exponents $\{ x^\lambda : \lambda \in \Lambda \}$ with multiplication $x^\lambda \cdot x^\mu := x^{\lambda + \mu}$.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$
\begin{align*}
    H_{\alpha,k} := \{ \lambda \in \mathfrak{h}^*_R : \langle \lambda, \alpha^\vee \rangle = k \}. 
\end{align*}
$$
These reflections generate the affine Weyl group $W_{\text{aff}}$ for the dual root system $\Phi^\vee := \{ \alpha^\vee : \alpha \in \Phi \}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_R$ into open regions, called alcoves. The fundamental alcove $A^\circ$ is given by

$$A^\circ := \{ \lambda \in \mathfrak{h}^*_R : 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+ \}.$$ 

2.2 Kirillov-Reshetikhin (KR) crystals

A $\mathfrak{g}$-crystal (for a symmetrizable Kac-Moody $\mathfrak{g}$) is a nonempty set $B$ together with maps $e_i, f_i : B \to B \cup \{0\}$ for $i \in I$ (I indexes the simple roots, as usual, and $0 \not\in B$), and $\text{wt} : B \to \Lambda$. We require $b' = f_i(b)$ if and only if $b = e_i(b')$, and $\text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i$. The maps $e_i$ and $f_i$ are called crystal operators and are represented as arrows $b \to b' = f_i(b)$ colored $i$; thus they endow $B$ with the structure of a colored directed graph.

Given two $\mathfrak{g}$-crystals $B_1$ and $B_2$, we define their tensor product $B_1 \otimes B_2$ as follows. As a set, $B_1 \otimes B_2$ is the Cartesian product of the two sets. For $b = b_1 \otimes b_2 \in B_1 \otimes B_2$, the weight function is simply $\text{wt}(b) := \text{wt}(b_1) + \text{wt}(b_2)$.

The highest weight crystal $B(\lambda)$ of highest weight $\lambda \in \Lambda^+$ is a certain crystal with a unique element $u_\lambda$ such that $e_i(u_\lambda) = 0$ for all $i \in I$ and $\text{wt}(u_\lambda) = \lambda$. It encodes the structure of the crystal basis of the $U_q(\mathfrak{g})$-irreducible representation with highest weight $\lambda$ as $q$ goes to 0.

A Kirillov-Reshetikhin (KR) crystal [5] is a finite crystal $B^{r,s}$ for an affine algebra, associated to a rectangle of height $r$ and width $s$, where $r \in I \setminus \{0\}$ and $s$ is any positive integer.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ be a partition, which encodes a dominant weight in classical types; let $\lambda'$ be the conjugate partition. We define

$$B^{\otimes \lambda} := \bigotimes_{i=1}^{\lambda_1} B^{\lambda_i',-1},$$

assuming that the corresponding column shape KR crystals exist. We denote such a tensor product generically by $B$. It is known that $B$ is connected as an affine crystal, but disconnected as a classical crystal (i.e., with the 0-arrows removed).

2.3 Quantum alcove model (QAM)

Next we describe the objects of the quantum alcove model. We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves $A$ and $B$, we write
$A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from $A$ to $B$.

**Definition 2.1.** [14] An *alcove path* is a sequence of alcoves $(A_0, A_1, \ldots, A_m)$ such that $A_{j-1}$ and $A_j$ are adjacent, for $j = 1, \ldots, m$. We say that an alcove path is *reduced* if it has minimal length among all alcove paths from $A_0$ to $A_m$.

Let $A_\lambda = A_\circ + \lambda$ be the translation of the fundamental alcove $A_\circ$ by the weight $\lambda$.  

**Definition 2.2.** [14] The sequence of roots $\Gamma := (\beta_1, \beta_2, \ldots, \beta_m)$ is called a *$\lambda$-chain* if

$$A_0 = A_\circ \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_{-\lambda}$$

is a reduced alcove path.

We now fix a dominant weight $\lambda$ and an alcove path $\Pi = (A_0, \ldots, A_m)$ from $A_0 = A_\circ$ to $A_m = A_{-\lambda}$. Note that $\Pi$ is determined by the corresponding $\lambda$-chain $\Gamma = (\beta_1, \ldots, \beta_m)$, which consists of positive roots. We let $r_i := s_{\beta_i}$, and let $\tilde{r}_i$ be the affine reflection in the hyperplane containing the common face of $A_{i-1}$ and $A_i$, for $i = 1, \ldots, m$; in other words, $\tilde{r}_i := s_{\beta_i} - l_i$, where $l_i := |\{j < i; \beta_j = \beta_i\}|$. We define $\tilde{l}_i := \langle \lambda, \beta_i^\vee \rangle - l_i = |\{j \geq i; \beta_j = \beta_i\}|$.

The objects of the quantum alcove model are defined next.

**Definition 2.3.** A subset $J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$ (possibly empty) is an *admissible subset* if we have the following path in the quantum Bruhat graph on $W$:

$$1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1}r_{j_2} \xrightarrow{\beta_{j_3}} \cdots \xrightarrow{\beta_{j_s}} r_{j_1}r_{j_2} \cdots r_{j_s}.$$  

(6) We denote the collection of admissible subsets of $\Gamma$ by $\mathcal{A}(\Gamma)$. When $\Gamma$ is fixed we will denote this set $\mathcal{A}(\mu)$.

We will use $J$, positions in the $\Gamma$-chain, and $T = (\beta_{j_1}, \beta_{j_2}, \cdots, \beta_{j_s})$, the corresponding sequence of roots interchangeably. Each admissible subset $J = \{j_1 < j_2 < \cdots < j_s\}$ can be identified with the sequence of Weyl group elements (6) which is denoted $\pi(J)$.

**Remark 2.4.** If we restrict to admissible subsets for which the path (6) has no down steps, we recover the classical alcove model in [14, 15].
Theorem 2.5. [12, 7] Consider a composition $p = (p_1, \ldots, p_k)$ and the corresponding crystal $B := \bigotimes_{i=1}^{k} B^{p_i}$. Let $\lambda := \omega_{p_1} + \ldots + \omega_{p_k}$, and let $\Gamma$ be a $\lambda$-chain. Then the (combinatorial) crystal $A(\Gamma)$ is isomorphic to the subgraph of $B$ consisting of the dual Demazure arrows, via a specific bijection.

In this thesis, we will have $p = (p_1 \geq \ldots \geq p_k)$ a permutation and we will select type specific $\lambda$-chains, $\Gamma$.

2.4 Type $A_{n-1}$

We begin with the essential facts about the type $A_{n-1}$ root system. We have $\mathfrak{h}^*_\mathbb{R} = V := \mathbb{R}^n/\mathbb{R}(1, \ldots, 1)$. The root system is $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j : i \neq j, 1 \leq i, j \leq n\}$ where $\varepsilon_1, \ldots, \varepsilon_n \in V$ are the images of the coordinate vectors in $\mathbb{R}^n$. The simple roots are $\alpha_i = \alpha_{i,i+1}$, for $i = 1, \ldots, n-1$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1, \ldots, n-1$. A dominant weight $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0)$ of length at most $n - 1$. We denote the conjugate partition of $\mu$ by $\mu'$. Thus a Young diagram corresponding to the dominant weight $\mu$ can be viewed as a concatenation of columns with heights $\mu_1, \mu_2', \ldots$ which corresponds to expressing $\mu$ as $\mu = \omega_{\mu_1'} + \omega_{\mu_2'} + \ldots$. Recall that $\rho = \sum \omega_i$, so $\rho = (n-1, n-2, \ldots, 0)$. A column $C$ of type $A$ is a Young diagram of column shape (corresponding to the partition $\mu = (1, 1, \ldots, 1, 1)$) filled increasingly by letters of $A_n = \{1 < 2 < \ldots < n\}$.

The Weyl group $W$ is the symmetric group $S_n$ which acts on $V$ by permuting the coordinates $\varepsilon_1, \ldots, \varepsilon_n$. Permutations $w \in S_n$ are represented in one-line notation: $w = w(1) \ldots w(n)$. We use $(i, j)$ with $1 \leq i, j \leq n$ to mean the root $\alpha_{ij}$ as well as the reflection $s_{\alpha_{ij}} = t_{ij}$. To describe the quantum Bruhat graph (QBG) on $W = S_n$ we need only provide a criterion for when an edge is in the graph. This criterion depends on the circular order $<_i$ on $[n]$ starting at $i$, defined by

$$i <_i i + 1 <_i \ldots <_i n <_i 1 <_i \ldots <_i i - 1.$$ 

To visualize the circular order it is convenient to imagine the numbers $1, \ldots, n$ arranged clockwise on a circle. If $i$ is not specifically indicated then $a < b < c$ means the circular order starting at $a$.

Proposition 2.6. [9] We have an edge in the quantum Bruhat graph $w \xrightarrow{(i,j)} w(i,j)$ if and only if there is no $k$ such that $i < k < j$ and $w(i) < w(k) < w(j)$. 

6
Next we discuss the KR crystal and the tableaux model for type $A$. We let $B^{k,1}$ be the traditional notation for the type $A_{n-1}$ KR crystal indexed by a column of height $k$, whose vertices are indexed by increasing fillings of the mentioned column of integers in $[n]$. Recall that for a partition corresponding to a dominant weight we define the KR crystal by

$$B^{\otimes \lambda} = \bigotimes_{i=1}^{1} B^{\lambda_i,1}$$

Via this definition, a vertex in the KR crystal is identified with a column strict filling of $\lambda$ with entries in $[n]$. We think of the Young diagram corresponding to $\lambda$ as a concatenation of columns $\sigma = C^1 \ldots C^{\mu_1}$ of heights $\mu'_1, \mu'_2, \ldots, \mu'_1$. The crystal operator $f_i$ acts on a column by changing $i$ to $i+1$ if there is an $i$ but no $i+1$ in the column and otherwise sends the column to $0$. $f_i$ can be then be extended to the tensor product by the tensor product rule [3].

Now we fix a $\mu$-chain in order to explore the type $A_{n-1}$ quantum alcove objects, $J \in A(\mu)$. We define the $\omega_k$-chain denoted by $\Gamma(k)$:

$$(k, k+1), \ldots, (k, n-1), (k, n), \ldots, (2, k+1), \ldots, (2, n-1), (2, n), (1, k+1), \ldots, (1, n-1), (1, n)$$

We let $\Gamma^\mu$ be the concatenation of $\omega_k$-chains, $\Gamma^\mu := \Gamma(\mu'_1) \Gamma(\mu'_2) \ldots$.

For $\Gamma^\mu = (\beta_1, \beta_2, \ldots)$, fix $J = \{j_1 < j_2 < \ldots < j_s\}$ an admissible subset according to definition 2.3 and we let $T = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_s})$ be the corresponding sequence of reflections. We will use $J$ and $T$ interchangeably. For notation purposes we separate $J$ and $T$ into factors by single bars, induced by the factorization of $\Gamma^\mu$: $J = J(\mu'_1)J(\mu'_2)\ldots$ and $T = T(\mu'_1)T(\mu'_2)\ldots$.

We now associate to $T$ a filling of the Young diagram $\mu$. Consider the permutations

$$\pi^j := T(\mu'_1)T(\mu'_2)\ldots T(\mu'_j), \quad j = 1, \ldots, \mu_1.$$ 

Here, as opposed to above, $T(\mu'_1)T(\mu'_2)\ldots T(\mu'_j)$ denotes the permutation obtained by multiplying left to right the transpositions in the sequence $T$ above.

The filling map is the map $fill$ from $J$ (i.e. $T$) in $A(\Gamma^\mu)$ to fillings $fill(J) = C^{\mu'_1}C^{\mu'_2}\ldots$ of shape $\mu$ defined by

$$C^{\mu'_i} := \pi^i[1, \mu'_i], \quad \text{for } i = 1, \ldots, \mu_1$$ 

(7)
Example 2.7. [9] Consider \( n = 4 \) and \( \mu = (3, 2, 1, 0) \), for which we have the following \( \mu \)-chain:

\[
\Gamma^{\mu} = \Gamma(3)\Gamma(2)\Gamma(1) = ((3, 4), (2, 4), (1, 4) | (2, 3), (2, 4), (1, 4), (1, 3), (1, 4) | (1, 2), (1, 3), (1, 4)) . 
\] (8)

Here the splitting of \( \Gamma^{\mu} \) into \( \Gamma(j) \) is shown by bars. In order to visualize this \( \mu \)-chain, let us represent the Young diagram of \( \mu \) inside a broken 3 \( \times \) 4 rectangle, with columns from longest to shortest (left to right), as shown below. In this way, a transposition \((i, j)\) in \( \Gamma^{\mu} \) can be viewed as swapping entries in the two parts of each column (in rows \( i \) and \( j \), where the row numbers are also indicated below).

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 \\
3 \\
\end{array}
\begin{array}{ccc}
2 \\
3 & 3 \\
4 & 4 & 4
\end{array}
\]

The underlined positions in (8) correspond to \( J = \{1, 2, 4, 5, 8\} \in \mathcal{A}(\Gamma^{\mu}) \). Thus, we have

\[
T = T(3)T(2)T(1) = ((3, 4), (2, 4) | (2, 3), (2, 4) | (1, 2)) .
\]

The corresponding chain is the following, where the swapped entries are shown in bold:

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & 4 & 4 & 2 \\
3 & < & 4 & < & 4 & < & > \\
4 & 3 & 2 & 2 & 4 & 4 & 4
\end{array}
\]

So we have the following filling of \( \mu \):

\[
\text{fill}(J) = \begin{array}{ccc}
1 & 1 & 2 \\
3 & 2 \\
4
\end{array}
\]

Consider the following map:

\[
\mathcal{A}(\mu) \xrightarrow{\text{fill}} \mathcal{F}^{-\mu} \xrightarrow{\text{sort}} B^{\otimes \mu}
\]

8
Here, we use $\mathcal{F}^\mu$ to denote the set of fillings of shape $\mu$, the map $\text{sort}$ means rewriting columns to be increasing and $s\text{fill}$ denotes the composite map $\text{sort} \circ \text{fill}$.

**Theorem 2.8.** [9, 10] The map $s\text{fill}$ is an affine crystal isomorphism between $\mathcal{A}(\Gamma)$ and a certain subgraph of the crystal $B^\otimes\lambda$.

The inverse map of $\text{fill}$ is based on two main steps: the reordering of columns and the greedy algorithm.

Let $CC'$ be a pair of adjacent columns. We will define $\text{reorder}_C(C') = D'$ one row at a time as follows:

**Definition 2.9.** $D'$ is the unique reordering of $C'$ satisfying

$$D'(i) = \min\{D'(l) : i \leq l \leq \#C'\} \quad (9)$$

for each $1 \leq i \leq \#C'$ where the minimum is with respect to the circular order $\prec_{C(i)}$ on $[n]$ starting at $C(i)$.

**Definition 2.10.** Let $C = C_1C_2\ldots C_{\mu_1}$ be a filled Young diagram of shape $\mu$. We define $\text{reorder}$ on $T$ as follows:

$$\text{reorder}(C) = C_1C_2'\ldots C_{\mu_1}' \quad \text{where} \quad C_2' := \text{reorder}_{C_1}(C_2), \quad C_i' := \text{reorder}_{C_{i-1}}(C_i) \quad (10)$$

**Example 2.11.** Given $\tau \in B_\mu$ we find $\sigma = \text{reorder}(\tau)$:

$\tau = \begin{array}{cccc}
3 & 2 & 1 & 2 \\
5 & 3 & 2 & \\
6 & 4 & 4 & \\
\end{array}$, \quad $\sigma = \begin{array}{cccc}
3 & 3 & 4 & 2 \\
5 & 2 & 2 & \\
6 & 4 & 1 & \\
\end{array}$.

We now construct the corresponding path in the quantum Bruhat graph based on a greedy procedure. Let $DD'$ be a pair of adjacent columns in the filling of a reordered Young diagram of shape $\mu$ with $\#D = l \leq \#D' = k$. We construct a chain from some $w$ with $w[1,l] = D$ for some $v$ with $v[1,k] = D'$. Note that $w = \text{id}$ for the first column of the filling. The sequence of edge labels in this chain is constructed of $((i,m_1),\ldots,(i,m_p))$ for $i = k,k-1,\ldots,1$ and is given by the procedure $\text{path-A}$. The algorithm is called with the parameters $u, i$, and $d = D'(i)$ and $L$ where $u$ is the starting permutation of this part of the chain and $L$ is the list of positions $(k+1,\ldots,n)$. The function $\text{next}(m,L)$ determines the successor of the element $m$ in the list $L$. 
Algorithm 2.12.

procedure path-A(u, i, d, L);
if u(i) = d then return \(\emptyset, u\)
let \(S := \emptyset, m := L(1), v := u\);
while \(v(m) \neq d\) do
if \(v(i) \prec v(m) \prec d\) then let \(S := S, (i, m), v := v(i, m)\);
end if;
let \(m := \text{next}(m, L)\);
end while;
let \(S := S, (i, m), v := v(i, m)\);
return \((S, v)\);
end if;
end.

Example 2.13. We continue example 2.7 above by constructing the path for

\[
C = \begin{array}{ccc}
1 & 1 & 2 \\
3 & 2 \\
4
\end{array}
\]

by calling path-A for each column and each row until we achieve the desired column.

We begin with the first column with list \(L = (4)\) for rows \(i = 3, 2, 1\):

path-A(id, 3, 4, (4)) = ((3, 4), 1243)
path-A(1243, 2, 3, (4)) = ((2, 4), 1342)

Note that it is not necessary to call path-A for \(i = 1\).

For the second column, we call path-A with list \(L = (3, 4)\) for rows \(i = 2, 1\):

path-A(1342, 2, 2, (3, 4)) = (((2, 3), (2, 4)), 1234)

For the third column, we call path-A with list \(L = (2, 3, 4)\) for row \(i = 1\):

path-A(1234, 1, 2, (2, 3, 4)) = ((1, 2), 2134)

The full quantum Bruhat graph path is

\[ T = ((3, 4), (2, 4)|(2, 3), (2, 4)|(1, 2)) \]

which is exactly the path we started with in example 2.7.
2.5 Type $C_n$

We begin with the essential facts about the type $C_n$ root system. We have $h^*_R = V := \mathbb{R}^n$, with coordinate vectors $\varepsilon_1, \ldots, \varepsilon_n$. The root system is $\Phi = \{\pm \varepsilon_i \pm \varepsilon_j : i \neq j, 1 \leq i, j \leq n\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq n\}$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, n - 1$ and $\alpha_n = 2\varepsilon_n$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1, \ldots, n$. A dominant weight $\mu = \mu_1 \varepsilon_1 + \ldots + \mu_n - 1 \varepsilon_{n-1}$ is identified with the partition $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n \geq 0)$ of length at most $n$. As in Type $A$, the Young diagram corresponding to the dominant weight $\mu$ can be viewed as a concatenation of columns of heights $\mu_1', \mu_2', \ldots$ which corresponds to expressing $\mu$ as a sum of fundamental weights.

We fix a dominant weight, $\mu$. Recall that $B^\otimes \mu$ is a tensor product of KR crystals corresponding to the columns of $\mu$ and $B^{r,1} \cong B(\omega_r)$ as classical crystals. Here $B(\omega_r)$, the fundamental crystal, is realized by type $C$ Kashiwara-Nakashima (KN) columns of height $r$ [4]. So KR crystals are realized by the corresponding KN columns and so a tensor product of KR crystals is a sequence of KN columns of various heights.

We now describe the type $C$ KN columns. A column of type $C_n$ is a Young diagram of column shape filled by letters of $[\bar{n}]$ such that $C$ is increasing.

**Definition 2.14.** [4] Let $C$ be a column of type $C_n$ and denote by $I_C = \{z_1 \ldots \succ z_s\}$ the set of letters $z \preceq n$ such that the pair $(z, \overline{z})$ occurs in $C$. We will say that $C$ can be split or will be called admissible when there exists a set of $s$ unbarred letters $J_C = \{t_1 \succ \ldots \succ t_s\} \subset [\bar{n}]$ such that:

- $t_1$ is the greatest letter of $[\bar{n}]$ satisfying: $t_1 \prec z_1, t_1 \not\in C$ and $\overline{t_1} \not\in C$,
- $t_i$ is the greatest letter of $[\bar{n}]$, for $i = 2, \ldots, s$, satisfying: $t_i \prec \min (t_{i-1}, z_i), t_i \not\in C, \overline{t_i} \not\in C$.

In this case we write:

- $rC$ for the column obtained by changing $\overline{z_i}$ to $t_i$ in $C$ for each letter $z_i \in I$, then reordering if necessary.
• \( lC \) for the column obtained by changing \( z_i \) to \( t_i \) in \( C \) for each letter \( z_i \in I \), then reordering if necessary.

The pair \( (lC, rC) \) will be called a split column. The KN columns are precisely the admissible columns.

**Example 2.15.** The following is a KN column of type \( C_6 \) with its corresponding split columns.\[
C = \begin{array}{c}
4 \\
5 \\
\text{5}
\end{array}, \quad (lC, rC) = \begin{array}{cc}
4 & 1 \\
5 & 2 \\
\text{3} & \text{5}
\end{array}.
\]

The following is NOT a KN-column.\[
C = \begin{array}{c}
2 \\
5 \\
\text{5}
\end{array}.
\]

Note that in the first column, the pairs \((4, \text{4})\) and \((5, \text{5})\) must be separated during the splitting process and this can be done because the smaller values 1 and 2 are available to use. However, in the second column, we must separate the pair \((2, \text{2})\) in the splitting process but there is no smaller value available to use so splitting is not possible.

The filling map from quantum alcove model objects to sequences of KN columns is defined similarly to type \( A \). We also use a similar reordering and a greedy algorithm to construct the inverse of the filling map.
2.6  Type $B_n$ setup

2.6.1  Root system & Weyl group

We begin with the essential facts about the type $B_n$ root system. As in type $C_n$, we have $h^*_R = V := \mathbb{R}^n$, with coordinate vectors $\varepsilon_1, \ldots, \varepsilon_n$. The roots are

$$\Phi = \{ \pm \varepsilon_i \pm \varepsilon_j | 1 \leq j \neq i \leq n \} \cup \{ \pm \varepsilon_i | 1 \leq i \leq n \}. \quad (11)$$

The positive roots are

$$\Phi^+ = \{ \varepsilon_i \pm \varepsilon_j | 1 \leq j \neq i \leq n \} \cup \{ \varepsilon_i | 1 \leq i \leq n \}. \quad (12)$$

We denote by $s_\alpha$ the reflection corresponding to the root $\alpha$. For the root $\alpha = \varepsilon_i - \varepsilon_j$ we have $s_\alpha = t_{ij}t_{\bar{j}}$ which we abbreviate as $(i, j)$. For the root $\alpha = \varepsilon_i + \varepsilon_j$ we have $s_\alpha = t_{\bar{i}}t_{\bar{j}}$ which we abbreviate as $(\bar{i}, \bar{j})$. For the root $\alpha = \varepsilon_i$ we have $s_\alpha = t_i$ which we abbreviate as $(i, i)$.

The fundamental weights are

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } i < n, \quad \omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n). \quad (13)$$

The corresponding $\rho$ vector is

$$\rho = \left( n - \frac{1}{2} \right) \varepsilon_1 + \left( n - \frac{3}{2} \right) \varepsilon_2 + \cdots + \frac{1}{2} \varepsilon_n. \quad (14)$$

The Weyl group $W$ is the group of signed permutations $B_n$, which acts on $V$ by permuting the coordinates $\varepsilon_1, \ldots, \varepsilon_n$ and changing their signs. A signed permutation is a bijection $w$ from $[\bar{n}] := \{ 1 < 2 < \ldots < n < \bar{1} < \ldots < \bar{2} < \bar{1} \}$ with $w(\bar{i}) = \overline{w(i)}$. We view $i$ as $-i$ so $\text{sgn}(\bar{i}) = -1$. Signed permutations are determined by $w = w(1) \ldots w(n)$ which we call the window notation but there are also times when it will be useful to write the whole signed permutation $w = w(1) \ldots w(n)w(\bar{1}) \ldots w(\bar{n})$ which we call the full one-line notation. The length of an element $w$ in $B_n$ is given by

$$\ell(w) = \# \{(k, l) \in [n] \times [\bar{n}] : k \leq |l|, w(k) > w(l)\}. \quad (15)$$

2.6.2  Crystals and tableau models

In this thesis we will restrict to $\mu$ which are positive integer combinations of fundamental weights $\omega_1, \ldots, \omega_{n-1}$. This means that $\mu$ is represented as a partition of at most $n - 1$ rows. Recall that
$B \otimes \mu$ is a tensor product of KR crystals corresponding to the columns of $\mu$. As opposed to type $C$, the KR crystal $B^{r,1}$ of type $B$ decomposes as a classical crystal into a direct sum

$$B^{r,1} \cong B(\omega_r) \oplus B(\omega_{r-2}) \oplus \ldots$$

where $B(\omega_r)$ is the fundamental crystal. Like type $C$, $B(\omega_r)$ is realized by type $B$ KN columns of height $r$ [4]. By combining this information, we can realize tensor products of KR crystals as a sequence of type $B$ KN columns of various heights.

A column of type $B$ is a Young diagram

\[
C = \begin{array}{c}
x_1 \\
\vdots \\
x_k
\end{array}
\]

of column shape filled by letters of $\mathcal{B}_n = \{1 < 2 < \ldots < n < 0 < \pi < \ldots < 2 < 1\}$ such that $C$ is increasing and $0$ is the only letter that can be repeated. This is similar to the definition of type $C$ columns except for the addition of the repeatable letter 0.

**Definition 2.16.** We define KN columns of type $B$ as in Definition 2.14 with the added condition that any occurrence of the letter 0 in the column is treated as a pair $(z, \bar{z})$. So, $I_C = \{z_1 = 0, \ldots, z_r = 0 > z_{r+1} > \ldots > z_s\}$ and everything else is the same.

If we are given a split column $(lC, rC)$ we can recover $C$ by the map $\text{merge}(lC, rC)$ by considering the positive entries of $rC$, denoted $rC_+$, and the negative entries of $lC$, denoted $lC_-$. Then $C$ consists of the entries $lC_-, rC_+$ and $v$ zeros where $v = \#|C| - |lC_-| - |rC_+|$.

As in Type $C_n$, we will write $KN_i$ to denote the set of KN columns of height $i$ and $KN_i$ to denote the set of split KN columns of height $i$.

**Example 2.17.** The following is a KN column in type $B_8$, together with its corresponding split column:

$$A = \begin{pmatrix} 5 \\ 0 \\ 8 \\ 5 \\ \bar{2} \end{pmatrix}, \quad (lA, rA) = \begin{pmatrix} 4 & 5 \\ 7 & \bar{8} \\ 8 & 7 \\ 5 & \bar{4} \\ \bar{2} & \bar{2} \end{pmatrix}.$$
We used the fact that

\[ |A| = \{2, 5, 8, 0\} \quad I_A \{0 > 5\} \quad J_A = \{8 > 4\}. \]

Observe that \( \text{merge}(lA, rA) \) recovers \( A \).

### 2.6.3 Quantum Bruhat graph criteria for type \( B_n \)

In this section, we describe the criteria for the edges of the quantum Bruhat graph for type \( B_n \).

The circular order \( \prec_i \) on \([n]\) starting at \( i \) in \([n]\) means

\[
i \prec_i i + 1 \prec_i \ldots \prec_i 1 \prec_i \ldots \prec_i i - 1.
\]

It is convenient to think of this order in terms of the numbers 1, \( \ldots \), \( \bar{1} \) arranged clockwise on a circle. We will also need \( N_{ab}(u) \) which denotes the number of letters of the word \( u \) which are strictly between \( a \) and \( b \).

**Fact 2.18.** Given a permutation \( w \) in \( B_n \) we have the following three facts:

1. Given \( 1 \leq i < j \leq n \) such that \( a := w(i) < b := w(j) \), we have

\[
\ell(w(i, j)) - \ell(w) = 2N_{ab}(w[i, j]) + 1
\]

2. Given \( 1 \leq i < j \leq n \) such that \( a := w(i) < b := w(\bar{j}) \), we have

\[
\ell(w(i, \bar{j})) - \ell(w) = 2N_{ab}(w[i, j - 1]) + 2N_{ab}(w[j + 1, \bar{j}]) + 2\delta_{\operatorname{sgn}(a), -\operatorname{sgn}(b)} + 1
\]

where \( \delta_{k,t} \) is the Kronecker delta.

3. Given \( 1 \leq i \leq n \) such that \( a := w(i) \in [n] \), we have

\[
\ell(w(i, \bar{i})) - \ell(w) = 2N_{a\bar{i}}(w[i, n]) + 1
\]

**Proposition 2.19.** Let \( 1 \leq i < j \leq n \)

1. We have an up edge \( w \xrightarrow{(i, j)} w(i, j) \) if and only if there is no \( k \) such that \( i < k < j \) and \( w(i) < w(k) < w(j) \).

2. We have a down edge \( w \xrightarrow{(i, j)} w(i, j) \) if and only if there is no \( k \) such that \( i < k < j \) and \( w(i) < w(k) < w(j) \).
3. We have an up edge, $w^{(i,j)} w(i,j)$ if and only if $\text{sgn}(w(i)) = \text{sgn}(w(j))$ with $w(i) < w(j)$ and there is no $k$ such that $i < k < j$ and $w(i) < w(k) < w(j)$.

4. We have a down edge, $w^{(i,j)} w(i,j)$ if and only if $\text{sgn}(w(i)) \neq \text{sgn}(w(j))$ with $w(i) > w(j)$ and there is no $k$ such that $i < k \neq j \leq n$ and $w(i) < w(k) < w(j)$.

5. We have an up edge $w^{(i,\bar{i})} w(i,\bar{i})$ if and only if there is no $k$ such that $i < k < \bar{i}$ and $w(i) < w(k) < w(\bar{i})$.

6. We have a down edge $w^{(i,\bar{i})} w(i,\bar{i})$ if and only if $w(i) > w(\bar{i})$ and $i = n$.

Note that we will never have an $(i,\bar{i})$ down edges because we always have $i < n$.

Proof. Note that up edges are the same as in type $C$ and so is the down edge for $(i,j)$ because $\langle \rho, \alpha^\vee \rangle$ is the same in type $B$. So the down edges for $(i,j)$ and $(i,\bar{i})$ are left.

Let $a := w(i) > b := w(j)$. For $\alpha = \epsilon_i + \epsilon_j$, we have $\langle \rho, \alpha^\vee \rangle = 2(n - i) + 1$ so $2\langle \rho, \alpha^\vee \rangle - 1 = 4n - 4i + 1$. Also,

$$\ell(w(i,\bar{i})) - \ell(w) = 2N_{\alpha\pi}(w[i,j - 1]) + 2N_{\alpha b}(w[j + 1,\bar{j}]) + 2\delta_{\text{sgn}(a),-\text{sgn}(b)} + 1$$

$$\leq 2(j - i - 1) + 4(n - j) + 2 + 1 = 4n - 2j - 2i + 1.$$

So $\ell(w(i,\bar{j})) - \ell(w) = 2\langle \rho, \alpha^\vee \rangle - 1$ only when $2N_{\alpha b}(w[i,j - 1]) + 2N_{\alpha b}(w[j + 1,\bar{j}]) + 2\delta_{\text{sgn}(a),-\text{sgn}(b)} + 1$ is maximized. This requires $a$ and $b$ to be opposite sign and since $a > b$ we must have that $a$ is negative and $b$ is positive. Additionally, we must have that $w(i) < w(k) < w(j)$ for $k \in \{i + 1, \ldots, j - 1, j + 1, \ldots, \bar{j} + 1\}$.

Let $a := w(i)$. For $\alpha = \epsilon_i$, we have $\langle \rho, \alpha^\vee \rangle = 4(n - i) + 1$ and $\ell(w(i,\bar{i})) - \ell(w) \leq 2(n - i) + 1$. So $2\langle \rho, \alpha^\vee \rangle - 1 = \ell(w(i,\bar{i})) - \ell(w)$ if and only if $n = i$. 

\qed
3 Combinatorial Data

In this section, we fix a \(\Gamma\)-chain in order to explore \(A(\mu)\), we define the filling map associated to \(J \in A(\mu)\) and the various maps between sets defined by the KN columns of type \(B\). This Combinatorial Data will be needed to state the main results relating the objects of the alcove model to the fillings of Young diagrams.

3.1 Quantum alcove model objects

Fix \(k\) with \(1 \leq k \leq n - 1\). The following is an \(\omega_k\)-chain. [8]

\[
\Gamma(k) := \Gamma_l(k) \Gamma_r(k),
\]

where

\[
\Gamma_l(k) := \Gamma_{kk} \ldots \Gamma_{k1}, \quad \Gamma_r(k) := \Gamma_{k} \Gamma_{k-1} \ldots \Gamma_1,
\]

\[
\Gamma_{ki} := ((i,k+1), (i,k+2), \ldots, (i,n), (i,\bar{i}), (i,\bar{n}), (i,\bar{n}-1), \ldots, (i,\bar{k}+1), (i,\bar{i}-1), (i,\bar{i}-2), \ldots, (i,\bar{1})).
\]

\[
\Gamma_i := ((i,\bar{i}),(i,\bar{i}-1),(i,\bar{i}-2),\ldots,(i,\bar{1}))
\]

We construct a \(\mu\)-chain by concatenation: \(\Gamma^\mu := \Gamma(\mu'_1) \Gamma(\mu'_2) \ldots\). As in Definition 2.3, the admissible subsets of \(\Gamma\) are denoted \(A(\mu)\). When we write \(\Gamma^\mu\) as a concatenation of transpositions we use a double bar between each \(\Gamma(\mu'_i)\) and single bars between the right and left chains \(\Gamma_l(\mu'_i) \& \Gamma_r(\mu'_i)\) within each \(\Gamma(\mu'_i)\).

We will refer to the four stages of \(\Gamma_{ki}\) as follows:

Stage I: \((i,k+1), (i,k+2), \ldots, (i,n)\)

Stage II: \((i)\)

Stage III: \((i,\bar{n}), (i,\bar{n}-1), \ldots, (i,\bar{k}+1)\)

Stage IV: \((i,\bar{i}-1), (i,\bar{i}-2), \ldots, (i,\bar{1})\)

Note that \(\Gamma_i\) consists of only stages II and IV, in that order.
Example 3.1. Let $n = 4$ and $\mu = (2, 2, 1, 0)$. So $\mu' = (3, 2)$.

$$\Gamma^\mu = \Gamma(3)\Gamma(2)$$

$$= ((3, 4), (3, 3), (3, 1), (3, 2), (2, 4), (2, 3), (2, 2), (2, 1), (1, 4), (1, 3), (1, 2), (1, 1))$$

3.2 Filling map

The filling map in type $B$ is defined similarly to the filling maps in types $A$ and $C$. For each $J \in A(\mu) = A(\Gamma^\mu)$ we associate a filling of the Young diagram of shape $2\mu$ which we view as a concatenation of columns, $\sigma = C_i^\mu_1 C_i^\mu_2 C_i^\mu_3 \cdots$ where $C^\mu_i$ and $C^\mu_j$ have height $\mu'_j$.

Recall that for every $J \in A(\mu)$ there is a corresponding sequence of reflections, $T$, viewed as a concatenation of transpositions and that there is a factorization of $T$ induced by the factorization of $\Gamma$. Given $T = T_l(\mu'_1)T_r(\mu'_2)T_l(\mu'_2)T_r(\mu'_2) \cdots$ we consider the permutations

$$\pi^1_1 := T_l(\mu'_1) \quad \pi^1_r := T_r(\mu'_1)$$

$$\pi^i_l := \pi^{i-1}_l T_l(\mu'_1) \quad \pi^i_r := \pi^{i-1}_r T_r(\mu'_1)$$

Definition 3.2. The filling map is the map fill from admissible chains $J$ in $A(\mu)$ to fillings

$\text{fill}(J) = C_i^\mu_1 C_i^\mu_2 C_i^\mu_3 \cdots$ of the shape $2\mu$ defined by

$$C_i^\mu_j := \pi^1_i [1, \mu'_1], \quad C_i^\mu_j := \pi^i_r [1, \mu'_1], \quad \text{for } i = 1, \ldots, \mu_1. \quad (16)$$

We use $F^\mu$ to denote the set of fillings of shape $2\mu$: $A(\mu) \xrightarrow{\text{fill}} F^\mu$.

Example 3.3. The underlined positions in Example 3.1 correspond to

$$J = \{1, 2, 3, 4, 5, 6, 7, 8, 13, 14, 19, 20, 21, 22, 31\} \in A(\mu).$$

Thus we have

$$T = T(3)T(2) = ((3, 4), (3, 3), (3, 1), (3, 2), (2, 4), (2, 3), (2, 2), (2, 1), (1, 4), (1, 3), (1, 2), (1, 1))$$

$$\quad (2, 3), (2, 4), (2, 2), (2, 1), (1, 1))$$
So we have the following filling of $2\mu$:

\[
\begin{array}{cccc}
2 & 2 & 2 & 3 \\
3 & 1 & 3 & 2 \\
1 & 3
\end{array}
\]

3.3 Fillings and related maps

In this section we discuss sets defined using KN columns of type $B_n$. The goal is to define the set of KN fillings of shape $2\mu$. Recall that $KN_i$ is the set of KN columns of height $i$ and $KN_i^+$ is the set of split KN columns of height $i$.

Here we define the set $KN^{\otimes \mu}$, KN fillings of Young diagrams of shape $\lambda$ where $\lambda$ is a sub-permutation of $\mu$ where each of its columns $\lambda'_i$ can be shorter than $\mu'_i$ by a multiple of two with $1 \leq i \leq \mu'_1$.

\[
KN^{\oplus \mu} := KN_k \oplus KN_{k-2} \oplus KN_{k-4} \oplus \ldots
\]

\[
KN^{\otimes \mu} := KN^{\oplus \mu}_{\mu'_1} \otimes KN^{\oplus \mu}_{\mu'_2} \otimes \ldots
\]

We can similarly define $KN^{\otimes \mu}$, the set of split KN fillings of Young diagrams of shape $2\lambda$.

\[
KN_{\oplus k} := KN_k \oplus KN_{k-2} \oplus KN_{k-4} \oplus \ldots
\]

\[
KN^{\otimes \mu} := KN_{\oplus \mu'_1} \otimes KN_{\oplus \mu'_2} \otimes \ldots
\]

Let $CC' \in KN_{\oplus k}$. We define $\text{extend}(CC')$ by choosing the smallest possible "inadmissible" pair, adding the negative value to the left column and the positive value to the right column until both columns are height $k$. Note that when we use a set and a column interchangeably, we mean the increasing column.

**Definition 3.4.** Let $\{x_1 < x_2 < \ldots < x_p : x_i \in [n] \setminus |C|\}$, where $p = k - |C|$. Then we define $\text{extend}(CC') = DD'$ where $D = C \cup \{x_1, \ldots, x_p\}$ and $D' = C' \cup \{x_1, \ldots, x_p\}$.

Now we can define the set of split and extended KN fillings.

\[
KN_{\oplus k} := \text{extend}(KN_{\oplus k})
\]

\[
KN^{\otimes \mu} := KN^{\oplus \mu}_{\mu'_1} \otimes KN^{\mu'_2} \otimes \ldots
\]
Recall the definition of reorder on a column in Definition 2.9 and on a Young diagram in Definition 2.10, where the minimum in 2.9 is now taken with respect to circular order on the alphabet \([n]\). Also, recall the definition of sort\((C)\) to mean sorting the column increasingly.

The following condition on a pair of adjacent columns \(CC'\) in a filling of a Young diagram is equivalent to Definition 2.9 [9].

**Condition 3.5.** For any pair of indices \(1 \leq i < l \leq \#C'\), both statements below are false:

\[
C(i) = C'(l), \quad (19)
\]

\[
C(i) \prec C'(l) \prec C'(i). \quad (20)
\]

where \(\prec = C(i)\) is the circular order on \([n]\) starting at \(C(i)\).

The inverse of the split map is merge as constructed in 2.6.2.

**Definition 3.6.** Let

\[
\text{contract} := \text{extend}^{-1}
\]

We summarize the above definitions of sets and maps:

\[
\begin{align*}
\mathcal{F}_\mu & \xrightarrow{\text{reorder}} K \otimes \mu \xrightarrow{\text{extend}} K \otimes \mu \xrightarrow{\text{split}} K \otimes \mu \\
\mathcal{A}(\mu) & \xrightarrow{\text{fill}} \mathcal{F}_\mu \xrightarrow{\text{sort}} K \otimes \mu \xrightarrow{\text{contract}} K \otimes \mu \xrightarrow{\text{merge}} K \otimes \mu \xrightarrow{} B \otimes \mu
\end{align*}
\]

(21)

(22)
4 Summary of Main Results

In this section we state the main Conjectures and Theorems.

Recall the following notation. Given a partition $\mu = (\mu_1 \geq \mu_2 \geq \ldots)$ then $B^\otimes \mu = B^{\mu_1,1} \otimes B^{\mu_2,1} \otimes \ldots$ is the KR crystal. $A(\Gamma^\mu)$ is the set of paths in the quantum Bruhat graph compatible with $\Gamma$ starting with the word $id$ and we use $A(\Gamma^\mu, w)$ to denote the chains in the quantum Bruhat graph, compatible with $\Gamma$ starting with the word $w$.

The following conjecture is the main conjecture.

**Conjecture 4.1.** Let $\mu = (\mu_1 \geq \mu_2 \geq \ldots)$. The maps defined in (21) and (22) in Section 3 are bijections.

Conjecture 4.1 is based on the following two conjectures which address the sequences of roots $\Gamma_r$ and $\Gamma_l$ separately.

The following conjecture refers to the $\Gamma_r$ chain which will correspond to a filling from a left column to its right column. Recall that $W^{\omega_k}$ is the lowest coset representative modulo the stabiliser of $\omega_k$ and $W_{\omega_k}$ is the stabiliser of the $k$th fundamental weight.

**Conjecture 4.2.** Fix $u \in W_{\omega_k}$. Then the filling map gives a bijection from $\bigcup_{w \in W^{\omega_k}} A(\Gamma_r(k), w \cdot u)$ to $B^{k,1} \cong \widehat{KN}_k$.

The following conjecture refers to the $\Gamma_l$ chain which will correspond to a filling from a right column to the next left column.

**Conjecture 4.3.** Consider a signed permutation $w \in B_n, C = w[1, k]$ and another column $C'$ of height $k$. Then there is a unique path in $A(\Gamma_l(k), w)$ ending with some $u \in B_n$ such that $u[1, k]$ is some permutation of $C'$.

The following are the main results of this thesis.

This is the special case of Conjecture 4.2 where $u = id$.

**Theorem 4.4.** Let $B^{k,1} \cong B(\omega_k) \oplus B(\omega_{k-2}) \oplus \ldots = \widehat{KN}_k \bigsqcup \widehat{KN}_{k-2} \bigsqcup \ldots$. Then:

a) the map $\text{extend} \circ \text{split}$ from $\widehat{KN}_k \bigsqcup \widehat{KN}_{k-2} \bigsqcup \ldots$ to $\widehat{KN}_k$ defines a bijection.

b) the map $\text{sort} \circ \text{fill}$ from $\bigcup_{w \in W^{\omega_k}} A(\Gamma_r(k), w)$ to $\widehat{KN}_k$ defines a bijection.
This is an important part of Conjecture 4.3 where the path in \( A(\Gamma_i(k), w) \) has no stage IV down steps. Recall that stage IV consists of the transpositions switching position \( i \) with positions \( i - 1, \ldots, 1 \).

**Theorem 4.5.** Let \( w = w_0, w_1, \ldots, w_p = u \) be a path in \( A(\Gamma_i(k), w) \) with no down steps in stage IV of each \( \Gamma_{ki} \) with \( 1 \leq i \leq k \). Let \( C = w[1, k] \) and \( C' = u[1, k] \). Then \( CC' \) are related by Condition 3.5.

**Conjecture 4.6.** Moreover, there is no other path in \( A(\Gamma_i(k), w) \) ending at some permutation \( u' \) with \( u'[1, k] = C' \).


5  The First Part of the Quantum Bruhat Path: Proving Theorem 4.4

In the first part of this section, we prove Theorem 4.4 part a). The goal is to develop an intrinsic description of $\hat{KN}_k$, which is exactly the set of columns realizing $B^{k,1}$ combinatorially. In the second part of this section, we prove Theorem 4.4, part b) by exhibiting the connection between the intrinsic description we developed in part a) and the quantum Bruhat graph paths. We provide an algorithm for the construction of the mentioned paths. We then conclude that the maps in (22) define a bijection between the $????$ combinatorial models for the KR crystal $B^{r,1}$, namely the KN columns and the paths in the quantum Bruhat graph coming from the quantum alcove model.

5.1  The $extend \circ split$ map

Fix two columns $CC'$ of height $k$ with entries in $[\bar{n}]$. We will describe specific conditions on the pair of columns $CC'$. Then we will show the set of columns satisfying the conditions is exactly the image of the $extend \circ split$ map applied to $\hat{KN}_{\leq k}$, the set of KN columns of height $k$ or shorter by two. In this section, we will describe several ways to “match” values in a pair of columns. Note that any pair of columns $CC'$ of equal height can be thought of as a matching by matching $C(i)$ to $C'(i)$ and that any matching of values can be thought of as a pair of columns by ordering the first column increasingly and defining $C'(i)$ to be the value $C(i)$ was matched with.

We begin by restating Condition 3.5, the reordering condition.

Condition 0. For any pair of indices $1 \leq i < l \leq k$, both statements below are false:

$$C(i) = C'(l), \quad \text{(23)}$$

$$C(i) \prec C'(l) \prec C'(i). \quad \text{(24)}$$

where $\prec_{C(i)}$ is the circular order on $[\bar{n}]$ starting at $C(i)$.

We now state four additional conditions on the pair $CC'$.

Condition 1. We have

$$\{|C(i)| : i = 1, \ldots, k\} = \{|C'(i)| : i = 1, \ldots, k\}.$$
Definition 5.1.

\[ \text{int}(C, C') := \left( \bigcup_{i=1}^{k} \{ j \in [n] : C(i) < j < C'(i) \} \right) \setminus \{ \pm C(i) : i = 1, \ldots, k \} \]  

(25)

**Condition 2.** We have \( \text{int}(C, C') = \emptyset \).

**Condition 3.** If \( C(i) \) and \( C'(i) \) are the same sign then \( C(i) < C'(i) \). Additionally, there are an even number of entries where \( C(i) \) is negative and \( C'(i) \) is positive.

**Condition 4.** \( CC' \) has the following four region structure, \( C \) is increasing and \( C' \) is increasing in the first three regions and also increasing in last region.

\[
CC' = \begin{array}{c|c|c|c}
+ & + & \\
+ & - & \\
- & - & \\
- & + & \\
\end{array}
\]

For any value in either column, \( C(i) \) or \( C'(i) \), we will say that it is in region I if \( C(i) \) and \( C'(i) \) are positive, region II if \( C(i) \) is positive and \( C'(i) \) is negative, region III if \( C(i) \) and \( C'(i) \) are negative and region IV if \( C(i) \) is negative and \( C'(i) \) is positive.

The following theorem is a refinement of Theorem 4.4, part a). This intrinsic description of the set \( \hat{KN}_k \) is an intermediate step in showing its connection to the set of quantum Bruhat graph paths which is the subject of Theorem 4.4, part b).

**Theorem 5.2.** The set \( \hat{KN}_k \) can be described by the set of columns \( CC' \) satisfying Conditions 0, 1, 2, 3 and 4.

Moreover, the map \( \text{extend} \circ \text{split} \) is a bijection between \( \overline{KN} \oplus_k \) and \( \hat{KN}_k \).

### 5.1.1 Properties of split and extended columns

In order to describe \( \hat{KN}_k \), we will first define an “initial matching” on the set of columns \( \hat{KN}_k \) in 5.3 where each value of the left column is matched to a value of the corresponding right column.

In Lemma 5.5, we show this matching satisfies Conditions 1, 2 and 3. We then define a second matching between the values of the left and right columns in Definition 5.6, called the “corrected matching” and then show this matching still satisfies Conditions 1, 2 and 3 and now also satisfies Conditions 0 and 4.
Fix $A \in \overline{KN}_{\otimes k}$. Recall that the map *split* from Definition 2.16 splits $A$, a KN column of height $k$ or shorter by a factor of two, into a left column and a right column, $lA$ and $rA$. The map *extend* from Definition 3.4 extends both $lA$ and $rA$ to columns of full height $k$ called $\hat{l}A$ and $\hat{r}A$. When we refer to the split pair of columns and the extended pair of columns we mean the entries in both columns are sorted increasingly.

The process of splitting and extending describes an intuitive matching between the values of $\hat{l}A$ and the values of $\hat{r}A$ which we will call the initial matching. All the entries in the extended columns come from one of four scenarios which will be the main criteria we use to define the initial matching. Either the entry was not involved in the splitting or extending, it is one of the four values involved in splitting a non-zero pair, it is the result of splitting a zero or it was added during extending.

**Definition 5.3.** We define a new pair of columns $BB'$ which we will call the initial matching. Let $B := \hat{l}A$ and define $B'$ by matching each value in $B$ from top to bottom with a value in $\hat{r}A$ as follows:

*Case 1.* If $a$ was not involved with the splitting or extending process, match $a$ to itself.

*Case 2.* If $a \in [n]$ is non-zero and required splitting, match $b$ to $a$ and $\overline{a}$ to $\overline{b}$ where $b \in [n]$ is the value used to split $a$.

*Case 3.* If $a \in [n]$ is the result of a zero splitting, match $a$ with $\overline{a}$.

*Case 4.* If $a \in [n]$ is a result of extending, match $\overline{a}$ with $a$.

Note that the initial matching is a result of the splitting and extending process where the first column is increasing so we will refer to an initial matching and a KN column that has been split and extended interchangeably.

**Example 5.4.** We continue Example 2.17 with $n = 8$ and $k = 7$. 

25
A is a KN column together with its corresponding split column \((lA, rA)\):

\[
\begin{array}{c|c|c|c}
  & 5 & 0 & 4 \\
 8 & 7 & 8 & 8 \\
 5 & 5 & 4 & 5 \\
 2 & 2 & 2 & 2 \\
\end{array}
\]

\(\hat{lA, \hat{rA}}\) is the extended column of full height \(k\) and \(BB'\) is the initial matching with \(B(i)\) matched to \(B'(i)\):

\[
\begin{array}{c|c|c|c}
  & 4 & 1 & 4 \\
 7 & 3 & 7 & 7 \\
 8 & 5 & 8 & 8 \\
 5 & 8 & 5 & 4 \\
 3 & 7 & 3 & 3 \\
 2 & 4 & 2 & 2 \\
 1 & 2 & 1 & 1 \\
\end{array}
\]

**Lemma 5.5.** Any initial matching satisfies Conditions 1, 2 and 3.

*Proof.* Note that Condition 1 is clearly satisfied because whenever a value is added to a column during splitting or extending its negative is added to the other column. Condition 2 is satisfied because of the tightness of the splitting and extending procedures. In case 1 we are matching \(a\) to itself so the matching does not have any values in between. Case 2 originates from splitting a non-zero entry \(a\) in which we choose the largest value not in the column with \(b < a\). So all the values between \(b\) and \(a\) are already in the columns and similarly for \(\overline{a}\) and \(\overline{b}\). Case 3 comes from splitting a zero entry in which we choose the largest possible value not in the column, \(a\). When we match \(a\) to \(\overline{a}\) the only values in between are already in the columns. Case 4 comes from extending a column in which the smallest possible values are chosen. Again, when we match \(\overline{a}\) to \(a\) the only values in between (on the circle) are values already in the columns. So \(BB'\) satisfies Condition 2.

Lastly, condition 3 states that for any pair in an initial matching \(B(i)B'(i)\) where \(B(i)\) and \(B'(i)\) are the same sign we have \(B(i) \leq B'(i)\). Note that the initial matching only matches values of the
same sign in cases 1 and 2. So either \( B(i) = B'(i) \) or the pair comes from a non-zero splitting. The splitting process takes an entry \( a \) with \( a \) and \( \pi \) in the column and introduces the largest \( b \notin B \) with \( b < a \). The initial matching says to match \( b \) with \( a \) and \( \pi \) with \( \pi \) so the condition is met for both pairs in the initial matching. Condition 3 also states that we have an even number of pairs \( B(i)B'(i) \) in the initial matching with \( B(i) \in [n] \) and \( B'(i) \in [\pi] \setminus [n] \). Note that this only happens in case 4 when we match entries that come from extending. Since extending contributes an even number of values to both columns there will always be an even number of such matchings. So \( BB' \) satisfies Condition 3.

Next we describe an algorithm that starts with an initial matching \( BB' \) and produces what we will call a corrected matching \( CC' \) where \( C = B \) and \( C' \) is a reordering of \( B' \). This is done iteratively from the top to the bottom of the increasing column \( B \). This new matching is equivalent to Condition 0, the reordering condition.

**Definition 5.6.** Given a pair of columns \( BB' \) defined by an initial matching, we produce a corrected matching \( CC' \) by the following algorithm:

Let \( C := B \) and \( C' := B' \);

for \( i \) from 1 to \( k - 1 \) do

let \( j \geq i \) be such that \( C'(j) = \min\{C'(\ell) : \ell = i, \cdot, k\} \) where the minimum is with respect to \( \prec_{C(j)} \);

swap the values in positions \( i \) and \( j \) of \( C' \);

end;

Note that a corrected matching is an initial matching that has been reordered so we will refer to a corrected matching and a KN column that has been split, extended and reordered interchangeably.
Example 5.7. Continuing Example 5.4, $CC'$ is the corrected matching for $BB'$.

$$
(C, C') = \begin{array}{|cc|}
4 & 5 \\
7 & 8 \\
5 & 4 \\
3 & 2 \\
\hline
2 & 1 \\
1 & 3 \\
\end{array}
$$

Note that we also have $C' = \text{reorder}_B(B')$.

Lemma 5.8. Any corrected matching $CC'$ defined from an initial matching $BB'$ still satisfies Conditions 1, 2 and 3. Additionally, $CC'$ satisfies Conditions 0 and 4.

Proof. $CC'$ clearly satisfies Condition 1 because $B$ and $C$ have the same entries, $B'$ and $C'$ have the same entries and $BB'$ satisfies this condition.

Next, note that $BB'$ satisfies Condition 2. Consider the step in the correcting procedure where $B(i)$ gets matched to $B'(j)$ and thus $B(j)$ gets rematched with $B'(i)$. Note that in the initial matching the values are nested: $B(i) \prec B(j) \prec B'(j) \prec B'(i)$. The corrected matching covers the same values as the initial match, namely every value between $B(i)$ and $B'(i)$. Since no new values are crossed, we have $\text{int}(C, C') = \text{int}(B, B') = \emptyset$.

Next we address Condition 3. Note that because $k < n$ there is an $a$ with $a \notin B$ and $\overline{a} \notin B$. Also, because $\text{int}(B, B') = \emptyset$ none of the pairs in the initial matchings $B(i)B'(i)$ have $a$ or $\overline{a}$ sitting between them. This is also true for the pairs in the corrected matchings $C(i)C'(i)$. Let $x = C(i)$, $x' = C'(i)$, $\text{sgn}(x) = \text{sgn}(x')$. Suppose $x' < x$. If $x \in [n]$ then we must have $x < \overline{a} < x'$, violating $\text{int}(C, C') = \emptyset$. So we must have $x \leq x'$. If $x \in [\overline{n}] \setminus [n]$, then we must have $x < a < x'$, again violating $\text{int}(C, C') = \emptyset$. Thus, we have $x \leq x'$ whenever $x$ and $x'$ are the same sign, preserving the first part of Condition 3.

Next we show that the number of negative entries matched to positive entries is even. Recall that in the initial matching there are an even number of negative to positive pairs because they correspond to the extended values which are added to the columns two at a time. I will show that
at each step in the correcting process, the number of negative to positive pairs in the corrected matching is preserved. Recall that \( a \) was previously chosen to be the smallest value with neither \( a \) nor \( \bar{\pi} \) in either column. Because of Condition 2, no matching can cross over \( a \) or \( \bar{\pi} \). So if an entry in \( C \) is corrected to a new value it can only switch sign once (from positive to negative or negative to positive). By checking the signs in the eight possible configurations involved in a correcting step, it is easy to see that every correcting step preserves the number of sign pairings and thus preserves the number of negative to positive pairs from the initial matching to the corrected matching.

Condition 0 is clearly satisfied because the algorithm to define a corrected matching is the same as the reordering algorithm.

Condition 4 is clearly true because any deviation from the structure of the regions would violate Condition 2 and any violation of the increasing conditions on \( B' \) would violate Condition 0.

5.1.2 The inverse of the extend \( \circ \) split map

In this section we will show that given a pair of columns \( CC' \) satisfying Conditions 0-4 we can find an initial matching whose corrected matching is precisely \( CC' \).

**Definition 5.9.** Let \( CC' \) satisfy Conditions 0, 1, 2, 3 and 4. Let \( a \in [n] \cap |C| \). We will define a new matching between the entries of \( C \) and \( C' \) by identifying the following cases for an entry in \( C \):

**Case 1.** \( a \in C \cap C' \).

**Case 2.** \( a \in C \) and \( \bar{\pi} \in C' \).

**Case 3.** \( \bar{\pi} \in C \cap C' \).

**Case 4.** \( \bar{\pi} \in C \) and \( a \in C' \).

We define the new matching using the following algorithm:

1. For every \( a \) in case 1, match \( a \) to itself. For every \( \bar{\pi} \) in case 3, match \( \bar{\pi} \) to itself.

2. For case 2, we begin with the smallest \( a \) and continue increasingly. If there is a \( b \) such that \( a < b \), \( \bar{b} \in C \) and \( b \in C' \) (i.e. \( b \) satisfies case 4), match \( a \) with the smallest such \( b \) and match \( \bar{b} \) with \( \bar{\pi} \). If no such \( b \) exists, match \( a \) with \( \bar{\pi} \). Continue with the next smallest \( a \) until all case 2 values are matched.
3. For every $\bar{a}$ in case 4 that has not already been matched in the previous step, match $\bar{a}$ with $a$.

Note that the new matching in 5.9 is well defined. Clearly, by Condition 1, any value in a column falls into one of the 4 cases. In cases 1, 3, and 4, the values are matched to themselves or to their negatives, which is always possible. In case 2, if a $b$ does not exist then $a$ is matched to $\bar{a}$, which is also always possible.

**Theorem 5.10.** The matching defined in 5.9 is an initial matching.

We will need several lemmas to prove this theorem.

**Lemma 5.11.** If we have $C(i) = \bar{b}$, $C'(i) = a$, then $\bar{a} \in C$.

*Proof.* Suppose $\bar{a} \notin C$. Then by Condition 1, $a \in C$. Clearly, $a < \bar{b}$. Let $a = C(m)$. Then by Condition 0, $C'(m) = \min\{C'(\ell) : m \leq \ell \leq k\}$ where the minimum is taken with respect to circular order starting with $a$. Note that either $a = C'(j)$ with $j < m$ or $a = C'(m)$ because if $a \in \{C'(\ell) : m \leq \ell \leq k\}$ then it is the minimum. So $C(i) = \bar{b}$, $C'(i) = a$ is a violation of Condition 0 and we cannot have $a \in C'$. Thus $\bar{a} \in C'$. 

**Lemma 5.12.** If $\bar{b}$ is matched to $b$ in the new matching, then $b$ is in region IV of $C'$.

*Proof.* Suppose $\bar{b}$ and $b$ are matched in the new matching and suppose $b$ is not in region IV of $C'$. Then we must have $b$ in region I of $C'$ because it is the only other region with positive $C'$ values. So $b = C'(i)$ for some $i$ and there is $x$ such that $x = C(i)$. We will consider $x' \in C'$, the value $x$ is matched to in the new matching. Either $x' = x$ when $x$ is in Case 1 or $x' = y$ for some $y$ such that $x < y < b$ when $x$ is in Case 2. In both cases, $x' < b$ so it must be that $x' = C'(j)$ for some $j < i$, otherwise we would have had $C'(i) = x'$ instead of $b$ by Condition 0. By a similar argument, $C(j) = x_1$ matches with $x_1' \in C'$ in the new matching such that $x_1'$ is either $x_1$ or some $y_1$ with $x_1 < y_1 < b$ and $x_1' = C'(j_1)$ with $j_1 < j$. This argument continues so we can find a smallest $x_p$ matched to $x_p'$ in the new matching where $x_p' = C'(j_p)$ with $j_p < j_{p-1}$ is either $x_p$ or $y_p$ with $x_p < y_p < b$. Now we consider $z = C(j_p)$. Since $x_p'$ is the smallest such value, we must have $z$ matched to $\bar{z}$ in the new matching. We also have $z < x_p < b$ by Condition 0. But then the order
of the new matching process would match \(z\) with the as of yet unmatched \(b\) instead of \(\overline{z}\). But this violates \(\overline{b}\) matched to \(b\). Therefore, \(b\) must be in region IV of \(C'\).

The next lemma utilizes distinct arcs on the circle defined by the ordering on the entries in \(CC'\). Note that there is at least one value \(z \in [\overline{n}]\) that does not appear in \(C\) or \(C'\) because \(k < n\). By Condition 1, \(\overline{z}\) is also not in \(C\) or \(C'\). Consider the set \(\{z_1 < z_2 < \cdots < z_p\}\) of such positive and negative values. By Condition 2, none of the \(z\) values sit between \(C(i)\) and \(C'(i)\) on the circle for \(i = 1, \cdots, k\). In fact, for each \(z_i\) there is a largest \(C'(j)\) with \(C'(j) < z < C(j + 1)\). So we can define dense, distinct arcs on the circle with all the values strictly between \(z_i\) and \(z_{i+1}\) for \(i = 1, \cdots, p - 1\) and either the arc between \(\overline{z}\) and \(z_1\) if \(CC'\) has any values in region IV or the two (possibly empty) arcs starting from 1 up to \(z_1\) and the arc strictly bigger than \(z_p = \overline{z_1}\) through \(\overline{1}\) if region IV of \(CC'\) is empty. Note that all of these arcs contain all the values in a block of \(CC'\) from some \(C(i)\) through some \(C'(j)\) for \(i \leq j\) except possibly the last arc which would contain all the values in the last block of \(CC'\) and the first block. Note that at most we can have one arc that starts with a positive value and ends with a negative value, which we will call the bottom arc since it sits at the bottom of the circle, and one arc that starts with a negative value and ends with a positive value, which we will call the top arc since it sits at the top of the circle. All other arcs are either all positive or all negative.

I will use the following notation to indicate different types of matchings from the new matching:

- **a** is matched to \(b\) and \(\overline{b}\) is matched to \(\overline{a}\).
- **c** is matched to \(\overline{c}\).
- **d** is matched to \(d\).
- **\(\overline{c}\)** is matched to \(\overline{c}\).
- **\(\overline{f}\)** is matched to \(\overline{f}\).

**Lemma 5.13.** If \(x\) is matched to \(y\) in the new matching then \(x\) and \(y\) are in the same arc defined by \(CC'\).

**Proof.** Note that the arcs are symmetric because if \(z \in [\overline{n}]\) is not in \(C\) or \(C'\) then neither is \(\overline{z}\). Also note that if \(d\) is matched to \(d\) or \(\overline{c}\) is matched to \(\overline{c}\) then they are clearly in the same arc. Lastly,
note that if there is a top arc and a bottom arc then they are distinct because if not then the whole circle would be one arc and there is at least one $z \in \overline{n}$ making the arcs distinct.

We will assume that region I of $CC'$ is not empty? First I will show that if $a$ is in the arc containing $C(1)$ and $a$ is matched to $b$ then $b$ is also in that arc. Suppose the arc ends at $C'(m)$. So the arc contains the block $C[1, m]C'[1, m]$ from region I. Either this arc is the top arc or is a positive arc. If this arc is the top arc then there might be $f$ values in it but by Lemma 5.12 they all sit in region IV. So the only values in $C'[1, m]$ are $d$ or $b$ values. For every $d$ in this arc, $d$ is in both $C$ and in $C'$. So every other value in $C'[1, m]$ must be a $b$ value because there are no other positive values in $C'$ which means the number of $b$ values in $C'$ of this arc is exactly the number of $a$ values and $c$ values in $C$ of this arc. Suppose we have a $b' = C'(i)$ in this arc without its matching $a'$ in this arc. Then we have $C(i) < b'$ by Condition 0. But also, by the matching, we have $a' < b'$. If $a' < C(i)$ it must be in $C$ in a position above $C(i)$ because $C$ is increasing and thus it is in the arc. If $C(i) < a' < b'$ then $a'$ is also in the arc because everything between $C(i)$ and $b'$ is in the arc. So every $b$ in the arc must have its matching $a$ in the arc and every $a$ must have its matching $b$ in the arc because then there would not be enough $C'$ values in the arc. Additionally, if there is a $c$ in the arc then every $a$ is in the arc because $a < c$ for every $a$ matched to $b$ in the new matching (otherwise $c$ would have matched to $b$). But then there are more $C$ values than $C'$ values in region I of the arc (because every $b$ value is in $C'$ of the arc, all the $d$ values in the arc are in $C'$ and we cannot take $d$ values from a different arc and none of the $f$ values are allowed in region I which is all of the available positive values). So there are no $c$ values in the arc containing $C(1)$ (this was already true if we were in the top arc but now also true if this is a positive arc). This same argument applies for each following positive arc, which sits entirely in region I. So each positive arc containing $a$ must also contain $b$ and it cannot contain any $c$.

We will now consider the bottom arc. Since $c$ cannot sit in the top arc or any positive arcs we must have that every $c$ is in the bottom arc. Since this arc crosses from positive values to negative values and the arcs are symmetric $\overline{a}$ is in the bottom arc as well. Notice that if $a$ is in the bottom arc then so is its matching $b$ since we have $a < b \leq n$ and the arc contains at least one negative $C'$ value.

So all $a$ in an arc have $b$ in the same arc and since the arcs are symmetric, every $\overline{b}$ in an arc has $\overline{a}$ in the same arc. We also have every $c$ and $\overline{c}$ are in the bottom arc. Lastly, since every $f$ with $\overline{f}$
matched to \( f \) is in region IV, every \( f \) is in the top arc. Again, by the symmetry of the arcs, \( \overline{f} \) is also in the top arc. So for every \( x \) matched to \( y \) in the new matching, \( x \) and \( y \) are in the same arc.

\[ \square \]

**Proof of Theorem 5.10.** Suppose \( CC' \) is used to define a new matching, \( DD' \), that is not an initial matching. Since \( DD' \) satisfies Condition 1, \( DD' \) must violate one of the following properties of an initial matching:

Tightness of \( c \) matched to \( \overline{c} \): Any \( c \) matched to \( \overline{c} \) is in the bottom arc and so there is no \( x \) with \( c < x < \overline{c} \) unless \( x \in C \cup C' \). Additionally, every \( c \) is bigger than the largest \( a \) so any \( c \) matched to \( \overline{c} \) only crosses over a \( d \) or an \( \overline{a} \) or a \( b \) matched to an \( a \) smaller than \( c \). This is exactly the criteria for a split zero in the initial matching so the set of such \( c \) values will be exactly the split zero values in the initial matching.

Tightness of \( a \) matched to \( b \) and \( \overline{b} \) matched to \( \overline{a} \): Let \( a \) be matched to \( b \) and \( \overline{b} \) be matched to \( \overline{a} \). These are in the same arc and \( b \) is chosen to be the smallest possible, so there is no \( x \) with \( a < x < b \) or \( \overline{b} < x < \overline{a} \) unless \( x \in C \cup C' \). This is exactly the criteria for a non-zero split value in the initial matching so the set of \( b \) and the set of \( a \) will be exactly the non-zero split values and the result of the non-zero split values, respectively.

Tightness of \( \overline{f} \) matched to \( f \): Let \( \overline{f} \) be matched to \( f \). They are both in the top arc and so there is no \( x \) with \( \overline{f} < x < f \) unless \( x \in C \cup C' \). Additionally there is no \( a < f < b \) because \( a \) would have been matched to \( f \) and \( \overline{f} \) matched to \( \overline{a} \), violating the new matching algorithm. This is exactly the criteria for extended values in the initial matching so the set of such \( f \) will be exactly the extended values in the initial matching.

\[ \square \]

**Proof of Theorem 5.2.** By Lemma 5.8 it is clear that every KN column has a corresponding pair of columns of height \( k \) satisfying Conditions 0-4 and by Theorem 5.10 it is clear that any pair of columns of height \( k \) satisfying Conditions 0-4 has a corresponding KN column.

In fact, it is clear that if \( CC' \) is the corrected matching of a KN column \( A \) then the matching in 5.9 defined on \( CC' \) is exactly the initial matching defined on \( A \). So \( \text{extend} \circ \text{split} \) is a bijection. \[ \square \]
5.2 The relationship between quantum Bruhat graph paths and KN columns

In this section we prove Theorem 4.4, part b). This follows immediately from Proposition 5.14 and Theorem 5.19. We will show that any quantum Bruhat graph path beginning with an increasing column produces a pair of columns $CC'$ that satisfy the reordering condition.

Recall the chain $\Gamma_r(k) := \Gamma_k \Gamma_{k-1} \ldots \Gamma_1$ where $\Gamma_i := ((i, \bar{i})(i, \bar{i} - 1), (i, \bar{i} - 2), \ldots, (i, \bar{1}))$. Recall that we refer to $(i, \bar{i})$ as stage II and the rest of $\Gamma_i$ as stage IV.

5.2.1 Necessity of conditions

Proposition 5.14. Given a signed permutation $u$ in $B_n$ with $u(1) < u(2) < \ldots < u(k)$ where $1 \leq k < n$ and a subsequence $T$ of $\Gamma_r(k)$ labeling a path $u = u_0, u_1, \ldots, u_p$ in the quantum Bruhat graph of type $B_n$. Define $C := u[1, k]$ and $C' := u_p[1, k]$. Then $CC'$ satisfies Conditions 0-4.

Note that for $1 \leq j \leq k$

$$C'(j) := u_0(i) < u_1(j) < \ldots < u_p(j) =: C'(j)$$

because otherwise the quantum Bruhat graph condition would be violated by switching across a value in $C[k + 1, k + 1]$.

First we will show that $CC'$ satisfies the reordering condition, Condition 0.

Proposition 5.15. Let $u$, $C$ and $C'$ be as in Proposition 5.14. Then $CC'$ satisfy Condition 0.

We will need two Lemmas, the first pertaining to the structure of columns for each $u_i[1, k]$ for $1 \leq i \leq p$ and the second pertaining to the structure of rows in $CC'$.

Lemma 5.16. For $u_i$ in the quantum Bruhat graph path in Theorem 5.14, let $B := u_i[1, k]$. Then $B$ has the following three region structure, with possible empty regions, where each region is increasing in the column.

$$B = \begin{array}{c}
+ \\
- \\
+
\end{array}$$

Throughout this proof we will refer to the regions as region I, II and III, respectively.

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Proof. We will analyze what happens to each fixed position $j$ of the column $1 \leq j \leq k$, one transposition at a time. Let $(j, \overline{m})$ be the first transposition in $T$ and let $B = u_{i}[1,k]$. Since $C$ is increasing, we begin with only regions I and II. Suppose $j = k$. Then $(k, \overline{m})$ cannot switch a positive value with a positive value because if $C(k)$ is positive then $C(\overline{k}, \overline{1})$ is negative. If $(k, \overline{m})$ switches a positive value with a negative value then by the quantum Bruhat graph criteria we must have $(k, \overline{m}) = (k, \overline{k})$ which clearly preserves the increasing property. If $(k, \overline{m})$ switches a negative value with a negative value then $m < k$. We clearly preserve the increasing property, because this is an up edge so both the values in positions $m$ and $k$ increase and we still have $B(m) < B(m + 1)$ by the quantum Bruhat graph criteria. Lastly, if $(k, \overline{m})$ switches a negative value with a positive value then it is a down edge. Applying this edge creates region III. In fact, we must have $m = k - 1$, otherwise we violate the quantum Bruhat graph criteria by switching across position $\overline{k-1}$. So we only need to check region III which consists of $B[k-1,k]$. The increasing property is still satisfied because $C(\overline{k}) = B(k - 1) < B(k) = C(\overline{k-1})$.

Now suppose $j < k$. Then $(j, \overline{m})$ must be an up edge and the increasing property is satisfied from positions $j$ to $j + 1$ and $m$ to $m + 1$, otherwise the quantum Bruhat graph criteria is violated by switching across position $j + 1$ or position $m + 1$.

Next we consider the case where $j = k$ but $(k, \overline{m})$ is the $i^{th}$ transposition in $T$, $1 < i \leq p$. Let $B = u_{i}[1,k]$. Then $(k, \overline{m})$ cannot switch a positive value with a negative value since that can only happen with the first available transposition, $(k, \overline{k})$. If it switches a negative value with a negative value or a positive value then the above arguments still hold. Lastly, $(k, \overline{m})$ can switch a positive value with a positive value which must be an up edge and the increasing property is still preserved by the same argument as switching a negative value with a negative value.

Suppose $A = u_{i}[1,k]$ and $A$ satisfies the increasing property. Suppose the next transposition in $T$ is $(j, \overline{m})$ where $m \leq j$. Let $B = u_{i+1}[1,k] = A(j, \overline{m})$. The proof depends on the regions of $A$ containing $j$ and $j + 1$.

Let $A(j)$ be in region I. Suppose $A(j + 1)$ is not in region III. $A(j)$ is positive so $(j, \overline{m})$ must be an up edge. Moreover, $A(j)$ cannot switch with a positive value because then $A(m)$ would be negative and sit above $A(j)$, violating the increasing property. So $A(j)$ must switch with a negative value and we have $(j, \overline{m}) = (j, \overline{j})$. We must have $A(j) < A(j) < A(j + 1)$ because $(j, \overline{j})$ is a quantum Bruhat graph edge. Thus we maintain the increasing property. Now suppose $A(j + 1)$ is in region
III. Then region II is empty since \( A(j) \) is in region I. Again, \( A(j) \) must switch with a negative value so we have \( (j, \overline{m}) = (j, j) \). This ensures the increasing property is still satisfied because region II becomes non-empty and consists just of position \( j \).

Now let \( A(j) \) be in region II. Suppose \( j + 1 \) is also in region II. So \( A(j) < A(j + 1) \) are both negative. Then \( (j, \overline{m}) \) cannot be a down edge or it would violate the quantum Bruhat graph criteria by switching across position \( j + 1 \) so \( (j, \overline{m}) \) must be an up edge and \( A(j) \) must switch with a negative value. The increasing property still holds after applying \( (j, \overline{m}) \) for positions \( j \) to \( j + 1 \) and \( m \) to \( m + 1 \), otherwise we would switch across position \( j + 1 \) or position \( \overline{m} + 1 \). Suppose \( j + 1 \) is in region III. If \( A(j) \) switches with a negative value the previous argument holds and the increasing property is still satisfied. Assume \( A(j) \) switches with a positive value. Then \( (j, \overline{m}) \) is a down edge.

Additionally, we must have \( m = j - 1 \), otherwise we would have \( A(m) < A(j - 1) < A(j) \) which are all negative values because they are in region II which would violate the quantum Bruhat graph criteria by switching across position \( j - 1 \). So we must show \( B(j - 1) < B(j) < B(j + 1) \) which are all in region III of \( B \) and thus all positive. We need not check that the increasing property holds for position \( j - 2 \) because \( B(j - 2) \) is in region II and \( B(j - 1) \) is in region III. Note that \( B(j - 1) < B(j) \) since \( A(j) < A(j - 1) \) since \( A \) satisfies the increasing property. Additionally, \( B(j) < B(j + 1) \), otherwise \( (j, j - 1) \) would violate the quantum Bruhat graph criteria by switching across position \( j + 1 \).

Lastly, let \( A(j) \) and \( A(j + 1) \) both be in region III. By the increasing property of \( A \) we have \( A(j) < A(j + 1) \), both positive. Note that any region III position must have originated in region II. So \( (j, \overline{m}) \) must be an up edge with \( m \neq j \), otherwise we would switch from a negative value to a positive value back to a negative value, violating the quantum Bruhat graph condition by switching across at least one positive value in \( C[k + 1, \overline{k} + 1] \) at some point in applying transpositions to position \( j \). Thus, \( A(j) \) switches with a positive value. The increasing property still holds after applying \( (j, \overline{m}) \) for positions \( j \) to \( j + 1 \) and \( m \) to \( m + 1 \), otherwise we would switch across position \( j + 1 \) or position \( \overline{m} + 1 \).

\[ \square \]

For the next lemma, we recall the language of 4. Given two columns \( CC' \) of height \( k \), we say that \( C(i) \) and \( C'(i) \) are in region I if \( C(i) \) and \( C'(i) \) are positive, region II if \( C(i) \) is positive and
$C'(i)$ is negative, region III if $C(i)$ and $C'(i)$ are negative and region IV if $C(i)$ is negative and $C'(i)$ is positive.

**Lemma 5.17.** $CC'$ has the following four region structure:

\[
CC' = \begin{array}{ccc}
+ & + \\
+ & - \\
- & - \\
- & +
\end{array}
\]

with $C(i) \leq C'(i)$ when $i$ is in regions I, II and II. For region IV, we have $C'(i) < M$ where $M = \min \{u(k+1), \ldots, u(k+1)\}$.

**Proof.** Note that any subsequence of $\Gamma_r(k)$ starts by adjusting position $k$ until it is fixed for the rest of the transpositions and then adjusts position $k-1$ until it is fixed for the rest of the transpositions and continues this way until position 1. By Lemma 5.16, we know that $C'$ has the three region column structure. Since $C$ is increasing it consists of regions I and II. Also note that by (26) each position can switch sign at most once.

We will begin with $i$ in region IV. Suppose $C(i) \prec M \prec C'(i)$. Note that $C(i)$ is negative while $M$ and $C'(i)$ are positive. There must be a stage IV transposition, $(i, m)$ that switches some $a$ and $a'$ across $M$, violating the Quantum Bruhat graph condition. So we must have $C(i) \prec C'(i) \prec M$.

Next we consider $i$ in region III. Suppose $C'(i) < C(i)$ which are both negative. So the transpositions applied to position $i$ must pass over all positive values. In particular, since $M \in C[k+1,k+1]$ is in a position that cannot be moved by the available transpositions, there must be one transposition that switches some $a$ and $a'$ across $M$, violating the Quantum Bruhat graph criteria.

Note $i$ in region I with $C'(i) < C(i)$ is also a contradiction to the Quantum Bruhat graph criteria because it would require switching two values across $\overline{M}$. Note also that region II always has $C(i) < C'(i)$ because $C(i)$ is positive and $C'(i)$ is negative.

\[\square\]

Note that the next Lemma is proved in [9] for $u$ a subsequence of $\Gamma_l(k)$ labeling a quantum Bruhat path in type $C$. However, the same proof holds for $u$ a subsequence $\Gamma_l(k)$ labeling a quantum
Bruhat path in type $B$. Additionally, since $\Gamma_r(k)$ is a subsequence of $\Gamma_l(k)$ it holds here as well. We will refer to this Lemma again in Section 6.

**Lemma 5.18.** [9] Fix $i$ and $l$ with $1 \leq i < l \leq k$. Let $a$ be the entry in position $i$ of the signed permutation obtained at some point in the process of applying to $u$ the reflections in $T_k, T_{k-1}, \ldots, T_{l+1}$. Then either $a$ appears in $C_{l+1}[1, i]$ or $\bar{a}$ appears in $C_{l+1}[l + 1, k]$.

**Proof of Theorem 5.15.** Note that our path in the quantum Bruhat graph can factored as $T_k, \ldots, T_1$. Assume $C(i) = C'(l) = a$ for $1 \leq i < l \leq k$. We will consider what happens to $a$ as we apply $T_k, \ldots, T_{l+1}$. From Lemma 5.18, either $a \in C_{l+1}[1, i]$ or $\bar{a} \in C_{l+1}[l + 1, k]$. Notice that $C_{l+1}[l + 1, k]$ is fixed for all subsequent $T_{l+1}, \ldots, T_1$. Since we have $C'(l) = a$ we cannot have $\bar{a} \in C'[l + 1, k] = C'_{l+1}[l + 1, k]$. So we must have $a \in C_{l+1}[1, i]$. But now $a$ can only be moved higher up in $C'$ or to $C'[\bar{r}, \bar{t}]$ by the transpositions in $T_l$. This means it can never be moved to position $l$ of $C'$, as needed.

Suppose $C(i) \prec C'(l) \prec C'(i)$ for $1 \leq i < l \leq k$. By Lemma 5.16, either $i$ is in region I and $l$ is in region IV or $i$ is in region II and $l$ is in region IV. In all other configurations we have $C(i) \preceq C'(i) \prec C'(l)$ by Lemmas 5.16 and 5.17.

Let $C'(l) = b$. Then we have three cases: $C(i) \prec b \prec C_{l+1}(i)$, $C_{l+1}(i) \preceq b \prec C_l(i)$ or $C_l(i) \preceq b \prec C'(i)$.

Note that $C(i)$ and $b$ are always positive but $C'(i)$ may be positive or negative. $C_l(i) \preceq b \prec C'(i)$ is easily ruled out because we will only apply transpositions from $T_{l-1}, \ldots, T_1$ with $b$ fixed in position $l$ so at some point position $i$ must switch across $b$, violating the QBG criteria.

Note that in the other two cases, $C_{l+1}(i)$ and $C_l(i)$ must both be positive because $C(i)$ is positive and position $i$ can only change from positive to negative during the application of $(i, \bar{r})$ in $T_l$. Suppose $C_{l+1}(i) = a \preceq b \prec C_l(i) = a'$. Since $a \neq a'$ we must have $(i, \bar{t})$ in $T_l$ since this is the only transposition that moves position $i$. This must be the edge that passes $b$ and it must be an up edge by the QBG criteria. Note that $C_{l+1}(l) \neq C_l(l) = b$ and after $(i, \bar{t})$ is applied $b$ is still not in position $l$ because $\bar{a}$ is in position $l$ and $\bar{a}$ is negative. So we must have $b \in C_{l+1}[\bar{r}, \bar{t}]$ and $\bar{b} \in C_{l+1}[1, i - 1]$. But $\bar{b}$ is negative and sits above $a'$ which is positive, violating the increasing column structure of $C_{l+1}$.
Suppose $C(i) = a < b < C_{l+1}(i) = a'$. Again, all three values are positive. Let $(i, \overline{m})$ be the transposition that passes $b$: $a \xrightarrow{(i, \overline{m})} c$ with $a < b < c$. Consider the sequence of transpositions in $T_m$: $(i, \overline{m}) = (i_0, \overline{m}), (i_1, \overline{m}), \ldots, (i_p, \overline{m})$ with $i_0 < i_1 < \ldots < i_p$. Let $C_{m+1}(j) = a_j$. Note that all $a_j$ are positive and $a_p < a_{p-1} < \ldots < a_1 < a_0$ by the increasing column structure of $C_{m+1}$.

Claim: No values between $a_p$ and $c$ are in $C_m[i, \overline{m}]$, including $b$.

Let $w$ be the word right before $(i, \overline{m})$ is applied. By the quantum Bruhat graph criteria, there is no values in $w[i, \overline{m}]$ between $a$ and $c$. The only values in that region that are changed by the rest of $T_m$ are those in positions $m$ and $\overline{m}$. All the values that get moved to position $\overline{m}$ come from above position $i$ and so must all be less than $a$ by the increasing column structure. Now consider the value in position $m$ which goes from $a$ to $\overline{a}_p$. These are all negative values but all the values between $a$ and $c$ are positive. Thus $b$ is not in $C_m[i, \overline{m}]$.

Claim: No values between $a_p$ and $c$ can be moved into positions $i$ through $\overline{m}$ by $T_{m-1}, \ldots, T_l$, including $b$.

Suppose there is a transposition that moves a value between $a_p$ and $c$ into one of the positions. But this would require switching a value NOT between $a_p$ and $c$ with a value that IS between $a_p$ and $c$. This would switch it across $a_p$ in position $\overline{m}$ which is a violation of the quantum Bruhat graph criteria. Thus $b$, the target for position $l$, can never be moved to position $l$.

\[ \square \]

Proof of Theorem 5.14. We begin by assuming we have a Quantum Bruhat graph chain and showing the resulting pair of columns $CC'$ satisfy Conditions 0-4 and is therefore a KN-column. By Proposition 5.15, we have that any Quantum Bruhat graph chain yields $CC'$ that satisfies the reordering condition, Condition 0. We also have that $CC'$ satisfies Condition 4 by Lemmas 5.16 and 5.17. Additionally, Condition 1 is clear because the available transpositions in the chain can only permute and negate the values in $u[1, k] = C$. The first part of Condition 3 is clear from Lemma 5.17. It is easy to see the second part of Condition 3 is clear because to get a value in region IV of $CC'$ we must apply a stage IV down edge. Specifically, we must first apply the edge $(k, \overline{k-1})$ at the bottom of the column. If we were to apply a down edge higher up in the column we would have the following situation. Let $B$ be the intermediate column and suppose we intend to apply the stage IV down edge $(i, \overline{m})$. Also suppose $B(k)$ is a negative value. So we must have $B(i) \prec B(k) \prec B(\overline{m})$ so
applying \((i, \overline{m})\) would violate the Quantum Bruhat graph criteria. A similar argument shows that if we apply a down edge to position \(k\) it must be \((k, \overline{k-1})\), otherwise we would switch across position \(k-1\). This transposition will change the negative values in positions \(k\) and \(k-1\) to positive values, creating region IV. This reasoning continues for the rest of region IV. Since this process happens in pairs, we have that the size of region IV is even. Lastly, to show Condition 2, assume there is \(C(i)C'(i)\) such that \(C(i) \prec x \prec C'(i)\) and \(x \notin C \cup C'\). Then \(x \in u[k+1, \overline{k+1}]\) and at some point in applying \(T\) there must be a transposition that switches a value in position \(i\) across \(x\). \(\Box\)

5.2.2 Constructing the quantum Bruhat graph path

In this section we will start with \(CC'\) of height \(k\) satisfying Conditions 0-4 and construct a subsequence of \(\Gamma_r\) in the quantum Bruhat graph.

Theorem 5.19. Given \(CC'\) of height \(k\) satisfying Conditions 0-4, we define \(u\) a permutation in \(B_n\) with \(u(i) = C(i)\) for \(i = 1, \cdots, k\) and \(u[k+1, n] = \text{sort}\{x \in [n] \setminus |C|\}\). Then there is a subsequence \(T = T_kT_{k-1}\cdots T_1\) of \(\Gamma_r(k)\) labeling a path \(u = u_0, u_1,\ldots, u_p\) in the quantum Bruhat graph of type \(B_n\). Moreover, this path is unique.

We will construct the path in Theorem 5.19 by a greedy algorithm. The algorithm inputs the signed permutation \(u\), the position \(i\), the target, \(c = C'(i)\) and \(L\), the list of positions \((\overline{i-1}, \cdots, \overline{1})\).

The function \(\text{next}(m, L)\) determines the successor of the element \(m\) in the list \(L\). The algorithm outputs the list of transpositions \(T_i\) and the end permutation, \(v\).

Algorithm 5.20. procedure path-B\((u, i, c, L)\);

\begin{align*}
\text{if } u(i) &= c \text{ then return } \emptyset, u \\
\text{else} & \\
\text{if } u(i) &\leq n \text{ and } \text{sign}(u(i)) \neq \text{sign}(c) \text{ then} \\
\text{let } S := (i, \overline{i}), \; v := u(i, \overline{i}); \\
\text{else let } S := \emptyset, \; v := u; \\
\text{end if;}
\end{align*}

\begin{align*}
\text{let } m := L(1): \\
\text{while } v(m) \neq c \text{ do} \\
\text{if } v(i) &\prec v(m) \prec c \text{ then let } S := S, (i, m), \; v := v(i, m); \\
\end{align*}
end if;
     let \( m := \text{next}(m, L) \);
end while;
let \( S := S, (i, m), v := v(i, m) \);
return \((S, v)\);
end if;
end.

We will need to refer to the intermediate columns obtained after each position \( i \) has reached its target. Let \( \pi_{k+1} = u \) and \( C_{k+1} = C \). Then define \( \pi_i := uT_kT_{k-1} \cdots T_i \) and \( C_i := \pi_i[1, k] \) for \( i = k, \cdots, 1 \). The next lemma shows that when the greedy algorithm is applied to position \( i \) with starting permutation \( \pi_{i+1} \) then the target \( C'(i) \) is in a position that can be reached, namely one of the positions \( \overline{t-1}, \ldots, \overline{1} \).

**Lemma 5.21.** \( C'(i) \), the target of position \( i \), is in \( \pi_{i+1}[\overline{r, \overline{1}}] \).

**Proof.** Note that \( C \) is height \( k \) and by Condition 1 we must have \( C'(i) \notin \pi_{i+1}[k, \overline{k}] \) for \( i = 1, \cdots, k \).
Also, note that \( \pi_{i+1}[i+1, k] = C'[i+1, k] \). So \( C'(i) \notin \pi_{i+1}[i+1, k] \) and we must have \( C'(i) \in \pi_{i+1}[1, i] \) or \( \pi_{i+1}[\overline{r, \overline{1}}] \). Whenever \( i \) is in regions I, II or III, we have \( C(i) \leq C_k(i) \leq \cdots \leq C_{i+1}(i) \leq C'(i) \) by Condition 3 and (26). Also \( C_{i+1}[1, i] \) is increasing so either position \( i \) has already reached the target or \( C'(i) \) sits lower in \( \pi_{i+1} \). So \( C'(i) \notin \pi_{i+1}[1, i-1] \) and thus \( C'(i) \in \pi_{i+1}[\overline{r, \overline{1}}] \). Now we consider \( i \) in region IV. So we have \( C(i) = \overline{a} \), \( C'(i) = b \) and \( \overline{b} \in C \). Since \( b \in \pi_{i+1}[1, i] \) or \( b \in \pi_{i+1}[\overline{r, \overline{1}}] \), the same is true for \( \overline{b} \). In fact, since \( \overline{b} \) started in \( C \) and is not the target for positions \( i+1, \cdots, k \) it must be that either \( b = C_{i+1}(i) \) or \( \overline{b} \in C_{i+1}[1, i-1] \). This is clear because even if \( b \) is moved to position \( l \) during \( T_i \) it is not the target of that position so it is moved again and \( \overline{b} \) moves higher up in the column. So we have \( b \in \pi_{i+1}[\overline{t-1}, \overline{1}] \). This means that at the beginning of applying transpositions from \( T_i \) we have the target in an accessible position.

**Proof of Theorem 5.19.** Suppose we have CC' with Conditions 0-4. We will show that Algorithm 5.20 produces \( T \) that labels a path in the quantum Bruhat graph from \( u \) to \( u_{p'} \). We will do this by the regions of CC'. Let positions 1, \cdots, \( r \) be region I, positions \( r+1, \cdots, s \) be region II, positions \( s+1, \cdots, t \) be region II and positions \( t+1, \cdots, k \) be region IV. Suppose region IV is not empty.
By Condition 3, region IV is at least size two. Note that \((k, \bar{k})\), the first available transposition is not a quantum Bruhat graph edge since \(C(k)\) is negative and \(C'(k)\) is positive. It is easy to see that \((k, k+1)\) is a quantum Bruhat graph (down) edge and does not pass the target, \(C'(k)\). Since \(\text{int}(CC') = \emptyset\) there are no values in \(\pi_{k+1}[k+1, k+1]\) between \(C(k)\) and \(C'(k)\). Since \(C\) is increasing, \(C'(k) \neq \overline{C(k)}\) and \(\overline{C'(k)} \in C\), we have \(\pi_{k+1}(k-1) \leq C'(k)\). So the algorithm will choose \((k, k-1)\). Furthermore, since \(C\) is increasing and \(\overline{C'(k-1)} \in C\) position \(k-1\) has not passed its target either.

Now, let \(B = C(k, k-1)\). If \(B(k) \neq C'(k)\) then the rest of the transpositions applied to position \(k\) must be up edges switching with positive values only. So we can only switch with positions \(\bar{k} - 2, \ldots, s + 1\) because these are the only positive values available. So, by Lemma 5.21, the target is an accessible position: \(C'(k) = B(\bar{i})\) with \(s + 1 \leq i \leq k - 2\). Since \(B[1, k-2] = C[1, k-2]\) is increasing, we have that \(\overline{C(k-1)} = B(k) < \overline{B(k-2)} < \cdots < \overline{B(i)}\). Since \(\text{int}(C, C') = \emptyset\) we have no values in \(\pi_{k+1}[k+1, k+1]\) between \(C(k)\) and \(C'(k)\) so this is also true for all values between \(B(k)\) and \(C'(k)\). This means we can do each transposition \((k, k-2), \ldots, (k, \bar{i})\) without violating the quantum Bruhat graph criteria and this string of transpositions will terminate with the target, \(C'(k)\). So the greedy algorithm will choose each of these transpositions. Additionally, if any \((k, \bar{j})\) was skipped we would not be able to reach the target without switching across position \(\bar{j}\).

Now we will consider \(C_k(k-1)\) which is positive and has positive target \(C'(k-1)\). So \(\overline{C'(k-1)} \in C_k[i, k-2]\). Let \(C'(k-1) = C_k(j)\). We already know there are no values in \(\pi_k[k+1, k+1]\) sitting between \(C_k(k-1)\) and \(C'(k-1)\) and we know that \(C_k(k-1) < C'(k-1) < C'(k)\) by Condition 4 so none of these positions will be switched across. Additionally, we have \(C_k(k-1) < \overline{C_k(k-2)} < \cdots < \overline{C_k(j)}\) by the increasing structure of \(C\) and the transpositions applied in \(T_k\). So by a similar argument as above, the transpositions \((k-1, k-2), \ldots, (k-1, \bar{j})\) can all be applied without violating the quantum Bruhat graph criteria and this string of transpositions will terminate with the correct target, \(C'(k-1)\). Again, if any transposition was skipped we would not be able to do the next transposition without violating the quantum Bruhat graph criteria. So these transpositions will be exactly \(T_{k-1}\) as chosen by the greedy algorithm.

If there are any further region IV positions note that we cannot have \(\overline{C(k-1)} = C'(k)\) or \(\overline{C(k)} = C'(k-1)\) because region IV of \(C'\) is increasing and these \(C\) values are the largest negatives and thus the smallest potential region IV target values. And since we applied all the transpositions in \(T_k\) and \(T_{k-1}\) until the target was attained, we have that \(C(k-1) = C_{k-1}(k-3)\) and \(C(k) =
$C_{k-1}(k - 2)$. Thus, $T_{k-2}$ will proceed similarly to $T_k$ first applying the down edge $(k - 2, k - 3)$ and then applying all up edges until the target is reached and $T_{k-3}$ will proceed similarly to and $T_{k-1}$, applying all up edges until the target is reached. By the same argument, the targets were not previously passed, the targets sit in accessible positions and each of the transpositions satisfy the quantum Bruhat graph conditions. If a transposition is skipped it will cause the next transposition to violate the quantum Bruhat graph criteria. If there are any other region IV positions they will be in pairs and will proceed in the same manner.

Next we consider region III and we will refer often to the dense, disjoint arcs previously defined by the new matching and showed to be equivalent to the initial matching in Section 5.1.2. Note that no region III values have passed their targets. The only positions that may have been affected by the region IV transpositions are those with $C(i)$ in the top arc. $T_k, \cdots, T_{t+1}$, the region IV transpositions, filter all the extended values to the bottom of the column and leave behind only values $\bar{b}$ such that $\bar{b}$ is matched to $\bar{a}$ or values $\bar{e}$ such that $\bar{e}$ is matched to itself. Additionally, in $C_{t+1}$ the leftover values are increasing in region III and their targets are exactly the values they were matched to in the new matching. So if $C_{t+1}(j) = \bar{b}$ then $C'(j) = \bar{a}$ and if $C_{t+1}(j) = \bar{e}$ then $C'(j) = \bar{e}$ where $j$ is in region III and $C(j)$ is in the top arc. In both cases, position $j$ has not passed its target. No other region III values have changed so they have not passed their targets. Now we will consider which transpositions are chosen for $T_t$. We will again do this by arc, starting with any values in the top arc, which are at the bottom of region III of $C_{t+1}$. Note that since all the region III values are negative with negative targets, we can only switch with other negative values, so positions $\bar{s}, \cdots, \bar{1}$ which are region II and I negative positions. Since $C_{t+1}[1, t]$ is increasing we will need to use every successive transposition in region II and I until position $\bar{j}$, where the target is sitting, otherwise we would violate the quantum Bruhat graph criteria. Additionally, these switches are all allowed because no values in $C[k + 1, k + 1]$ sit between $C_{t+1}(t)$ and $C'(t)$ by Condition 2. Additionally, all the values in $C_{t+1}[t+1, k]$ are extended values and by the tightness of split values, no extended values can sit between $\bar{b}$ and $\bar{a}$. This is true for all the region III values that are in the top arc. So next we consider $C(j) = C_{j+1}(j)$ in a negative arc. Its target is in $\pi_{j+1}[\bar{s}, \bar{1}]$. Note that by reordering, none of the $C_{j+1}[j + 1, l]$ in the same negative arc sit between $C(j)$ and $C'(j)$. So none of the values in $\pi_{j+1}[j + 1, \bar{j} + 1]$ sit between $C(j)$ and $C'(j)$. So there is a smallest value $\pi_{j+1}(\bar{\tau}) \in C_{j+1}[\bar{s}, \bar{1}]$ bigger than $C(j)$. We will do the transpositions starting at $(j, \bar{\tau})$ and every
successive transposition until we reach the target. This can be done because \(\pi_{j+1}[\overline{s}, \overline{1}]\) is increasing and the target is in this part of \(\pi_{j+1}\). Lastly, we will consider a \(C(j) = C_{j+1}(j)\) in the bottom arc. Similar to the top arc, the only values in \(\pi_{j+1}[s + 1, j]\) are \(e\) such that \(e\) is matched to itself. These values are increasing in \(C_{j+1}\) since the have not been affected by \(T_k \cdots T_{j+1}\). While this arc can contain \(e\) values such that \(c\) is matched to \(e\), since \(\overline{e} > \overline{c}\) for all such \(c\), by the tightness of split zeros, all the \(e\) values in \(C'\) sit above the \(\overline{e}\) values. In fact, any \(e\) in region III of \(C'\) must sit next to an \(e\). Let \(e\) be the largest such value in region III of \(C'\). Until this \(c\) value interferes, all the \(C_{j+1}\) values are \(e\) or \(c\) with the corresponding \(\overline{e}\) or \(\overline{c}\) as their targets in \(C'\) and these targets can be reached by the same means as previous region III values. We now consider the remaining region III values which are all \(e\) next to some \(e\) or a \(c\). But, again, by the increasing structure of \(C_{j+1}\) and Lemma 5.21, the target is in an accessible position and we pick the first value sitting between \(C_{j+1}(j)\) and the target and then pick all the successive transpositions until we reach the target. None of the values sitting below in the column sit between the two values being switched. So region III terminates correctly and the transpositions selected are exactly those selected by the greedy algorithm and none of the transpositions can be skipped.

Next we consider region II. Since the sign of these positions has not changed, none of the region II values have passed their targets. In fact, we must change the sign of the position from positive to negative by using the up edge \((j, \overline{j})\). The only positive values that sit below are the extended values in region IV and since the bottom arc is disjoint from the top arc, all of these values are smaller than \(C_{j+1}(j)\). Also, by the reordering condition, none of the negative values in \(C'[j + 1, k]\) sit between \(C(j)\) and \(C'(j)\) and this still holds for \(C_{j+1}(j)\) since it sits between \(C(j)\) and \(C'(j)\). Lastly, note that \(C_{s+1}[1, s]\) contains the positive values from \(C'[1, s]\) since the rest of the positions in the column have reached their targets. In fact, \(x = C_{s+1}(s)\) is the largest value in \(|C'[1, s]|\) so is the smallest possible region II value of \(C'\) and so we can always switch \(x\) with \(\overline{x}\) without passing the target of any of the positions in region II. In fact, \(x\) will be \(C_{j+1}(j)\) for all \(j\) in region II since it is the largest value in \(C_{j+1}\) which is increasing and because at worst it is the target for position \(r + 1\), the top most position in region II. So we can, and will, always apply \((j, \overline{j})\) at the beginning of \(T_j\). So \((j, \overline{j})\) is a quantum Bruhat graph edge and should be applied and will not take position \(j\) past its target. After \((j, \overline{j})\) is applied, position \(j\) is negative. So we can proceed as in region III by switching with values in \(\pi_{j+1}[\overline{j}, \overline{1}]\) first with the smallest value bigger than \(\overline{x}\) and then with
all the successive transpositions until we reach the target, which is in this part of \( \pi_{j+1} \) by Lemma 5.21.

Lastly, note that when we are done with region II, the only values left in \( C_{r+1}[1,r] \) are exactly the \( C'[1,r] \) values and they are all positive, because these positions never changed sign. Lastly, because of the transpositions applied in \( T_k, \cdots, T_{r+1} \) we have that \( C_{r+1}[1,r] \) is increasing and so each of these positions has already reached its target.

\[ \square \]

This concludes the proof of Theorem 4.4 since we have now shown both parts a) and b).
6 The Second Part of the Quantum Bruhat path: Proving Theorem 4.5

Recall $\Gamma_l(k) := \Gamma_{kk} \ldots \Gamma_{k1}$ where each $\Gamma_{kj}$ was divided into the following four stages.

Stage I: $(i, k + 1), (i, k + 2), \ldots, (i, n)$

Stage II: $(i)$

Stage III: $(i, n), (i, n - 1), \ldots, (i, k + 1)$

Stage IV: $(i, \bar{i} - 1), (i, \bar{i} - 2), \ldots, (i, 1)$

**Theorem 6.1.** Let $w = w_0, w_1, \ldots, w_p = u$ be a path in $A(\Gamma_l(k), w)$ with no down steps in stage IV. Let $C = w[1, k]$ and $C' = u[1, k]$. Then $CC'$ satisfies Condition 3.5.

**Lemma 6.2.** $C(i) \prec C_k(i) \prec C_{k-1}(i) \prec \cdots \prec C_1(i) = C'(i)$ for $i = 1, \ldots, k - 1$

**Proof.** To rephrase, this says that the only position that can pass its target is position $k$. First, suppose a transposition in $T_k$ moves a higher up position past its target with $(k, \bar{i})$ and let $B$ be the column with $T_k$ applied up until that transposition. This is in stage IV of $T_k$ so it must be an up edge and so we must have that $B(i) = a$ and $B(k) = x$ are the same sign. Let $b$ be the target of position $i$ and $\bar{y}$ be the target of position $k$. Then $a < x < b$. Also position $i$ is not changed again during $T_k$. So $C_k(k) = \bar{y}$ and does not change again. But position $i$ must eventually switch across this position in a later $T_j$ in order to pass from $x$ to its target $b$ which would violate the QBG criteria. So $T_k$ cannot move positions $1, \ldots, k - 1$ past their targets. Additionally, now that position $k$ is fixed, no other position can pass its target because it would force a later transposition to switch two values across position $k$. So the only position that can pass its target is position $k$.

**Proof of Theorem 6.1.** We will need to show both (19) and (20). For contradiction to (19), suppose $C(i) = C'(l) = a$ for some $1 \leq i < l \leq k$. Apply to $u$ the reflections $T_k, T_{k-1}, \ldots, T_{l+1}$. By Lemma 5.18, either $a \in \pi_{l+1}[1, i]$ or $\bar{a} \in \pi_{l+1}[l + 1, k]$. If $a \in \pi_{l+1}[1, i]$, the only reflection in $T_l$ that moves position $i$ is $(i, l)$ which moves $\pi$ to position $l$ and so we can never move $a$ to position $l$. If $\bar{a} \in \pi_{l+1}[l + 1, k]$ then both $a$ and $\bar{a}$ are in positions that will not be moved by any subsequent reflections so we would have both $a$ and $\bar{a}$ in $C'$ which is a contradiction.
Next, for contradiction to (20), assume \( C(i) \prec C'(l) = b \prec C'(i) \) for some \( i \leq l \leq k \). Since \( C(i) \prec C_k(i) \prec C_{k-1}(i) \prec \cdots \prec C_1(i) = C'(i) \ i = 1, \cdots, k-1 \) by Lemma 6.2, we must have \( C(i) \prec b \prec C_{l+1}(i) \) or \( C_{l+1}(i) \prec b \prec C_l(i) \) or \( C_l(i) \prec b \prec C'(i) \).

Let \( a = C_{l+1}(i) \) and \( a' = C_l(i) \). It is clear that \( C_l(i) = a' \prec b \prec C'(i) \) is not possible because in order to change \( a' \) to \( C'(i) \) there will be a reflection in \( T_{l-1}, \cdots, T_1 \) that switches two values across \( b \) which is fixed in position \( l \). Next consider \( C_{l+1}(i) = a \prec b \prec C_l(i) = a' \). In order for position \( i \) to change while applying \( T_{l+1} \) we must use the transposition \((i, \overline{l})\). This is a stage IV transposition so it must be an up edge so the value in position \( l \) at the time \((i, \overline{l})\) is applied must have opposite sign to \( a \) and \( a' \). But since we are in stage IV of \( T_{l+1} \) we must also have that position \( l \) does not change sign before reaching its target, \( b \). So \( b \) is also opposite sign of \( a \) and \( a' \). But then it is a contradiction for \( b \) to sit between \( a \) and \( a' \).

Lastly, consider \( C(i) \prec b \prec C_{l+1}(i) \). So there is some \((i, \overline{m})\) applied to some \( w \), the permutation at some point in applying \( T_k, T_{k-1}, \cdots, T_{l+1} \) with \( w(i) = a < b < w(\overline{m}) = c \). Note that \( w(i) \neq b \) because by Lemma 5.18 \( b \in C_{l+1}[1, i] \) or \( b \in C_{l+1}[l + 1, k] \) which means \( b \) can only be moved to position \( \overline{l} \) in \( T_l \) or \( b \) is already fixed in some position \( l + 1, \cdots, k \) which contradicts \( b = C'(l) \). In fact, \((i, \overline{m})\) is in stage IV of \( T_m \). Consider the string of transpositions applied in stage IV of \( T_m \) starting with \((i, \overline{m}): (i, \overline{m}), (i_1, \overline{m}), \cdots, (i_p, \overline{m})\) with \( i > i_1 > \cdots > i_p \). Let \( a' = w(i_p) \), the value in position \( i_p \). This will be the last position to switch in \( T_m \) and so \( a' \) will be the target of position \( m \). We have \( a' \leq a < b < c \) and their all the same sign, by the QBG criteria, the fact that stage IV has no down edges and position \( m \) is increasing for each transposition by Lemma 6.2. Note that \( \pi_m[i, \overline{m}] \) contains no entry between \( a' \) and \( c \) because each transposition in our list from stage IV of \( T_m \) was a QBG edge and so there were no such values between the values being switched, which cover everything from \( a' \) to \( c \). So, in particular, \( b \notin \pi_m[i, \overline{m}] \). Furthermore, no such value can be moved into this region with reflection from \( T_{m-1}, \cdots, T_l \). Stages I, II and III only switch values in positions already between \( i \) through \( \overline{m} \) so nothing new can be moved into these positions. Lastly, we can never use stage IV transpositions to switch a value in position \( j \) with \( l \leq j \leq m \) which is NOT between \( a' \) and \( c \) with a value between \( a' \) and \( c \) because we would be switching across \( a' \) in position \( \overline{m} \). But this means we cannot move \( b \) to position \( l \) with transpositions in \( T_k, \cdots, T_l \) so position \( l \) can never reach its target.

\( \square \)
7 Future Work

It remains to show Conjectures 4.3 and 4.2 which together imply Conjecture 4.1.

In particular for Conjecture 4.2, we have shown the case where $u[1, k]$ is increasing. It remains to show that a quantum Bruhat graph path can be found for any starting permutation, $u$. The main difficulty is with finding the correct reordering process.

For Conjecture 4.3, we have shown that a quantum Bruhat graph path that is a subsequence of $\Gamma_l$ with no stage IV down steps produces a pair of columns, $CC'$ that satisfy the normal reordering condition. It remains to show that any subsequence of $\Gamma_l$ is a quantum Bruhat graph path if and only if it corresponds to a pair of columns $CC'$ with $C$ the split and extended right column of some KN column and $C'$ the split and extended left column of some KN column.

Additionally, Theorem 4.5 proves that the resulting columns satisfy the normal reordering condition. It remains to show that two columns satisfying the normal reordering result in a quantum Bruhat graph path with no stage IV down steps. In type $C$, this is accomplished with a greedy algorithm on $\Gamma_l$. There are two additional complications in type $B$. First, it is possible for a value in $C$ to pass its target value, but we have shown in Lemma 6.2 that this can only happen at the bottom of the column, in position $k$. Second, the algorithm is greedy most of the time but there is a scenario when the algorithm should skip a greedy step.

The next conjecture describes the conditions on a pair of columns $CC'$ that would require the quantum Bruhat graph path to pass the target.

Conjecture 7.1. Let $w, C$ and $C'$ be as in Conjecture 4.3. Any quantum Bruhat graph path from $C$ to $C'$ must go twice around the circle in position $k$ if the following criteria are met:

- $C(k)$ is negative
- $C'(k)$ is positive
- $|C(k)| \leq C'(k)$
- There is no $l$ such that $k + 1 \leq l \leq k + 1$ and $C(k) \prec w(l) \prec C'(k)$

Next, we describe the conjectured criteria to skip a transposition in $\Gamma_l$ that is a quantum Bruhat edge.
Conjecture 7.2. Let $w$, $C$ and $C'$ be as in Conjecture 4.3. Let $(i,j_0), (i,j_1), \ldots, (i,j_p)$ be a quantum Bruhat graph path in $\Gamma_l(i)$ and let $B = C_{i+1}(i,j_0), (i,j_1), \ldots, (i,j_p)$. Suppose $(i,j) \in \Gamma_l(i)$ is the next transposition in $\Gamma_l(i)$ that satisfies the quantum Bruhat graph criteria. Then $(i,j)$ should be skipped if the following criteria are met:

- $(i,j)$ is in stage I of $\Gamma_l(i)$
- $C'(i) \in B[i-1,1]$.
- $C'(i) < C(i) < C'(i)$.
- $C(j)$ is the only value in $B[k+1,n]$ with $C(j) \in \{C'(i) - 1, \ldots, \overline{1}\}$.
- there are no values from the set $\{C'(i) - 1, \ldots, \overline{1}\}$ in $B[n,\ldots,k+1]$. 
Bibliography


