On combinatorial formulas for non-symmetric Macdonald polynomials

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On Combinatorial Formulas for
Non-Symmetric Macdonald Polynomials

by

Kevin Ramer

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Abstract

In 1988, Macdonald introduced two remarkable families of polynomials which bear his name. The first family is the symmetric Macdonald polynomials, which generalize the irreducible characters of semisimple Lie algebras. There are two well-known combinatorial formulas for the symmetric Macdonald polynomials — the Haglund-Haiman-Loehr formula, expressed in terms of certain tableaux but only defined in Lie type $A$, and the Ram-Yip formula, expressed in terms of alcove paths and defined in all Lie types. The connection between these two formulas has been established by Lenart.

The second family is the non-symmetric Macdonald polynomials, which generalize the Demazure characters of semisimple Lie algebras. Again, there are two well-known combinatorial formulas for these polynomials — the Haglund-Haiman-Loehr formula, expressed in terms of certain tableaux but defined only in Lie type $A$, and the Ram-Yip formula, expressed in terms of alcove paths and defined in all Lie types. We establish the connection between these two combinatorial formulas with the long-term goal of finding tableau formulas for the non-symmetric Macdonald polynomials beyond type $A$. We also describe a special case in which this connection recovers the correspondence for the symmetric Macdonald polynomials.
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1 Introduction

In [Mac95, Mac03], Macdonald defines two families of orthogonal polynomials associated with
finite root systems of semisimple Lie algebras – the symmetric Macdonald polynomials $P_\lambda(x; q, t)$
(symmetric under the action of the Weyl group $W$) and the non-symmetric Macdonald polynomials $E_\mu(x; q, t)$.

The symmetric Macdonald polynomials $P_\lambda$ are indexed by dominant weights $\lambda$ and the non-
symmetric Macdonald polynomials $E_\mu$ are indexed by arbitrary weights $\mu$. The $P_\lambda(x; q, t)$ generalize
the characters of the irreducible representations of the corresponding Lie algebra; in fact, we obtain
these irreducible characters by specializing $q = t = 0$. In Lie type $A$, $P_\lambda(x; 0, 0)$ gives us the Schur
polynomials, the characters of the irreducible representations of the general linear group $GL_n$. The $E_\mu(x; q, t)$ generalize the characters of Demazure modules; by specializing $q = t = 0$, we obtain the
Demazure characters exactly.

The Macdonald polynomials have deep connections to affine Lie algebras, double affine Hecke al-
gebras, Hilbert schemes, quantum integrable systems, and conformal field theory, so it is of interest
to have nice methods for computing them. The nicest computational formulas are tableau formulas,
described in terms of fillings of certain diagrams. Consider the aforementioned Schur polynomi-
als, denoted by $s_\lambda(x)$. These can be described combinatorially in terms of semi-standard Young
tableaux, which are fillings of partition diagrams with weakly increasing rows and strictly increas-
ing columns. To illustrate, consider the partition $\lambda = (2, 1, 0)$. The semi-standard Young tableaux
corresponding to this partition shape are given below, along with the corresponding monomials:

\[
\begin{align*}
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\
2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\end{align*}
\]

\[x_1^2x_2 \quad x_1x_2^2 \quad x_1x_2x_3 \quad x_1^2x_3 \quad x_1x_2x_3 \quad x_1x_3^2 \quad x_2^2x_3 \quad x_2x_3^2.\]

Summing these monomials, we get the Schur polynomial

\[s_{(2,1,0)}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.\]

Haglund, Haiman, and Loehr give similar tableau formulas for both the symmetric [HHL05] and
non-symmetric [HHL08] Macdonald polynomials, but these are only defined for Lie type $A$. 

There are other Macdonald polynomial formulas which exist for arbitrary Lie type — those based on the *alcove model* [LP07]. Given an arbitrary Lie type, the alcove model is described via the corresponding Weyl group and alcove picture. From the work of Ram and Yip, we have alcove path formulas for both the symmetric and non-symmetric Macdonald polynomials [RY11]. While these are defined for all Lie types, they are not as nice computationally as the tableau formulas of Haglund, Haiman, and Loehr.

For the type $A$ symmetric Macdonald polynomials $P_\lambda(x; q, t)$, the connection between the Haglund-Haiman-Loehr and Ram-Yip formulas has been established for a dominant regular weight $\lambda$ [Len09]. In the symmetric case, the formula in terms of alcoves paths compresses to one in terms of tableaux, giving us a nice combinatorial formula with considerably fewer terms.

We describe such a connection for the non-symmetric Macdonald polynomials $E_\mu(x; 0, 0)$ where $\mu$ is a regular weight. In this case, there is no compression. In fact, we define a bijection between Ram-Yip alcove paths and Haglund-Haiman-Loehr tableaux. We also show that a certain specialization of this bijection recovers the connection for the symmetric Macdonald polynomials $P_\lambda(x; 0, 0)$.

The motivation for establishing this connection is twofold: (1) to explain how the intricate, non-transparent statistics in the Haglund-Haiman-Loehr formula follow from more general concepts, and (2) to derive tableau formulas beyond type $A$, where no such formulas currently exist.

In the future, we will consider the case of a general weight $\mu$ and the case of $E_\mu(x; q, t)$ for general $q, t$, both of which display additional complexity. We also believe that the connection described here can be extended to give analogs of the Haglund-Haiman-Loehr formula in other Lie types.
2 Background

2.1 Lie Algebras and Their Representations

A Lie algebra is a vector space $L$ over a field $F$ with a bilinear map $L \otimes L \to L$ given by $(a, b) \to [a, b]$ such that

1. $[a, a] = 0$
2. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

for all $a, b, c \in L$. We refer to the map $[\cdot, \cdot]$ as the commutator of $L$.

A subspace $K \subset L$ is a Lie subalgebra if $K$ is a Lie algebra with the commutator induced from $L$. A Lie subalgebra $K \subset L$ is an ideal of $L$ if $[K, L] \subset K$. A Lie algebra homomorphism between Lie algebras $L_1$ and $L_2$ is a linear map $f$ such that

$$f([a, b]) = [f(a), f(b)]$$

for all $a, b \in L_1$. For subalgebras $I, J \subset L$, let

$$[I, J] = \{[i, j] : i \in I, j \in J\}.$$ 

Define inductively $L^1 := L$ and $L^{n+1} = [L, L^n]$. This sequence of ideals is known as the lower central series. A Lie algebra $L$ is said to be nilpotent if $L^n = 0$ for some $n$. Also define inductively $L^{(0)} := L$ and $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. This sequence of ideals is known as the derived series. A Lie algebra $L$ is said to be solvable if $L^{(n)} = 0$ for some $n$.

A Lie algebra $L$ is simple if has no proper ideals. A Lie algebra $L$ is semisimple if it is a direct sum of simple Lie algebras. The normalizer of a subalgebra $K \subset L$ is

$$N(K) = \{a \in L : [a, K] \subset K\}.$$ 

A Cartan subalgebra $K$ of a Lie algebra $L$ is a nilpotent subalgebra such that $N(K) = K$.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of rank $n$, and $\Phi \subset \mathfrak{h}^*$ the irreducible root system of $\mathfrak{g}$. Let $\mathfrak{h}_\mathbb{R}$ be the real span of the roots, $\Phi^+ \subset \Phi$ the positive roots, and $\Phi^- = \Phi - \Phi^+$ the negative roots. We have a Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$
where $g_\alpha$ is a subalgebra such that $[h, g_\alpha] = g_\alpha$. For $\alpha \in \Phi$, define

$$sgn(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Phi^+ \\ -1 & \text{if } \alpha \in \Phi^- \end{cases}$$

and let $|\alpha| = sgn(\alpha)\alpha$. Define

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$ 

Let $\alpha_1, \ldots, \alpha_n \in \Phi^+$ be the simple roots, which generate $\Phi$ and therefore generate $h^*_R$. We have a non-degenerate scalar product $\langle \cdot, \cdot \rangle$ on $h^*_R$. Given a root $\alpha$, consider its corresponding coroot $\alpha^\vee$ given by

$$\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$ 

For each $\alpha \in \Phi$, we define the reflection $s_\alpha$ by

$$s_\alpha(\lambda) = \lambda - \alpha^\vee \langle \lambda, \alpha \rangle$$

for $\lambda \in h^*_R$. Each $\alpha \in \Phi$ can be written as a sum of simple roots. If $\alpha = \sum_i c_i \alpha_i$, then the height of $\alpha$ is $ht(\alpha) = \sum_i c_i$. We denote the highest root in $\Phi^+$ by $\tilde{\alpha}$ and let $\theta = \alpha_0 = -\tilde{\alpha}$.

For $\alpha \in \Phi$, the reflection $s_\alpha$ is the reflection in the hyperplane

$$H_\alpha := \{ \lambda \in h^*_R : \langle \lambda, \alpha^\vee \rangle = 0 \}.$$ 

The group generated by the reflections $s_\alpha$ for $\alpha \in \Phi$ is called the Weyl group, denoted by $W$. We denote the Coxeter generators of $W$ by $s_i = s_{\alpha_i}$ and the length function on $W$ by $\ell(\cdot)$. The Bruhat order on $W$ is defined by covers

$$w \lessdot ws_\alpha$$

such that $\ell(ws_\alpha) = \ell(w) + 1$ for $\alpha \in \Phi^+$. We will refer to the Hasse diagram of the Bruhat order on $W$ as the Bruhat graph on $W$. In other words, the Bruhat graph consists of edges of the form

$$w \xrightarrow{\alpha} ws_\alpha$$

where $\ell(ws_\alpha) = \ell(w) + 1$. The weight lattice $\Lambda$ is given by

$$\Lambda = \{ \lambda \in h^*_R : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi^+ \}.$$
Λ is generated by the fundamental weights ω₁, . . . , ωₙ which form a basis dual to the basis of simple coroots, i.e.

⟨ωᵢ, αⱼ∨⟩ = δᵢⱼ.

The set of dominant weights Λ⁺ is defined by

Λ⁺ = {λ ∈ ℱₗ⁺ : ⟨λ, αᵢ∨⟩ ≥ 0 for all α ∈ Φ⁺}

and the set of dominant regular weights Λ⁺reg is defined by

Λ⁺reg = {λ ∈ ℱₗ⁺ : ⟨λ, αᵢ∨⟩ > 0 for all α ∈ Φ⁺}.

Denote the subgroup of W stabilizing a weight λ by Wₗ and the set of minimal coset representatives in W/Wₗ by Wₗ. Let ℤ[Λ] be the group algebra of the weight lattice Λ, which has a ℤ-basis of formal exponents xᵢ such that

xᵢxⱼ = xᵢ₊ⱼ

for λ, µ ∈ Λ.

For α ∈ Φ and k ∈ ℤ, we define sₜₜ,ₜ to be the affine reflection in the hyperplane

Hₜₜ := \{λ ∈ ℱₗ⁺ : ⟨λ, αᵢ∨⟩ = k}\}.

In other words,

sₜₜ = λ - (⟨λ, αᵢ∨⟩ - k) α.

Reflections of this form generate the affine Weyl group \(\widetilde{W}\) for the dual root system

\(\Phi mundane := \{αᵢ∨ : α ∈ Φ\}\).

The hyperplanes Hₜₜ divide the vector space ℱₗ⁺ into alcoves, with the fundamental alcove given by

\(A₀ := \{λ ∈ ℱₗ⁺ : 0 < ⟨λ, αᵢ∨⟩ < 1\) for all α ∈ Φ⁺}\)

Let V be a vector space over a field F and let End(V) be the set of linear maps from V to V. A representation of a Lie algebra g on V is a Lie algebra homomorphism π : g → End(V). A representation of g on V defines a g-module structure by

\(g \cdot v = π(g)(v)\)
for \( g \in \mathfrak{g} \) and \( v \in V \). A \( \mathfrak{g} \)-module \( V \) is called a weight module if \( V \) decomposes into a direct sum of eigenspaces for \( \mathfrak{h} \)

\[
V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda
\]

where

\[
V_\lambda = \{ v \in V : hv = \lambda(h)v \text{ for all } h \in \mathfrak{h} \}.
\]

If \( V_\lambda \neq 0 \) for some \( \lambda \in \mathfrak{h}^* \) then \( \lambda \) is called a weight of \( V \) and \( V_\lambda \) is called the \( \lambda \)-weight space with weight multiplicity \( \dim(V_\lambda) \). A basis of \( V \) is a weight basis if each basis vector is an element of a weight space. The irreducible representations of \( \mathfrak{g} \) are the weight modules \( V(\lambda) = V_\lambda \) for \( \lambda \in \Lambda^+ \).

Let \( v_\lambda \) be the highest weight vector in \( V(\lambda) \) for \( \lambda \in \Lambda^+ \), \( B \subset \mathfrak{g} \) a Borel subalgebra, and \( w \in W \). The Demazure module \( V_w(\lambda) \) is defined by the action of \( B \) on the extremal weight vector \( w(v_\lambda) \).

### 2.2 The Alcove Path Model

We will now introduce the alcove path model of [LP07]. We say that two distinct alcoves \( A_1 \) and \( A_2 \) are adjacent if they share a common wall. Given such a pair of adjacent alcoves, we write

\[
A_1 \xrightarrow{\beta} A_2
\]

if the common wall is of the form \( H_{\beta,k} \) and \( \beta \in \Phi \) points from \( A_1 \) to \( A_2 \).

**Definition 2.1.** An alcove path is a sequence of alcoves \( (A_0, A_1, \ldots, A_m) \) such that \( A_{j-1} \) and \( A_j \) are adjacent for \( j = 1, \ldots, m \).

We say that an alcove path is reduced if it has minimal length among all alcove paths from \( A_0 \) to \( A_m \).

**Definition 2.2.** For a reduced alcove path \( (A_0, A_1, \ldots, A_m) \) such that

\[
A_0 = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_0 + \lambda
\]

the sequence of roots \( (\beta_1, \beta_2, \ldots, \beta_m) \) is called a \( \lambda \)-chain.

Let \( \lambda \in \Lambda^+ \). We define the collection of admissible subsets \( A(\lambda) \) of our \( \lambda \)-chain to consist of subsets \( J = (j_1 < j_2 < \cdots < j_s) \subset [m] = \{1, \ldots, m\} \) which correspond to a path in the Bruhat graph.
starting at the identity in which each reflection increases the length of the word by exactly 1. Such a path has the form

\[ \text{id} \xrightarrow{\beta_{j_1}} ws_{\beta_{j_1}} \xrightarrow{\beta_{j_2}} ws_{\beta_{j_1}}s_{\beta_{j_2}} \xrightarrow{\beta_{j_3}} \cdots \xrightarrow{\beta_{j_s}} ws_{\beta_{j_1}}s_{\beta_{j_2}} \cdots s_{\beta_{j_s}}. \]

**Theorem 2.3** ([LP07]). For \( \lambda \in \Lambda^+ \)

\[ \text{ch}(V(\lambda)) = \sum_{J \in A(\lambda)} x^{\text{wt}(J)}. \]

### 2.3 Macdonald Polynomials for Arbitrary Root Systems

Let \( \mathbb{Q}(q,t) \) be the field of rational functions in \( q \) and \( t \) and let \( \mathbb{Q}(q,t)[\Lambda] \) be the group algebra of the weight lattice \( \Lambda \) over \( \mathbb{Q}(q,t) \). In [Mac95, Mac03], Macdonald defines a scalar product \( \langle \cdot, \cdot \rangle_{q,t} \) on \( \mathbb{Q}(q,t)[\Lambda] \).

For each \( \lambda \in \Lambda^+ \) let

\[ m_{\lambda} = \sum_{\mu \in W\lambda} x^\mu \]

be the orbit-sum corresponding to \( \lambda \).

**Theorem 2.4** ([Mac03]). For each \( \lambda \in \Lambda^+ \) there is a unique element \( P_\lambda \in \mathbb{Q}(q,t)[\Lambda]^W \) such that

1. \( P_\lambda = m_{\lambda} + \text{lower terms} \)

2. \( \langle P_\lambda, m_\mu \rangle_{q,t} = 0 \) for all \( \mu \in \Lambda^+ \) such that \( \mu < \lambda \)

where “lower terms” means a \( \mathbb{Q}(q,t) \)-linear combination of the orbit sums \( m_\mu \) such that \( \mu \in \Lambda^+ \) and \( \mu < \lambda \).

**Remark 2.5.** The symmetric Macdonald polynomials \( P_\lambda \) describe the following:

1. Let \( \lambda \in \Lambda^+ \). Then

\[ P_\lambda(x;0,0) = \text{ch}(V(\lambda)). \]

2. The \( P_\lambda(x;0,t) \) are the Hall-Littlewood polynomials, also known as the spherical functions for \( p \)-adic groups.

3. \( P_\lambda(x; q, 0) \) are graded characters of Kirillov-Reshitikin modules.
4. The $P_\lambda(x; q, t)$ are Euler characteristics of certain complexes of bigraded modules for current Lie superalgebras [Kho15].

**Theorem 2.6** ([Mac03]). For each $\mu \in \Lambda$ there is a unique element $E_\mu \in \mathbb{Q}(q, t)[\Lambda]$ such that

1. $E_\mu = x^\mu + \text{lower terms}$

2. $\langle E_\mu, x^\lambda \rangle_{q, t} = 0$ for all $\lambda < \mu$

where “lower terms” means a $\mathbb{Q}(q, t)$-linear combination of the $x^\lambda$ such that $\lambda \in \Lambda$ and $\lambda < \mu$.

**Remark 2.7.** The non-symmetric Macdonald polynomials $E_\mu$ describe the following:

1. Let $\mu \in \Lambda$ and $w \in W$ such that $w(\lambda) = \mu$ for some $\lambda \in \Lambda^+$. Then

   $$E_\mu(x; 0, 0) = \text{ch}(V_w(\lambda)).$$

2. The $E_\mu(x; q, 0)$ are graded characters of Demazure-type submodules of Kirillov-Reshitikin modules [LNS+15].

3. If $\alpha_0$ is short then the $E_\mu(x; q, 0)$ are characters of certain Demazure modules for corresponding affine Lie algebras [Ion03].

### 2.4 The Ram-Yip Formula for Macdonald Polynomials

The Ram-Yip formula, first introduced in [RY11], gives an expression for the non-symmetric Macdonald polynomial $E_\mu(x; q, t)$ using the alcove path model. We will describe the Ram-Yip formula using the conventions in [LNS+15].

Let $\mu = v(\lambda)$ where $\lambda$ is a dominant weight and let $w := vw_0$. Identifying alcoves with their corresponding Weyl group elements, consider a reduced alcove path $\overline{\Gamma}$ from the fundamental alcove $A_o$ to $A_o + \mu$:

$$\overline{\Gamma} := (\overline{A}_0 = A_o \overset{-\gamma_1}{\longrightarrow} \overline{A}_1 \overset{-\gamma_2}{\longrightarrow} \cdots \overset{-\gamma_N}{\longrightarrow} \overline{A}_N = A_o + \mu).$$  \hspace{1cm} (1)

Now consider $\Gamma$, a reduced alcove path from the fundamental alcove $A_o$ to the alcove of smallest length in the orbit $WA_o + \mu$. It can be shown that this alcove is $wA_o + \mu$.

$$\Gamma := (A_0 = A_o \overset{-\gamma_1}{\longrightarrow} A_1 \overset{-\gamma_2}{\longrightarrow} \cdots \overset{-\gamma_n}{\longrightarrow} A_n = wA_o + \mu).$$  \hspace{1cm} (2)
Let \( \hat{w} \) denote the affine Weyl group element such that \( \hat{w}(A_\circ) = wA_\circ + \mu \). We will identify \( \Gamma \) with the \( \mu \)-chain of roots \((\gamma_1, \gamma_2, \ldots, \gamma_n)\). From [LP07, Lemma 5.3], we know that this alcove path corresponds to a reduced decomposition \( \hat{s}_{i_1} \cdots \hat{s}_{i_n} \) of \( \hat{w} \) and that

\[
A_j = \hat{s}_{i_1} \cdots \hat{s}_{i_j} A_\circ, \quad \gamma_j = s_{i_1} \cdots s_{i_j} (\alpha_{i_j})
\]

where \( s_{i_j} \) is the projection onto the finite Weyl group and \( \alpha_{i_j} \) is the projection onto the root system corresponding to the finite Weyl group.

We will now consider an alcove path closely related to (2). We call this new alcove path \( \Gamma' \). It is obtained by applying \( w^{-1} \) to \( \Gamma \):

\[
\Gamma' := w^{-1} \Gamma = (A'_0 = w^{-1} A_\circ \xrightarrow{-\delta_1} A'_1 \xrightarrow{-\delta_2} \cdots \xrightarrow{-\delta_n} A'_n = A_\circ + w_\circ \lambda).
\]

Notice that all of the roots \( \delta_i \) are positive, unlike the \( \gamma_i \) which may be positive or negative. Denote the hyperplane separating \( A_{i-1} \) and \( A_i \) (resp. \( A'_{i-1} \) and \( A'_i \)) by \( H_{\gamma_i, -m_i} \) (resp. \( H_{\delta_i, -l_i} \)).

Consider \( J := (j_1 < j_2 < \cdots < j_s) \subset [n] := \{1, \ldots, n\} \) as a folding of the alcove path \( \Gamma \) along the hyperplanes \( H_{\gamma_i, -m_i} \) for \( i \in J \). Define \( \hat{r}_i := s_{\gamma_i, -m_i} \). We define the weight of \( J \) by

\[
wt(J) := \hat{r}_{j_1}' \cdots \hat{r}_{j_s}' (\mu).
\]

Now consider \( J = (j_1 < j_2 < \cdots < j_s) \subset [n] \) as a folding of the alcove path \( \Gamma' \) along the hyperplanes \( H_{\delta_i, -l_i} \) for \( i \in J \). Define \( r'_i := s_{\delta_i} \) and \( \hat{r}_i := s_{\delta_i, -l_i} \). Because \( H_{\gamma_i, -m_i} = wH_{\delta_i, -l_i} \), we must have that \( \hat{r}_i = w\hat{r}_i' w^{-1} \). Notice the following:

\[
wt(J) = \hat{r}_{j_1}' \cdots \hat{r}_{j_s}' (\mu)
\]

\[
= w\hat{r}_{j_1}' w^{-1} \hat{r}_{j_2}' w^{-1} \cdots \hat{r}_{j_s}' w^{-1} (\mu)
\]

\[
= w\hat{r}_{j_1}' \cdots \hat{r}_{j_s}' (w_\circ \lambda).
\]

We define the collection of admissible subsets of \( \Gamma' \), denoted by \( A_\prec (\Gamma') \), to consist of those subsets \( J = (j_1 < j_2 < \cdots < j_s) \subset [n] \) for which we have a path in the Bruhat graph starting at \( w \) in which each reflection increases the length of our word by exactly 1. Such a path has the form

\[
w \xrightarrow{\delta_{j_1}} w\hat{r}_{j_1}' \xrightarrow{\delta_{j_2}} w\hat{r}_{j_1}'\hat{r}_{j_2}' \xrightarrow{\delta_{j_3}} \cdots \xrightarrow{\delta_{j_s}} w\hat{r}_{j_1}'\hat{r}_{j_2}' \cdots \hat{r}_{j_s}'.
\]

We can now describe our first combinatorial formula for the non-symmetric Macdonald polynomials \( E_\mu \).
Theorem 2.8 ([OS13]).

\[ E_\mu(x; 0, 0) = \sum_{J \in A_<(\Gamma')} x^{\text{wt}(J)}. \]  

(7)

Remark 2.9. There is a close analog of Theorem 2.8 which holds for general \( q \) with \( t = 0 \) [OS13].

We will now consider another alcove path related to (2). We call this new alcove path \( \Gamma'' \). It is obtained by applying \( -v^{-1} \) to \( \Gamma \):

\[ \Gamma'' := -v^{-1}\Gamma = (A_0'' = -v^{-1}A_0 \xrightarrow{-\beta_1} A_1'' \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_n} A_n'' = A_0 - \lambda). \]

(8)

Notice that the roots \( \beta_i = -v^{-1}(\gamma_i) = -w_o(\delta_i) \) are positive, just as the roots \( \delta_i \) are positive in (3).

Denote the hyperplane separating \( A_{i-1}' \) and \( A_i'' \) by \( H_{\beta_i, -k_i} \).

Consider \( J = (j_1 < j_2 < \cdots < j_s) \subset [n] \) as a folding of the alcove path \( \Gamma'' \) along the hyperplanes \( H_{\beta_i, -k_i} \) for \( i \in J \). Define \( r_i'' := s_{\beta_i} \) and \( \hat{r}_i'' := s_{\beta_i, -k_i} \). Since \( H_{\delta_i, -l_i} = -w_o H_{\beta_i, -k_i} \), we must have \( \hat{r}_i' = (-w_o)\hat{r}_i''(-w_o) \). Notice the following based on (5):

\[ \text{wt}(J) = w\hat{r}_{j_1}' \cdots \hat{r}_{j_s}'(w_o \lambda) \]

\[ = -v(-w_o)\hat{r}_{j_1}'(-w_o)(-w_o)\hat{r}_{j_2}'(-w_o) \cdots (-w_o)\hat{r}_{j_s}'(-w_o)(-\lambda) \]

\[ = -v\hat{r}_{j_1}'' \cdots \hat{r}_{j_s}''(-\lambda). \]

(9)

Define the collection of admissible subsets of \( \Gamma'' \), denoted by \( A_{\prec}(\Gamma'') \), to consist of those subsets \( J := (j_1 < j_2 < \cdots < j_s) \) for which we have a path in the Bruhat graph starting at \( v \) in which each reflection decreases the length of the word by exactly 1.

Remark 2.10. Notice that for any \( J \in A_{\prec}(\Gamma') \), we have a corresponding path \( J \in A_{\prec}(\Gamma'') \). We obtain this via right multiplication with \( w_o \):

\[
\begin{array}{cccccccccc}
  w & \xrightarrow{\delta_{j_1}} & w\hat{r}_{j_1}' & \xrightarrow{\delta_{j_2}} & \cdots & \xrightarrow{\delta_{j_s}} & w\hat{r}_{j_1}' \cdots \hat{r}_{j_s}' \\
  u_o & \xrightarrow{-\beta_{j_1}} & u\hat{r}_{j_1}'' & \xrightarrow{-\beta_{j_2}} & \cdots & \xrightarrow{-\beta_{j_s}} & u\hat{r}_{j_1}'' \cdots \hat{r}_{j_s}''.
\end{array}
\]

(10)
We obtain this correspondence because $w_\circ(\delta_i) = -\beta_i$ and for $1 \leq m \leq s$

\[
wr'_{j_1} \cdots r'_{j_m} w_\circ = ws_{\delta_{j_1}} \cdots s_{\delta_{j_m}} w_\circ =
\]

\[
= ws_{\delta_{j_1}} \cdots s_{\delta_{j_m}} w_\circ s_{\delta_{j_m}}(\delta_{j_m})
\]

\[
= w w_\circ s_{\delta_{j_1}} \cdots s_{\delta_{j_m}} w_\circ = w_\circ s_{\delta_{j_1}} \cdots s_{\delta_{j_m}} w_\circ
\]

\[
= vr''_{j_1} \cdots r''_{j_m}.
\]

We now obtain the following corollary of Theorem 2.8.

**Corollary 2.11.**

\[
E_\mu(x; 0, 0) = \sum_{J \in A_3(\Gamma''')} x^{\text{wt}(J)}. \tag{11}
\]

**2.5 The Alcove Model in Type A**

We will now specialize the alcove path model to type $A$. We begin with some basic facts about the root system of type $A_{n-1}$. We identify $\mathfrak{h}^*_R$ with the quotient space $V := \mathbb{R}^n/\mathbb{R}(1, \ldots, 1)$ where $\mathbb{R}(1, \ldots, 1)$ is the real span of the vector $(1, \ldots, 1)$ in $\mathbb{R}^n$. Let $\epsilon_1, \ldots, \epsilon_n \in V$ to be the the images of the coordinate vectors of $\mathbb{R}^n$ under this quotient.

The corresponding root system is

\[
\Phi = \{\alpha_{ij} = \epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n\}
\]

with positive roots

\[
\Phi^+ = \{\alpha_{ij} = \epsilon_i - \epsilon_j : 1 \leq i < j \leq n\}.
\]

The simple roots are $\alpha_i = \epsilon_{i,i+1}$ for $i = 1, \ldots, n-1$. The highest root is $\tilde{\alpha} = \alpha_1, n$ and $\alpha_0 = \theta = -\tilde{\alpha} = \alpha_{n,1}$. The weight lattice is $\Lambda = \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$ and the fundamental weights are

\[
\omega_i = \epsilon_1 + \cdots + \epsilon_i
\]

for $i = 1, \ldots, n-1$. A dominant weight $\lambda \in \Lambda^+$ of the form

\[
\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_{n-1} \epsilon_{n-1}
\]
is identified with the partition \((\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0)\). A dominant regular weight \(\lambda \in \Lambda_{\text{reg}}^+\) of the form
\[
\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_{n-1} \epsilon_{n-1}
\]
is identified with the partition with no repeated parts \((\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} \geq \lambda_n = 0)\). The weight \(\rho\) is
\[
\rho = (n-1)\epsilon_1 + (n-2)\epsilon_2 + \cdots + \epsilon_{n-1}
\]
which is identified with the partition \((n-1, n-2, \ldots, 1, 0)\).

The Weyl group \(W\) is the symmetric group on \(n\) letters, denoted by \(S_n\), which acts on \(V\) by permuting \(\epsilon_1, \ldots, \epsilon_n\). We write permutations \(w \in S_n\) in the following one-line notation
\[
w = w(1)w(2) \cdots w(n).
\]
For simplicity, we use the notation \((i, j)\) to denote both the root \(\alpha_{ij}\) and the reflection \(s_{\alpha_{ij}}\).

### 2.6 The Haglund-Haiman-Loehr Formula

**Definition 2.12.** Let \(\mu\) be a weak composition \((\mu_1, \ldots, \mu_n) \in \mathbb{N}^n\). A weak composition is a sequence of positive integers where terms in the sequence are allowed to be 0. To a weak composition, we associate a **diagram** consisting of \(n\) columns with \(\mu_i\) boxes in column \(i\). In Cartesian coordinates, this is the set
\[
dg(\mu) = \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq n, 1 \leq j \leq \mu_i\}.
\]

**Example 2.13.** Let \(\mu = (3, 1, 4, 0)\). We visualize the diagram as follows:

\[
dg(\mu) = \begin{array}{c}
\Hline
| & | & | & | \\
\Hline
| & | & | \\
\Hline
| & | \\
\Hline
\end{array}
\]

**Definition 2.14.** We may also associate an **augmented diagram**, constructed by adding one box to the bottom of each column. We will refer to this additional row as the **basement** of the augmented diagram. In Cartesian coordinates, this is the set
\[
\hat{\dg}(\mu) = \dg(\mu) \cup \{(i, 0) \mid 1 \leq i \leq n\}.
\]
Example 2.15. Let $\mu = (3, 1, 4, 0)$. We visualize the augmented diagram as follows:

For $u = (i, j) \in \text{dg}(\mu)$, define

$$\text{leg}(u) = \{ (i', j') \in \text{dg}(\mu) : j' > j \}$$

$$\text{arm}_L(u) = \{ (i', j) \in \text{dg}(\mu) : i' < i, \mu_{i'} \leq \mu_i \}$$

$$\text{arm}_R(u) = \{ (i', j - 1) \in \widehat{\text{dg}}(\mu) : i' > i, \mu_{i'} < \mu_i \}$$

$$\text{arm}(u) = \text{arm}_L(u) \cup \text{arm}_R(u).$$

Example 2.16. Let $\mu = (3, 1, 4, 0)$. For $u = (3, 1) \in \text{dg}(\mu)$, $\text{leg}(u)$, $\text{arm}_L(u)$, and $\text{arm}_R(u)$ are denoted by $a$, $b$, and $c$ respectively.

Definition 2.17. For $u = (i, j) \in \text{dg}(\mu)$, define the leg length $l(u)$ by

$$l(u) = |\text{leg}(u)|$$

$$= \mu_i - j.$$

Definition 2.18. For $u = (i, j) \in \text{dg}(\mu)$, define the arm length $a(u)$ by

$$a(u) = |\text{arm}(u)|.$$
**Definition 2.20.** A filling of \(\text{dg}(\mu)\) is a function
\[
\tau : \text{dg}(\mu) \rightarrow [n]
\]
where \([n] = \{1, \ldots, n\}\). Such a map assigns a number \(\{1, \ldots, n\}\) to each box in \(\text{dg}(\mu)\).

**Definition 2.21.** To each filling \(\tau\) of \(\text{dg}(\mu)\), we associate an augmented filling of \(\hat{\text{dg}}(\mu)\)
\[
\hat{\tau} : \hat{\text{dg}}(\mu) \rightarrow [n]
\]
defined by
\[
\hat{\tau}(u) = \begin{cases} 
\tau(u) & \text{if } u \in \text{dg}(\mu) \\
i & \text{if } u = (i, 0)
\end{cases}
\]

**Example 2.22.** Let \(\mu = (3, 1, 4, 0)\). Consider the following augmented filling \(\hat{\tau}\) of \(\hat{\text{dg}}(\mu)\):

\[
\hat{\tau} = \begin{bmatrix}
1 & 2 \\
1 & 2 \\
1 & 2 & 3 \\
1 & 2 & 3 & 4
\end{bmatrix}
\]

**Definition 2.23.** Let \(u = (i, j), v = (i', j') \in \hat{\text{dg}}(\mu)\). The boxes \(u\) and \(v\) attack each other if either of the following conditions is met:

1. They are in the same row, i.e. \(j = j'\).
2. They are in consecutive rows and the box in the lower row is to the right of the one in the upper row, i.e. \(j' = j - 1\) and \(i' > i\).

**Definition 2.24.** A filling \(\hat{\tau}\) is non-attacking if \(\hat{\tau}(u) \neq \hat{\tau}(v)\) for every pair of attacking boxes \(u, v \in \hat{\text{dg}}(\mu)\).

Notice that the filling described in Example 2.22 is non-attacking.

For a box \(u = (i, j) \in \hat{\text{dg}}(\mu)\), denote the box directly below it by \(d(u) = (i, j - 1) \in \hat{\text{dg}}(\mu)\).
**Definition 2.25.** A *descent* of a filling $\hat{\tau}$ is a box $u \in \hat{\text{dg}}(\mu)$ such that $d(u) \in \hat{\text{dg}}(\mu)$ and $\hat{\tau}(u) > \hat{\tau}(d(u))$. In other words, a descent is any instance where the number filled in a given box is greater than the number filled in the box directly below it.

For an augmented filling $\hat{\tau}$, we denote the set of descents of $\hat{\tau}$ by $\text{Des}(\hat{\tau})$. We may now define the statistic $\text{maj}(\hat{\tau})$, given by

$$\text{maj}(\hat{\tau}) = \sum_{u \in \text{Des}(\hat{\tau})} (l(u) + 1).$$

**Definition 2.26.** We will now consider *triples* of boxes $(u, v, w) \in \hat{\text{dg}}(\mu)$ such that

$$u = (i, j)$$
$$v = (i', j')$$
$$w = (i, j - 1).$$

1. If $i < i'$ and $j' = j - 1$, where the column containing $u, w$ is strictly taller than the column containing $v$, we have a *Type 1 triple*:

```
  u
 /|
 / |
 /  |
/    | v
   u
```

2. If $i > i'$ and $j' = j$, where the column containing $u, w$ is weakly taller than the column containing $v$, we have a *Type 2 triple*:

```
  v
 /|
 / |
 /  |
/    | u
   w
```

**Definition 2.27.** Let $\hat{\tau}$ be a filling of $\hat{\text{dg}}(\mu)$. A Type 1 or Type 2 triple $(u, v, w) \in \hat{\text{dg}}(\mu)$ is an *inversion triple* if $\hat{\tau}(u) > \hat{\tau}(v)$ or $\hat{\tau}(v) > \hat{\tau}(w)$. Otherwise, $(u, v, w) \in \hat{\text{dg}}(\mu)$ is said to be a *coinversion triple*. If $(u, v, w)$ is a coinversion triple, then

$$\hat{\tau}(u) \leq \hat{\tau}(v) \leq \hat{\tau}(w).$$

We may now define the statistic $\text{coinv}(\hat{\tau})$, given by

$$\text{coinv}(\hat{\tau}) = \# \text{ of coinversion triples in } \tau.$$
For a filling $\tau$ of $\text{dg}(\mu)$ let

$$x^{\tau} = x_1^\# \text{ of 1's} \cdot x_2^\# \text{ of 2's} \cdots x_n^\# \text{ of } n's.$$ 

The $n$-tuple ($\#$ of 1’s, ..., $\#$ of $n$’s) is often referred to as $\text{content}(\tau)$. Let

$$\text{HHL}(\mu) := \{\text{non-attacking fillings of } \widehat{\text{dg}}(\mu)\}.$$

**Theorem 2.28** ([HHL08]).

$$E_\mu(x; q, t) = \sum_{\widehat{\tau} \in \text{HHL}(\mu)} x^{\tau} q^{\text{maj}(\widehat{\tau})} t^{\text{coinv}(\widehat{\tau})} \prod_{u \in \text{dg}(\mu), \widehat{\tau}(u) \neq \widehat{\tau}(d(u))} \frac{1 - t}{1 - q^{d(u)} + q^{\mu(u)} + 1}.$$ 

Define $\text{SF}(\mu)$, the set of **skyline fillings of shape** $\mu$, to be the set of non-attacking fillings with no descents such that every Type 1 or Type 2 triple is an inversion triple.

**Corollary 2.29.**

$$E_\mu(x; 0, 0) = \sum_{\widehat{\tau} \in \text{SF}(\mu)} x^{\tau}.$$ 

**Proof.** When $q = t = 0$, the non-zero terms in $E_\mu(x; 0, 0)$ are those from non-attacking fillings $\widehat{\tau}$ with

$$\text{maj}(\widehat{\tau}) = \text{coinv}(\widehat{\tau}) = 0.$$ 

By the definitions of $\text{maj}(\widehat{\tau})$ and $\text{coinv}(\widehat{\tau})$, these are non-attacking fillings with no descents such that every triple is an inversion triple. \hfill \Box

**Example 2.30.** Let $\mu = (3, 1, 4, 0)$. We wish to compute $E_\mu(x; 0, 0)$. To do so, we must find all non-attacking fillings with no descents such that every triple is an inversion triple. Recall $\widehat{\text{dg}}(\mu)$:

There are exactly seven fillings $\widehat{\tau} \in \text{SF}(\mu)$. They are given below. Verification that these fillings satisfy the necessary conditions is left to the reader.
1. \( \hat{\tau} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \leftrightarrow x^\tau = x_1^4 x_2^3 x_3^1. \)

2. \( \hat{\tau} = \begin{pmatrix} 2 \\ 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \leftrightarrow x^\tau = x_1^3 x_2^4 x_3^1. \)

3. \( \hat{\tau} = \begin{pmatrix} 1 \\ 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \leftrightarrow x^\tau = x_1^4 x_2^2 x_3^2. \)

4. \( \hat{\tau} = \begin{pmatrix} 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \leftrightarrow x^\tau = x_1^3 x_2^3 x_3^2. \)
Thus, we have

\[ E_{\mu}(x; 0, 0) = x^4 x_2 x_3 + x_1^3 x_2 x_3 + x_1^2 x_2 x_3 + x_1^3 x_2^2 x_3 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2. \]  

For each skyline filling \( \hat{\tau} \in \text{SF}(\mu) \), we wish to associate a permuted basement skyline filling of the partition shape \( \lambda \). These are the fillings we will use to define the bijection between Ram-Yip alcove paths and Haglund-Haiman-Loehr fillings. Permuted basement skyline fillings are similar to the fillings defined in [Mas09].

For any weak composition \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n \), \( \mu = v(\lambda) \) for some partition \( \lambda \) and permutation \( v \in S_n \). Choose \( v \) to be the highest coset representative such that \( v(\lambda) = \mu \).
Example 2.31. Let $\mu = (3, 1, 4, 0)$. The corresponding partition is $\lambda = (4, 3, 1, 0)$ and the highest coset representative $v \in S_n$ such that $v(\lambda) = \mu$ is $v = 3124$.

We retain a similar notion of diagram and augmented diagram for $\lambda$.

Definition 2.32. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ be a partition. To this, we associate a diagram consisting of $n$ columns with $\mu_i$ boxes in column $i$. In Cartesian coordinates, this is the set

$$\text{dg}(\lambda) = \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq n, 1 \leq j \leq \lambda_i\}.$$ 

We may also associate an augmented diagram, constructed by adding one box to the bottom of each column. We will refer to this additional row as the basement of the augmented diagram. In Cartesian coordinates, this is the set

$$\widehat{\text{dg}}(\lambda) = \text{dg}(\lambda) \cup \{(i, 0) \mid 1 \leq i \leq n\}.$$ 

Example 2.33. Let $\lambda = (4, 3, 1, 0)$. We visualize the diagram and augmented diagram as follows:

\[ \text{dg}(\lambda) = \quad \widehat{\text{dg}}(\lambda) = \]

We retain the same notion of a filling for $\text{dg}(\lambda)$ while the definition of an augmented filling must be altered slightly.

Definition 2.34. A filling of $\text{dg}(\lambda)$ is a function

$$\sigma : \text{dg}(\lambda) \to [n].$$

Definition 2.35. To each filling $\sigma$ of $\text{dg}(\lambda)$, we associate an augmented filling of $\widehat{\text{dg}}(\lambda)$

$$\widehat{\sigma} : \widehat{\text{dg}}(\lambda) \to [n]$$

defined by

$$\widehat{\sigma}(u) = \begin{cases} 
\sigma(u) & \text{if } u \in \text{dg}(\lambda) \\
v(i) & \text{if } u = (i, 0).
\end{cases}$$
For any augmented filling $\hat{\tau}$ (resp. filling $\tau$) of $\hat{\text{dg}}(\mu)$ (resp. $\text{dg}(\mu)$), we can define an augmented filling $\hat{\sigma}$ (resp. filling $\sigma$) of $\hat{\text{dg}}(\lambda)$ (resp. $\text{dg}(\lambda)$) by permuting the columns with $v^{-1}$.

Similarly, for any augmented filling $\hat{\sigma}$ (resp. filling $\sigma$) of $\hat{\text{dg}}(\lambda)$ (resp. $\text{dg}(\lambda)$), we can define an augmented filling $\hat{\tau}$ (resp. filling $\tau$) of $\hat{\text{dg}}(\mu)$ (resp. $\text{dg}(\mu)$) by permuting the columns with $v$.

This defines a bijection between augmented fillings $\hat{\sigma}$ (resp. fillings $\sigma$) of $\lambda$ and augmented fillings $\hat{\tau}$ (resp. fillings $\tau$) of $\mu$. Notice that the content of the fillings is preserved by this bijection as we are only permuting the order of the columns.

**Example 2.36.** Let $\mu = (3, 1, 4, 0)$. Then $\lambda = (4, 3, 1, 0)$, $v = 3124$, and $v^{-1} = 2314$. Consider the following augmented filling $\hat{\tau}$ of $\hat{\text{dg}}(\mu)$ and the corresponding augmented filling $\hat{\sigma}$ of $\hat{\text{dg}}(\lambda)$:

\[
\hat{\tau} = \begin{array}{cccc}
1 & 2 & & \\
1 & 2 & 3 & 4 \\
\end{array} \quad \leftrightarrow \quad \hat{\sigma} = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & & \\
3 & 1 & 2 & \\
\end{array}.
\]

Notice that the basement for any augmented filling $\hat{\sigma}$ of $\hat{\text{dg}}(\lambda)$ is filled by $v(1) v(2) \cdots v(n)$.

In our new diagram shape, we retain exactly the same notion of a descent as in Definition 2.25. We must, however, adjust our definition of attacking boxes from Definition 2.23 for this new diagram shape. Since our partition shape arranges the columns from tallest to shortest, starting from the left, and $v$ is a highest coset representative, we get the following definition.

**Definition 2.37.** Let $a = (i, j), b = (i', j') \in \hat{\text{dg}}(\lambda)$ and assume without loss of generality that $i < i'$. We say that $a$ and $b$ attack each other if:

0. They are in the same row, i.e. $j = j'$. We call these Type 0 attacking pairs.
1. They are in consecutive rows with \( j' = j - 1 \) and \( v(i) < v(i') \). This gives the following arrangement:

\[
\begin{array}{c}
  a \\
  \text{ } \\
  v(i) \\
\end{array}
\begin{array}{c}
  b \\
  \text{ } \\
  v(i')
\end{array}
\]

We call these Type 1 attacking pairs.

2. They are in consecutive rows with \( j = j' - 1 \) and \( v(i') < v(i) \). This gives the following arrangement:

\[
\begin{array}{c}
  b \\
  \text{ } \\
  v(i') \\
\end{array}
\begin{array}{c}
  a \\
  \text{ } \\
  v(i)
\end{array}
\]

We call these Type 2 attacking pairs.

We retain the same notion of non-attacking fillings as Definition 2.23.

**Definition 2.38.** A filling \( \hat{\sigma} \) of \( \hat{\text{dg}}(\lambda) \) is non-attacking if \( \hat{\sigma}(a) \neq \hat{\sigma}(b) \) for any pair of attacking boxes \( a, b \).

Because our partition shape arranges the columns from tallest to shortest, starting from the left, and \( v \) is a highest coset representative, we get the following definitions for triples.

**Definition 2.39.** We will now consider triples of boxes \( (a, b, c) \in \hat{\text{dg}}(\lambda) \) such that

\[
\begin{align*}
  a &= (i, j) \\
  b &= (i', j') \\
  c &= (i, j - 1)
\end{align*}
\]

where \( i < i' \).
1. If \( j' = j - 1 \) and \( v(i) < v(i') \) then we have a *Type 1 triple*:

\[
\begin{array}{c}
\hat{\sigma} \\
\hline
a \\
\hline
c
\end{array}
\begin{array}{c}
\hat{\sigma} \\
\hline
b
\end{array}
\begin{array}{c}
v(i) \\
\hline
v(i')
\end{array}
\]

2. If \( j' = j \) and \( v(i) > v(i') \) then we have a *Type 2 triple*:

\[
\begin{array}{c}
\hat{\sigma} \\
\hline
a \\
\hline
c
\end{array}
\begin{array}{c}
\hat{\sigma} \\
\hline
b
\end{array}
\begin{array}{c}
v(i) \\
\hline
v(i')
\end{array}
\]

We retain the same definition of an inversion triples and coinversion triples as Definition 2.27.

**Definition 2.40.** Let \( \hat{\sigma} \) be a filling of \( \hat{\text{dg}}(\lambda) \). We say that a Type 1 or Type 2 triple \( (a, b, c) \in \hat{\sigma} \) is an *inversion triple* if \( \hat{\sigma}(a) > \hat{\sigma}(b) \) or \( \hat{\sigma}(b) > \hat{\sigma}(c) \). Otherwise, we say that \( (a, b, c) \) is a *coinversion triple*. If \( (a, b, c) \) is a coinversion triple then

\[
\hat{\sigma}(a) \leq \hat{\sigma}(b) \leq \hat{\sigma}(c).
\]

Let PBSF(\( \mu \)), the set of *permuted basement skyline fillings* of \( \mu \), denote the set of non-attacking fillings \( \hat{\sigma} \) of \( \hat{\text{dg}}(\lambda) \) with no descents and no coinversion triples.

**Corollary 2.41.**

\[
E_{\mu}(x; 0, 0) = \sum_{\hat{\sigma} \in \text{PBSF}(\mu)} x^\sigma.
\]

**Proof.** This is a simple consequence of Corollary 2.29 because \( v \) defines a bijection between SF(\( \mu \)) and PBSF(\( \mu \)) that has no effect on the content of the filling. \( \square \)
Example 2.42. Let $\mu = (3, 1, 4, 0)$. Then $\lambda = (4, 3, 1, 0)$ and $v = 3124$. Any $\hat{\sigma} \in \text{PBSF}(\mu)$ must have no descents, which leaves us with the following:

$$
\hat{\sigma} = \begin{array}{c}
  c \\
  b & 1 \\
  a & 1 \\
  3 & 1 & 2 \\
  3 & 1 & 2 & 4
\end{array}
$$

Since there are no descents, we must have $1 \leq c \leq b \leq a \leq 3$. Regardless of the particular values of $a$, $b$, and $c$, all Type 1 triples and Type 2 triples are inversion triples. We get the following seven fillings in $\text{PBSF}(\mu)$.

1. 

$$
\hat{\sigma} = \begin{array}{c}
  1 \\
  2 & 1 \\
  2 & 1 \\
  3 & 1 & 2 \\
  3 & 1 & 2 & 4
\end{array} \leftrightarrow x^\sigma = x_1^4 x_2^3 x_3^4.
$$

2. 

$$
\hat{\sigma} = \begin{array}{c}
  2 \\
  2 & 1 \\
  2 & 1 \\
  3 & 1 & 2 \\
  3 & 1 & 2 & 4
\end{array} \leftrightarrow x^\sigma = x_1^3 x_2^4 x_3^1.
$$

3. 

$$
\hat{\sigma} = \begin{array}{c}
  1 \\
  2 & 1 \\
  3 & 1 \\
  3 & 1 & 2 \\
  3 & 1 & 2 & 4
\end{array} \leftrightarrow x^\sigma = x_1^4 x_2^2 x_3^2.
$$
Thus, we have

\[
E_{\mu}(x; 0, 0) = x_4^4 x_2^3 x_3^1 + x_1^3 x_2^4 x_3^1 + x_1^3 x_2^2 x_3^2 + x_1^3 x_2^3 x_3^2 + x_1^4 x_2^1 x_3^3 + x_1^3 x_2^3 x_3^1 + x_1^2 x_2^3 x_3^1 + x_1^1 x_2^4 x_3^1.
\]

(13)

This matches the expression in (12).
3 \(\mu\)-chains in type \(A\)

Our goal in this section is to give precise descriptions of the chains \(\bar{\Gamma}, \Gamma,\) and \(\Gamma''\) introduced in Section 2.4 for Lie type \(A\). These are the chains necessary to describe the Ram-Yip formula via the conventions in [LNS+15].

Let \(\lambda \in \Lambda^+_{\text{reg}}\) be a dominant regular weight and \(\mu = v(\lambda)\) for some \(v \in W = S_n\). The \(\mu\)-chain is a sequence of roots orthogonal to the hyperplanes separating the fundamental alcove \(A_o\) from the translated alcove \(A_o + \mu\). Recall that for a dominant regular weight \(\lambda \in \Lambda^+_{\text{reg}}\)

\[
\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_{n-1} \varepsilon_{n-1}
\]

where \((\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > 0)\). We can also write \(\lambda\) in terms of fundamental weights \(\omega_i\)

\[
\lambda = \omega_{\lambda'_1} + \cdots + \omega_{\lambda'_{\lambda_1}}
\]

where \((\lambda'_1 \geq \cdots \geq \lambda'_{\lambda_1})\) is the conjugate partition to \(\lambda\). Notice that:

1. \(\lambda'\) can have repeated parts.

2. Because \(\lambda\) has no repeated parts, the difference between any two parts of \(\lambda'\) is at most 1.

We obtain the following expression for \(\mu\):

\[
\mu = v(\omega_{\lambda'_1} + \cdots + \omega_{\lambda'_{\lambda_1}})
\]

\[
= v(\omega_{\lambda'_1}) + \cdots + v(\omega_{\lambda'_{\lambda_1}}).
\]

We can describe the \(\mu\)-chain as a concatenation of smaller chains corresponding to the weights \(v(\omega_k)\). Let \(1 \leq k \leq n\). Recall that in type \(A\), a fundamental weight \(\omega_k\) has the form

\[
\omega_k = \varepsilon_1 + \cdots + \varepsilon_k.
\]

(14)

Rename \(v(1), \ldots, v(k)\) as \(i_1, \ldots, i_k\) with

\[
i_1 < i_2 < \cdots < i_k.
\]

We have the following expression for \(v(\omega_k)\):

\[
v(\omega_k) = \varepsilon_{i_1} + \cdots + \varepsilon_{i_k}.
\]

(15)
Now rename $v(k + 1), \ldots, v(n)$ as $j_1, \ldots, j_{n-k}$ where

$$j_1 < j_2 < \cdots < j_{n-k}.$$ 

**Proposition 3.1.** The chain of roots from the fundamental alcove $A_0$ to the translated alcove $A_0 + v(\omega_k)$ is given by

\[
\begin{pmatrix}
    \ldots, (j_1, i_p), (j_1, i_{p+1}), (j_1, i_{p+2}), \ldots \\
    \ldots, (j_{l-1}, i_p), (j_{l-1}, i_{p+1}), (j_{l-1}, i_{p+2}), \ldots \\
    \ldots, (j_{l}, i_p), (j_{l}, i_{p+1}), (j_{l}, i_{p+2}), \ldots \\
    \ldots, (j_{q}, i_1), (j_{q-1}, i_1), (j_{q-2}, i_1), \ldots \\
    \ldots, (j_{q}, i_2), (j_{q-1}, i_2), (j_{q-2}, i_2), \ldots \\
    \ldots, (j_{q}, i_k), (j_{q-1}, i_k), (j_{q-2}, i_k), \ldots \\
\end{pmatrix}
\]

where $j_l < i_p$ for all $p$  where $j_{l-1} < i_p$ for all $p$

where $j_1 < i_p$ for all $p$  where $j_{q} > i_1$ for all $q$

where $j_q > i_2$ for all $q$  where $j_q > i_k$ for all $q$

\[(16)\]

**Proof.** For simplicity, let $\xi = v(\omega_k)$. We use a construction from [LP07] with the weight $-\xi$. This construction depends on the choice of a total order on the simple roots. In Type $A_{n-1}$, the total order has the form

$$\alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}$$

where $\alpha_i = (i, i+1)$. The roots in our $\xi$-chain depend on the set

$$\mathcal{R} = \bigcup_{\alpha \in \Phi^+} = \begin{cases} 
    \{s_{\alpha, m} | 0 \geq m > (\xi, \alpha^\vee)\} & \text{if } (\xi, \alpha^\vee) < 0 \\
    \{s_{\alpha, m} | 0 < m \leq (\xi, \alpha^\vee)\} & \text{if } (\xi, \alpha^\vee) > 0 \\
    \emptyset & \text{if } (\xi, \alpha^\vee) = 0.
\end{cases}$$

Any $\alpha \in \Phi^+$ must have one of the following forms:

1. $\alpha = (i_p, i_q)$ with $i_p < i_q$. Here, $(\xi, \alpha^\vee) = 0$ and we get no contribution to $\mathcal{R}$.

2. $\alpha = (i_p, j_q)$ with $i_p < j_q$. Here, $(\xi, \alpha^\vee) = 1$ and we add $s_{\alpha, 1}$ to $\mathcal{R}$.

3. $\alpha = (j_q', i_p')$ with $j_q' < i_p'$. Here, $(\xi, \alpha^\vee) = -1$ and we add $s_{\alpha, 0}$ to $\mathcal{R}$.

4. $\alpha = (j_p, j_q)$ with $j_p < j_q$. Here, $(\xi, \alpha^\vee) = 0$ and we get no contribution to $\mathcal{R}$.
Consider the map \( h : \mathcal{R} \to \mathbb{R}^n \) given by

\[
h : s_{\alpha,m} \mapsto (\lambda, \alpha^\vee)^{-1}(m, -(\omega_1, \alpha^\vee), \ldots, -(\omega_{n-1}, \alpha^\vee)).
\]

If \( \alpha = (i_p, j_q) \), with \( i_p < j_q \), then \( m = 1 \) and

\[
h : s_{\alpha,1} \mapsto (1, -(\omega_1, \alpha^\vee), \ldots, -(\omega_{n-1}, \alpha^\vee)).
\]

If \( \alpha = (j_{q'}, i_{p'}) \), with \( j_{q'} < i_{p'} \), then \( m = 0 \) and

\[
h : s_{\alpha,0} \mapsto (0, (\omega_1, \alpha^\vee), \ldots, (\omega_{n-1}, \alpha^\vee)).
\]

Now we define a total order on \( \mathcal{R} = \{s_1 < s_2 < \cdots < s_r\} \) via the lexicographic order on \( h(s_1), \ldots, h(s_r) \) in \( \mathbb{R}^n \). Notice

1. Due to the values of \( m \), the \((j_{q'}, i_{p'})\) come before the \((i_p, j_q)\) under the lexicographic order on the image of \( h \).

2. Under the lexicographic order on the image of \( h \), the \((j_{q'}, i_{p'})\) are ordered by the following rules:
   a) Given two roots, the root with the highest value of \( j_{q'} \) comes first.
   b) If two roots have the same value of \( j_{q'} \), the root with the lowest value of \( i_{p'} \) comes first.

3. Under the lexicographic order on the image of \( h \), the \((i_p, j_q)\) are ordered as follows:
   a) Given two roots, the root with the lowest value of \( i_p \) comes first.
   b) If two roots have the same value of \( i_p \), the root with the highest value of \( j_q \) comes first.

Now consider the map \( b : \mathcal{R} \to \Phi \) given by

\[
b : s_{\alpha,k} \mapsto \begin{cases} 
\alpha & \text{if } m \leq 0 \text{ and } \alpha \in \Phi^+ \\
-\alpha & \text{if } m > 0 \text{ and } \alpha \in \Phi^+.
\end{cases}
\]

If \( \alpha = (i_p, j_q) \) with \( i_p < j_q \) then \( m = 1 \) and

\[
b : s_{\alpha,1} \mapsto -\alpha = (j_q, i_p).
\]
If \( \alpha = (j_q', i_{p'}) \) with \( j_q' < i_{p'} \) then \( m = 0 \), and

\[
b : s_{\alpha,0} \mapsto \alpha = (j_q', i_{p'}).
\]

Our \( \nu(\omega_k) \)-chain consists of the roots \((b(s_1), \ldots, b(s_r))\). Using the ordering rules from above, we get the expression in (16).

**Remark 3.2.** Define \( \hat{\Gamma}(k) \) as follows, by permuting non-overlapping roots in (16)

\[
\hat{\Gamma}(k) = \begin{pmatrix}
\ldots, (j_p, i_1), (j_{p-1}, i_1), (j_{p-2}, i_1), \ldots \\
\ldots, (j_p, i_2), (j_{p-1}, i_2), (j_{p-2}, i_2), \ldots \\
\ldots, (j_p, i_k), (j_{p-1}, i_k), (j_{p-2}, i_k), \ldots \\
\ldots, (j_q, i_1), (j_{q-1}, i_1), (j_{q-2}, i_1), \ldots \\
\ldots, (j_q, i_2), (j_{q-1}, i_2), (j_{q-2}, i_2), \ldots \\
\ldots, (j_q, i_k), (j_{q-1}, i_k), (j_{q-2}, i_k), \ldots
\end{pmatrix}
\]

where \( j_p < i_1 \) for all \( p \)

where \( j_p < i_2 \) for all \( p \)

where \( j_p < i_k \) for all \( p \)

where \( j_q > i_1 \) for all \( q \)

where \( j_q > i_2 \) for all \( q \)

where \( j_q > i_k \) for all \( q \).

Define the *positive part* of \( \hat{\Gamma}(k) \) by

\[
\hat{\Gamma}^+(k) = \begin{pmatrix}
\ldots, (j_p, i_1), (j_{p-1}, i_1), (j_{p-2}, i_1), \ldots \\
\ldots, (j_p, i_2), (j_{p-1}, i_2), (j_{p-2}, i_2), \ldots \\
\ldots, (j_p, i_k), (j_{p-1}, i_k), (j_{p-2}, i_k), \ldots
\end{pmatrix}
\]

where \( j_p < i_1 \) for all \( p \)

where \( j_p < i_2 \) for all \( p \)

and define the *negative part* of \( \hat{\Gamma}(k) \) by

\[
\hat{\Gamma}^-(k) = \begin{pmatrix}
\ldots, (j_q, i_1), (j_{q-1}, i_1), (j_{q-2}, i_1), \ldots \\
\ldots, (j_q, i_2), (j_{q-1}, i_2), (j_{q-2}, i_2), \ldots \\
\ldots, (j_q, i_k), (j_{q-1}, i_k), (j_{q-2}, i_k), \ldots
\end{pmatrix}
\]

where \( j_q > i_1 \) for all \( q \)

where \( j_q > i_2 \) for all \( q \)

where \( j_q > i_k \) for all \( q \).

For \( k = 1, \ldots, n - 1 \), define

\[
\hat{\Gamma}_s(k) := \hat{\Gamma}^-(k + 1)\hat{\Gamma}^+(k)
\]

(20)

\[
\hat{\Gamma}_t(k) := \hat{\Gamma}^-(k)\hat{\Gamma}^+(k)
\]

(21)
where \( \hat{\Gamma}^-(n) := \emptyset \). Further define
\[
\hat{\Gamma}_j = \begin{cases} 
\hat{\Gamma}_s(\lambda'_j) & \text{if } j = \min \{ i : \lambda'_i = \lambda'_j \} \\
\hat{\Gamma}_l(\lambda'_j) & \text{otherwise.} 
\end{cases}
\] (22)

Since \( \hat{\Gamma}^-(n) = \emptyset \), we have
\[
\hat{\Gamma} = \hat{\Gamma}_1 \hat{\Gamma}_2 \cdots \hat{\Gamma}_{\lambda_1} \hat{\Gamma}^-(\lambda_1). 
\] (23)

**Remark 3.3.** We use the notation \( \hat{\Gamma}_s(k) \) to denote a short chain, which corresponds to the first part of the conjugate partition of length \( k \), and the notation \( \hat{\Gamma}_l(k) \) to denote a long chain, which corresponds to a repeated part of length \( k \). Why \( \hat{\Gamma}_s(k) \) is ultimately “short” and why \( \hat{\Gamma}_l(k) \) is ultimately “long” will be made clear by the end of this section.

**Example 3.4.** Let \( \mu = (3, 1, 4, 0) \). Then \( \lambda = (4, 3, 1, 0) \), \( \lambda' = (3, 2, 2, 1) \), and \( v = 3124 \). We have
\[
\mu = v(\omega_3) + v(\omega_2) + v(\omega_2) + v(\omega_1) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + (\varepsilon_1 + \varepsilon_2) + (\varepsilon_1 + \varepsilon_2) + (\varepsilon_1).
\]
The corresponding \( \hat{\Gamma}(k) \) are
\[
\hat{\Gamma}(3) = (| (4, 1), (4, 2), (4, 3))
\]
\[
\hat{\Gamma}(2) = ((2, 3), | (4, 1), (2, 1), (4, 3))
\]
\[
\hat{\Gamma}(2) = ((2, 3), | (4, 1), (2, 1), (4, 3))
\]
\[
\hat{\Gamma}(1) = ((2, 3), (1, 3) | (4, 3))
\]
where the positive and negative parts of each are separated by a vertical line. From (20), and (21) we have
\[
\hat{\Gamma}_s(3) = \hat{\Gamma}^- (4) \hat{\Gamma}^+ (3) = \emptyset
\]
\[
\hat{\Gamma}_s(2) = \hat{\Gamma}^- (3) \hat{\Gamma}^+ (2) = ((4, 1), (4, 2), | (4, 3), (2, 3))
\]
\[ \tilde{\Gamma}_l(2) = \hat{\Gamma}^-(2) \hat{\Gamma}^+(2) \]
\[ = ((4,1), (2,1), (4,3), (2,3)) \]
\[ \tilde{\Gamma}_s(1) = \hat{\Gamma}^-(2) \hat{\Gamma}^+(1) \]
\[ = ((4,1), (2,1), (4,3), (2,3), (1,3)) \]
\[ \tilde{\Gamma}^-(1) = ((4,3)) \]

with \( \hat{\Gamma}^- \) and \( \hat{\Gamma}^+ \) separated by a vertical line in each. Concatenating these, as in (23), we have
\[ \tilde{\Gamma} = (\]
\[ (4,1), (4,2), (4,3), (2,3), \]
\[ (4,1), (2,1), (4,3), (2,3), \]
\[ (4,1), (2,1), (4,3), (2,3), (1,3) \]
\[ (4,3)) \]

To obtain a reduced \( \mu \)-chain \( \Gamma \) we must remove the last occurrence of each negative root in \( \tilde{\Gamma} \). We need a reduced \( \mu \)-chain \( \Gamma \) because the Ram-Yip formula requires a chain of roots from the fundamental alcove \( A_\circ \) to the alcove of smallest length in the orbit \( WA_\circ + \mu \). In Example 3.4, the last occurrence of each negative root is underlined.

We remove negative roots by moving them to the end of \( \tilde{\Gamma} \) via Coxeter moves then dropping them entirely. For roots \( \beta_1, \beta_2 \in \Phi \), a Coxeter move is a move of the form
\[ \beta_1, \beta_1 + \beta_2, \beta_2 \longrightarrow \beta_2, \beta_1 + \beta_2, \beta_1. \]

These are similar to the Yang-Baxter moves described in [Len07].

Notice that \( \tilde{\Gamma}^- (\lambda_1) \) is already at the end. Define \( \Gamma_s(k) \) (resp. \( \Gamma_l(k) \)) to be the chain of transpositions obtained from \( \tilde{\Gamma}_s(k) \) (resp. \( \tilde{\Gamma}_l(k) \)) after we have moved the last occurrence of each negative root in \( \tilde{\Gamma} \) to the end via Coxeter moves.

Let \( 1 \leq k \leq n - 1 \). Rename \( v(1), \ldots, v(k) \) as \( i_1, \ldots, i_k \) with
\[ i_1 < i_2 < \cdots < i_k. \]

Also rename \( v(k + 2), \ldots, v(n) \) as \( j_1, \ldots, j_{n-k-1} \) where
\[ j_1 < j_2 < \cdots < j_{n-k-1}. \]
We will refer to $v(k+1)$ as $v(k+1)$. Let

$$m := \min \{ p : i_p > v(k+1) \}$$

and notice that because $\mu$ is a regular weight, $\tilde{\Gamma}_s(k)$ must appear exactly once in $\tilde{\Gamma}$ for all $1 \leq k \leq n - 1$. Also notice that the final occurrence of a negative root always occurs in a $\tilde{\Gamma}_s(k)$ for $1 \leq k \leq n - 1$.

**Theorem 3.5.** For $1 \leq k \leq n - 1$

$$\Gamma_s(k) = \begin{pmatrix} (v(k+2), i_1), \ldots, (v(n), i_1) \\ (v(k+2), i_2), \ldots, (v(n), i_2) \\ \vdots \quad \vdots \quad \vdots \\ (v(k+1), i_m), (v(k+2), i_m), \ldots, (v(n), i_m) \\ \vdots \quad \vdots \quad \vdots \\ (v(k+1), i_k), (v(k+2), i_k), \ldots, (v(n), i_k) \end{pmatrix}$$

(25)

where we get no contribution to $\Gamma_s(k)$ if $k + 2 > n$.

**Proof.** We proceed by induction on $k$.

**Base Case:** Suppose $k = n - 1$. Recall that $\tilde{\Gamma}_s(n - 1) = \tilde{\Gamma}^-(n)\tilde{\Gamma}^+(n - 1)$ where $\tilde{\Gamma}^-(n) = \emptyset$. We have

$$\tilde{\Gamma}_s(n - 1) = \begin{pmatrix} (v(n), i_m) \\ (v(n), i_{m+1}) \\ \vdots \\ (v(n), i_{n-1}) \end{pmatrix}.$$ 

No negative roots must be removed from this section of the chain so this is the same as $\Gamma_s(n - 1)$. Notice that since $k + 2 = n + 1 > n$, this matches the expression in (25).

**Inductive Step:** Suppose that all of the negative roots from $\tilde{\Gamma}_s(\ell)$ have been moved forward in the chain to the beginning of $\tilde{\Gamma}_s(k)$ for all $k < \ell \leq n - 1$. 

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Notice that without considering all of these negative roots at the beginning, $\tilde{\Gamma}_s(k)$ is of the form

$$\begin{pmatrix}
\ldots & (j_q, i_1), & (j_{q-1}, i_1), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, v(k+1)), & (j_{q-1}, v(k+1)), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}$$

where $j_q > i_1$ for all $q$

$$\begin{pmatrix}
\ldots & (j_q, v(k+1)), & (j_{q-1}, v(k+1)), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, v(k+1)), & (j_{q-1}, v(k+1)), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}$$

where $j_q > v(k+1)$ for all $q$

$$\begin{pmatrix}
\ldots & (j_q, i_1), & (j_{q-1}, i_1), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}$$

where $j_q > i_k$ for all $q$

$$\begin{pmatrix}
\ldots & (j_q, i_1), & (j_{q-1}, i_1), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}$$

where $j_p < i_1$ for all $p$

$$\begin{pmatrix}
\ldots & (j_q, i_1), & (j_{q-1}, i_1), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}$$

where $j_p < i_k$ for all $p$

where the last occurrences of negative roots from within $\tilde{\Gamma}_s(k)$ are underlined. Our strategy for moving negative roots to the end of this section of the chain is as follows:

**Step 1.** Permute non-overlapping roots until all roots with the same $i$ are grouped together.

**Step 2.** Move the negative roots from within $\tilde{\Gamma}_s(k)$ to the end via Coxeter moves.

**Step 3.** Move the negative roots from $\tilde{\Gamma}_s(\ell)$ for $k < \ell \leq n - 1$ to the end via Coxeter moves.

**Step 1.** Define $j_a = \min\{j_i : j_i > v(k+1)\}$. We permute non-overlapping roots as follows:

$$\begin{pmatrix}
\ldots & (j_q, i_1), & (j_{q-1}, i_1), & \ldots , & (j_p, i_1), & (j_{p-1}, i_1) & \ldots \\
\ldots & (j_q, i_2), & (j_{q-1}, i_2), & \ldots , & (j_p, i_2), & (j_{p-1}, i_2) & \ldots \\
\ldots & (j_q, v(k+1)), & \ldots, & (j_a, v(k+1)), \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots , & (j_p, i_k), & (j_{p-1}, i_k) & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}$$

**Step 2.** Since $v(k+1) < i_j$ for all $i_j$ to the right, we can move the root $(j_a, v(k+1))$ forward until we encounter a root $(j_a, i_j)$. By the definition of $j_a$, the next root in the sequence must be $(v(k+1), i_j)$. We make the Coxeter move

$$(j_a, v(k+1)), (j_a, i_j), (v(k+1), i_j) \rightarrow (v(k+1), i_j), (j_a, i_j), (j_a, v(k+1)).$$

We continue to move $(j_a, v(k+1))$ forward until we encounter $(j_a, i_{j+1})$. Again, by the definition of $j_a$, the next root in the sequence must be $(v(k+1), i_{j+1})$. We make the Coxeter move

$$(j_a, v(k+1)), (j_a, i_{j+1}), (v(k+1), i_{j+1}) \rightarrow (v(k+1), i_{j+1}), (j_a, i_{j+1}), (j_a, v(k+1)).$$
We continue in this fashion until \((j_a, v(k + 1))\) is moved to the end of the chain, giving us the following arrangement:

\[
\begin{pmatrix}
\ldots & (j_q, i_1), & (j_{q-1}, i_1), & \ldots, & (j_p, i_1), & (j_{p-1}, i_1) & \ldots \\
\ldots & (j_q, i_2), & (j_{q-1}, i_2), & \ldots, & (j_p, i_2), & (j_{p-1}, i_2) & \ldots \\
\ldots & (j_q, v(k + 1)), & \ldots, & (j_{a+1}, v(k + 1)), & \ldots \\
\ldots & (j_q, i_k), & (j_{q-1}, i_k), & \ldots, & (j_p, i_k), & (j_{p-1}, i_k) & \ldots \\
\ldots & (j_q, v(k + 1)), & \ldots, & (j_{a}, v(k + 1))
\end{pmatrix}
\]

By the way the \(j_i\) are ordered, we can now move the rest of the \((j_q, v(k + 1))\) to the end in a similar fashion, starting with \((j_{a+1}, v(k + 1))\). After all of these Coxeter moves have been made, we get the following:

\[
\begin{pmatrix}
\ldots & \ldots & (j_q, i_1), & \ldots, & (j_p, i_1), & \ldots \\
\ldots & \ldots & (j_q, i_2), & \ldots, & (j_p, i_2), & \ldots \\
\ldots & (v(k + 1), i_m), & \ldots, & (j_q, i_m), & \ldots, & (j_p, i_m), & \ldots \\
\ldots & (v(k + 1), i_k), & \ldots, & (j_q, i_k), & \ldots, & (j_p, i_k), & \ldots \\
\ldots & \ldots & (j_q, v(k + 1)), & \ldots, & (j_{a}, v(k + 1))
\end{pmatrix}
\]

**Step 3.** For all \(k < \ell \leq n - 1\), define

\[j^\ell_a = \min\{j_i : j_i > v(\ell + 1)\}.
\]

For \(k < \ell \leq n - 1\), the roots that have been moved to the beginning of \(\widetilde{\Gamma}_s(k)\) are

\[
\begin{pmatrix}
\ldots & (j_q^{n-1}, v(n)), & (j_{q-1}^{n-1}, v(n)), & \ldots, & (j_p^{n-1}, v(n)), & (j_{p-1}^{n-1}, v(n)), & \ldots \\
\ldots & (j_q^{n-2}, v(n-1)), & (j_{q-1}^{n-2}, v(n-1)), & \ldots, & (j_p^{n-2}, v(n-1)), & (j_{p-1}^{n-2}, v(n-1)), & \ldots \\
\ldots & (j_q^{k+1}, v(k+2)), & (j_{q-1}^{k+1}, v(k+2)), & \ldots, & (j_p^{k+1}, v(k+2)), & (j_{p-1}^{k+1}, v(k+2)), & \ldots \\
\end{pmatrix}
\]

We begin moving these roots through starting from the last root \((j_{a}^{k+1}, v(k + 2))\). We move this root forward until we encounter \((j_{a}^{k+1}, i_j)\). By the definition of \(j_{a}^{k+1}\), the next root in the sequence
must be \((v(k + 2), i_j)\). We make the Coxeter move
\[
(j^{k+1}_a, v(k + 2)), (j^{k+1}_a, i_j), (v(k + 2), i_j) \rightarrow (v(k + 2), i_j), (j^{k+1}_a, i_j), (j^{k+1}_a, v(k + 2)).
\]
We can continue to move \((j^{k+1}_a, v(k + 2))\) forward until we encounter \((j^{k+1}_a, i_{j+1})\). Again, by the definition of \(j^{k+1}_a\), the next root in the sequence must be \((v(k + 2), i_{j+1})\). We make the Coxeter move
\[
(j^{k+1}_a, v(k + 2)), (j^{k+1}_a, i_{j+1}), (v(k + 2), i_{j+1}) \rightarrow (v(k + 2), i_{j+1}), (j^{k+1}_a, i_{j+1}), (j^{k+1}_a, v(k + 2)).
\]
We continue in this fashion until we have the following arrangement:
\[
\begin{pmatrix}
\cdots & \cdots & (j_q, i_1), & \cdots, & (j_p, i_1), & \cdots \\
\cdots & \cdots & (j_q, i_2), & \cdots, & (j_p, i_2), & \cdots \\
(v(k + 1), i_m), & \cdots, & (j_q, i_m), & \cdots, & (j_p, i_m), & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(v(k + 1), i_k), & \cdots, & (j_q, i_k), & \cdots, & (j_p, i_k), & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(v(k + 1), i_{m}), & (v(k + 2), i_{m}), & \cdots, & (j_q, i_{m}), & \cdots, & (j_p, i_{m}), & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(v(k + 1), i_{k}), & (v(k + 2), i_{k}), & \cdots, & (j_q, i_{k}), & \cdots, & (j_p, i_{k}), & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(j^{k+1}_a, v(k + 2)), & \cdots, & (j^{k+1}_a, v(k + 2)) & \cdots & (j^{k+1}_a, v(k + 2)) & \cdots & (j^{k+1}_a, v(k + 2)) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(j_q, v(k + 1)), & \cdots, & (j_q, v(k + 1)) & \cdots & (j_q, v(k + 1)) & \cdots & (j_q, v(k + 1))
\end{pmatrix}
\]
By the way the \(j_i\) are ordered, we can now move the rest of the \((j^{k+1}_q, v(k + 2))\) to the end in a similar fashion. After all of these Coxeter moves have been made, we get
\[
\begin{pmatrix}
(v(k + 2), i_1), & \cdots, & (j_q, i_1), & \cdots, & (j_p, i_1), & \cdots \\
(v(k + 2), i_2), & \cdots, & (j_q, i_2), & \cdots, & (j_p, i_2), & \cdots \\
(v(k + 1), i_m), & (v(k + 2), i_m), & \cdots, & (j_q, i_m), & \cdots, & (j_p, i_m), & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(v(k + 1), i_k), & (v(k + 2), i_k), & \cdots, & (j_q, i_k), & \cdots, & (j_p, i_k), & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(j^{k+1}_q, v(k + 2)), & \cdots, & (j^{k+1}_q, v(k + 2)) & \cdots & (j^{k+1}_q, v(k + 2)) & \cdots & (j^{k+1}_q, v(k + 2)) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(j_q, v(k + 1)), & \cdots, & (j_q, v(k + 1)) & \cdots & (j_q, v(k + 1)) & \cdots & (j_q, v(k + 1))
\end{pmatrix}
\]
Repetition of this argument shows we can move the rest of the \((j_q^\ell, v(\ell + 1))\) to the end in a similar fashion, starting with \(\ell = k + 2\). After moving all the roots through, we obtain

\[
\begin{pmatrix}
(v(k + 2), i_1), & \ldots, & (v(n), i_1) \\
(v(k + 2), i_2), & \ldots, & (v(n), i_2) \\
\vdots \\
(v(k + 1), i_m), & (v(k + 2), i_m), & \ldots, & (v(n), i_m) \\
\vdots \\
(v(k + 1), i_k), & (v(k + 2), i_k), & \ldots, & (v(n), i_k) \\
\vdots \\
\vdots \\
(j_q^{n-1}, v(n)), & \ldots, & (j_q^{n-1}, v(n)) \\
\vdots \\
\vdots \\
(j_q^{k+1}, v(k + 2)), & \ldots, & (j_q^{k+1}, v(k + 2)) \\
\vdots \\
\vdots \\
(j_q, v(k + 1)), & \ldots, & (j_q, v(k + 1))
\end{pmatrix}
\]

(27)

Dropping the underlined roots gives us the expression in (25).

\[\square\]

**Theorem 3.6.** For all \(1 \leq k \leq n - 1\)

\[
\Gamma_l(k) = \begin{cases}
(v(k + 1), i_1), & (v(k + 2), i_1), & \ldots, & (v(n), i_1) \\
(v(k + 1), i_2), & (v(k + 2), i_2), & \ldots, & (v(n), i_2) \\
\vdots & \vdots & \vdots & \vdots \\
(v(k + 1), i_k), & (v(k + 2), i_k), & \ldots, & (v(n), i_k)
\end{cases}
\]

(28)

**Proof.** We proceed by induction on \(k\).

**Base Case:** Suppose \(k = n - 1\). Recall that \(\Gamma_l(n - 1) = \Gamma^{-}(n - 1)\Gamma^{+}(n - 1)\). We have

\[
\Gamma_l(n - 1) = \begin{pmatrix}
(v(n), i_1) \\
(v(n), i_2) \\
\vdots \\
(v(n), i_{n-1})
\end{pmatrix}
\]

No negative roots must be removed from this section of the chain so this is the same as \(\Gamma_l(n - 1)\). This matches the expression in (28).

**Inductive Step:** Suppose that all of the negative roots from \(\tilde{\Gamma}_s(\ell)\) have been moved forward to the beginning of \(\Gamma_l(k)\) for all \(k \leq \ell \leq n - 1\). Notice that, without considering all of these negative
roots at the beginning, $\Gamma_1(k)$ is of the form

\[
\begin{pmatrix}
\ldots, (j_q, i_1), (j_{q-1}, i_1), \ldots \\
\ldots, (j_q, i_2), (j_{q-1}, i_2), \ldots \\
\ldots, (j_q, i_k), (j_{q-1}, i_k), \ldots \\
\ldots, (j_p, i_1), (j_{p-1}, i_1), \ldots \\
\ldots, (j_p, i_2), (j_{p-1}, i_2), \ldots \\
\ldots, (j_p, i_k), (j_{p-1}, i_k), \ldots 
\end{pmatrix}
\]

where $j_q > i_1$ for all $q$

where $j q > i_2$ for all $q$

where $j q > i_k$ for all $q$

where $j p < i_1$ for all $p$

where $j p < i_2$ for all $p$

where $j p < i k$ for all $p$.

Notice that there are no final occurrences of negative roots in this section of the chain so nothing has to be removed from within. We will, however, need to move the last occurrences of negative roots from previous sections of the chain through this section. Our strategy for moving particular negative roots to the end is as follows:

**Step 1.** Permute non-overlapping roots until that all roots with the same $i$ are grouped together.

**Step 2.** Move the negative roots from $\Gamma_s(\ell)$ for $k \leq \ell \leq n - 1$ to the end via Coxeter moves.

**Step 1.** We permute non-overlapping roots as follows:

\[
\begin{pmatrix}
\ldots, (j_q, i_1), (j_{q-1}, i_1), \ldots, (j_p, i_1), (j_{p-1}, i_1), \ldots \\
\ldots, (j_q, i_2), (j_{q-1}, i_2), \ldots, (j_p, i_2), (j_{p-1}, i_2), \ldots \\
\ldots, (j_q, i_k), (j_{q-1}, i_k), \ldots, (j_p, i_k), (j_{p-1}, i_k), \ldots 
\end{pmatrix}
\]

**Step 2.** For all $k \leq \ell \leq n - 1$, define

\[j^\ell_a = \min\{j_i : j_i > v(\ell + 1)\}\]

We can describe the negative roots that must be moved through as follows:

\[
\begin{pmatrix}
\ldots, (j_q^{n-1}, v(n)), (j_{q-1}^{n-1}, v(n)), \ldots, (j_a^{n-1}, v(n)), \\
\ldots, (j_q^{n-2}, v(n-1)), (j_{q-1}^{n-2}, v(n-1)), \ldots, (j_a^{n-2}, v(n-1)), \\
\ldots, (j_q^k, v(k + 1)), (j_{q-1}^k, v(k + 1)), \ldots, (j_a^k, v(k + 1)) 
\end{pmatrix}
\]

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We begin moving these roots forward the last root \((j^k_a, v(k+1))\). We move this forward until we encounter a root \((j^k_a, i_j)\). By the definition of \(j^k_a\), the next root in the sequence must be \((v(k+1), i_j)\).

We make the Coxeter move

\[
(j^k_a, v(k+1)), (j^k_a, i_j), (v(k+1), i_j) \rightarrow (v(k+1), i_j), (j^k_a, i_j), (j^k_a, v(k+1)).
\]

We can continue to move \((j^k_a, v(k+1))\) forward until we encounter \((j^k_a, i_{j+1})\). Again, by the definition of \(j^k_a\), the next root in the sequence must be \((v(k+1), i_{j+1})\). We make the Coxeter move

\[
(j^k_a, v(k+1)), (j^k_a, i_{j+1}), (v(k+1), i_{j+1}) \rightarrow (v(k+1), i_{j+1}), (j^k_a, i_{j+1}), (j^k_a, v(k+1)).
\]

We continue in this fashion until we have

\[
\begin{align*}
\ldots & (j_q, i_1), \quad (j_{q-1}, i_1), \quad \ldots, \quad (j_p, i_1), \quad \ldots \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \八大洲
After moving all the roots through, we obtain

\[
\left( (v(k+1), i_1), (v(k+2), i_1), \ldots, (v(n), i_1) \right),
\left( (v(k+1), i_2), (v(k+2), i_1), \ldots, (v(n), i_2) \right),
\ldots,
\left( (v(k+1), i_k), (v(k+2), i_1), \ldots, (v(n), i_k) \right),
\left( (v(k+1), i_{k+1}), (v(k+2), i_{k+1}), \ldots, (v(n), i_{k+1}) \right),
\ldots,
\left( (v(k+1), i_{n-1}), (v(k+2), i_{n-1}), \ldots, (v(n), i_{n-1}) \right),
\left( (v(k+1), i_{n}), (v(k+2), i_{n}), \ldots, (v(n), i_{n}) \right),
\ldots,
\left( (v(k+1), i_{n-k}), (v(k+2), i_{n-k}), \ldots, (v(n), i_{n-k}) \right)
\right) .
\] (29)

Dropping the underlined roots gives us the expression in (28).

\[\Box\]

**Remark 3.7.** Notice now that \( \Gamma_s(k) \) is in fact shorter than \( \Gamma_l(k) \). Certain roots present in \( \Gamma_l(k) \) are not present in \( \Gamma_s(k) \).

Define

\[\Gamma_j = \begin{cases} 
\Gamma_s(\lambda'_j) & \text{if } j = \min\{i : \lambda'_i = \lambda'_j\} \\
\Gamma_l(\lambda'_j) & \text{otherwise}
\end{cases} \] (30)

By the way we defined \( \tilde{\Gamma} \) earlier, we can easily see that \( \Gamma \) is given by the following concatenation:

\[\Gamma = \Gamma_1 \Gamma_2 \cdots \Gamma_{\lambda_1}.\] (31)

**Example 3.8.** Let \( \mu = (3,1,4,0) \). Recall that

\[\tilde{\Gamma} = (\)

\[\begin{array}{c}
(4,1), (4,2), | (4,3), (2,3), \\
(4,1), (2,1), | (4,3), (2,3), \\
(4,1), (2,1), | (4,3), (2,3), (1,3) \\
(4,3)
\end{array}\)

We will now move the underlined roots to the end via Coxeter moves, then drop them completely to obtain \( \Gamma \). First, notice that \( (4,3) \) is already at the end. Now we move \( (4,2) \) forward via the
moves

\[(4, 2), (4, 3), (2, 3) \rightarrow (2, 3), (4, 3), (4, 2)\]
\[(4, 2), (4, 1), (2, 1) \rightarrow (2, 1), (4, 1), (4, 2)\]
\[(4, 2), (4, 3), (2, 3) \rightarrow (2, 3), (4, 3), (4, 2)\]

which gives us

\[
(4, 1), (2, 3), (4, 3),
(2, 1), (4, 1), (2, 3), (4, 3),
(4, 2), (4, 1), (2, 1), (4, 3), (2, 3), (1, 3),
(4, 3)).
\]

Now we move \((2, 1)\) through via the moves

\[(2, 1), (4, 3) \rightarrow (4, 3), (2, 1)\]
\[(2, 1), (2, 3), (1, 3) \rightarrow (1, 3), (2, 3), (2, 1)\]

which gives us

\[
(4, 1), (2, 3), (4, 3),
(2, 1), (4, 1), (2, 3), (4, 3),
(4, 2), (4, 1), (4, 3), (1, 3), (2, 3)
(2, 1), (4, 3)).
\]

Now we move \((4, 1)\) through via the moves

\[(4, 1), (4, 3), (1, 3) \rightarrow (1, 3), (4, 3), (4, 1)\]
\[(4, 1), (2, 3) \rightarrow (2, 3), (4, 1)\]
leaving us with

\[
(4, 1), (2, 3), (4, 3), \\
(2, 1), (4, 1), (2, 3), (4, 3), \\
(4, 2), (1, 3), (4, 3), (2, 3) \\
(4, 1), (2, 1), (4, 3)).
\]

Now we move \((4, 2)\) to the end via

\[
(4, 2), (1, 3) \rightarrow (1, 3), (4, 2) \\
(4, 2), (4, 3), (2, 3) \rightarrow (2, 3), (4, 3), (4, 2)
\]

which gives

\[
(4, 1), (2, 3), (4, 3), \\
(2, 1), (4, 1), (2, 3), (4, 3), \\
(1, 3), (2, 3), (4, 3) \\
(4, 2), (4, 1), (2, 1), (4, 3)).
\]

Finally, dropping the negative roots, we get Π:

\[
Π = ( | (4, 1), (2, 3), (4, 3), | (2, 1), (4, 1), (2, 3), (4, 3) | (1, 3), (2, 3), (4, 3)).
\]

We can now describe Π″. Recall that

\[
Π″ = -v^{-1}(Π).
\]

Let

\[
Π_s″(k) = -v^{-1}(Π_s(k)) \tag{32}
\]

\[
Π_l″(k) = -v^{-1}(Π_l(k)). \tag{33}
\]

Recall from (24)

\[
m := \min\{p : i_p > v(k + 1)\}. \tag{34}
\]
Corollary 3.9.

\[
\Gamma''(k) = \begin{pmatrix}
(v^{-1}(i_1), k + 1), & \ldots, & (v^{-1}(i_1), n) \\
(v^{-1}(i_2), k + 1), & \ldots, & (v^{-1}(i_2), n) \\
\ldots & \ldots & \ldots \\
(v^{-1}(i_m), k + 1), & (v^{-1}(i_m), k + 2), & \ldots, & (v^{-1}(i_m), n) \\
(v^{-1}(i_{m+1}), k + 1), & (v^{-1}(i_{m+1}), k + 2), & \ldots, & (v^{-1}(i_{m+1}), n) \\
\ldots & \ldots & \ldots \\
(v^{-1}(i_k), k + 1), & (v^{-1}(i_k), k + 2), & \ldots, & (v^{-1}(i_k), n)
\end{pmatrix}.
\] (35)

**Proof.** We get this by applying \(-v^{-1}\) to \(\Gamma_s(k)\). See (25). \qed

Corollary 3.10.

\[
\Gamma''(k) = \begin{pmatrix}
(v^{-1}(i_1), k + 1), & \ldots, & (v^{-1}(i_1), n) \\
(v^{-1}(i_2), k + 1), & \ldots, & (v^{-1}(i_2), n) \\
\ldots & \ldots & \ldots \\
(v^{-1}(i_k), k + 1), & (v^{-1}(i_k), k + 2), & \ldots, & (v^{-1}(i_k), n)
\end{pmatrix}.
\] (36)

**Proof.** We get this by applying \(-v^{-1}\) to \(\Gamma_l(k)\). See (28). \qed

Define

\[
\Gamma''_j = \begin{cases} 
\Gamma''_s(\lambda'_j) & \text{if } j = \min\{i : \lambda'_i = \lambda'_j\} \\
\Gamma''_l(\lambda'_j) & \text{otherwise.}
\end{cases}
\] (37)

Now we can define \(\Gamma''\):

\[
\Gamma'' = -v^{-1}(\Gamma) = -v^{-1}(\Gamma_1 \Gamma_2 \cdots \Gamma_{\lambda_1}) = (-v^{-1}(\Gamma_1))(-v^{-1}(\Gamma_2)) \cdots (-v^{-1}(\Gamma_{\lambda_1})) = \Gamma''_1 \Gamma''_2 \cdots \Gamma''_{\lambda_1}.
\]

**Example 3.11.** Let \(\mu = (3, 1, 4, 0)\). Recall that \(v = 3124, v^{-1} = 2314\), and

\[
\Gamma = ( (4, 1), (2, 3), (4, 3), (2, 1), (4, 1), (2, 3), (4, 3) | (1, 3), (2, 3), (4, 3) ).
\]
Then

\[ \Gamma'' = -v^{-1}(\Gamma) \]

\[ = ( (2, 4), (1, 3), (1, 4), (2, 3), (2, 4), (1, 3), (1, 4), (1, 2), (1, 3), (1, 4)) \).}
4 Combinatorial Maps

4.1 The Filling Map

We will now describe the bijection between alcove paths in the Ram-Yip formula and fillings in the Haglund-Haiman-Loehr formula. Specifically, we will demonstrate the bijection between \( A_{\geq} (\Gamma'') \) and PBSF(\( \mu \)).

Recall that \( J \in A_{\geq} (\Gamma'') \) gives us a list of positions within the chain of roots \( \Gamma'' \). We will identify such a \( J \) with \((v,T)\) where \( v \) is our starting word and \( T \) is the set of transpositions corresponding to \( J \). A filling of \( \hat{\text{dg}}(\lambda) \) is viewed as a concatenation of columns \( \hat{\sigma} = R^0 R^1 \ldots R^{\lambda_1} \) of lengths \( n, \lambda'_1, \lambda'_2, \ldots, \lambda'_\lambda_1 \). The cells in each row are filled with distinct numbers from 1 to \( n \), with no restriction on the order within a given row.

Define
\[
T_i = \{ t^i_j \in \Gamma'' : j \in J \}.
\]

Given \((v,T) \in A_{\geq} (\Gamma'')\), consider the permutations
\[
u^i := v T^1 T^2 \ldots T^i.
\]
for \( i = 0, 1, \ldots, \lambda_1 \), with the right-hand side denoting the permutation obtained from \( v \) via right multiplication by the transpositions in \( T^1, \ldots, T^i \) considered from left to right. Notice that \( u^0 = v \).

**Definition 4.1.** The filling map is the map \( f \) from \((v,T) \in A_{\geq} (\Gamma'')\) to fillings
\[
f(v,T) = R^0 R^1 \ldots R^{\lambda_1}
\]
of shape \( \lambda \) (plus basement) with rows defined by
\[
R^i := \begin{cases} 
u^i[1, \lambda'_i] & \text{for } i = 1, \ldots, \lambda_1 \\ u^0[1,n] & \text{for } i = 0. \end{cases}
\]

In words, we obtain a filling \( f(v,T) \) by considering the left parts of the permutations \( u^i \), which are the permutations with which the subchains of \((v,T)\) end.
Example 4.2. Let \( \mu = (3, 1, 4, 0) \). Then \( \lambda = (4, 3, 1, 0), \lambda' = (3, 2, 2, 1), \) and \( v = 3124. \) Recall from Example 3.11 that our reduced \( \mu \)-chain \( \Gamma'' \) is
\[
\Gamma'' = \Gamma''_1 \Gamma''_2 \Gamma''_3 \Gamma''_4 \\
= \Gamma''_s(3) \Gamma''_s(2) \Gamma''_s(2) \Gamma''_s(1) \\
= ( | (2, 4), (1, 3), (1, 4), | (2, 3), (2, 4), (1, 3), (1, 4) | (1, 2), (1, 3), (1, 4)).
\]
Let \( J = \{2, 8\} \). We can visualize the corresponding \( T \) by underlining the roots in those positions:
\[
T = ( | (2, 4), (1, 3), (1, 4), | (2, 3), (2, 4), (1, 3), (1, 4) | (1, 2), (1, 3), (1, 4)).
\]
We apply these transpositions starting at \( v \):
\[
\begin{array}{c|c|c|c|c|c|c|c}
3 & 3 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & 2 & 2 \\
4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]
This gives us the following filling \( \tilde{\sigma} \in \text{PBSF}(\mu) \):
\[
\tilde{\sigma} = \begin{array}{c|c}
1 & \\
2 & 1 \\
3 & 1 \ \\
3 & 1 \ \\
\end{array} \quad \leftrightarrow \quad x^\sigma = x_1^4 x_2^3 x_3.
\]

4.2 Main Theorem

Theorem 4.3. The map \( f \) is a bijection between \( A_{\succ}((\Gamma'')) \) and \( \text{PBSF}(\mu) \).

We need a few lemmas to prove that \( f \) is a bijection.

Lemma 4.4. For an arbitrary \((v, T) \in A_{\succ}((\Gamma''))\), \( f(v, T) \) has no descents.

Proof. Elements of \( A_{\succ}((\Gamma'')) \) consist entirely of down by 1 steps in the Bruhat graph. Since we record the left parts of each ending permutation and start our recording at the bottom of the diagram, it’s clear that our filling with have no descents. \( \square \)
**Lemma 4.5.** For an arbitrary \((v, T) \in A_\nu(\Gamma'')\), \(f(v, T)\) is a non-attacking filling.

*Proof.* For a Type 0 attacking pair (i.e. two attacking boxes in the same row), the definition of \(f\) assures us that these boxes will be filled with two distinct numbers from \([n]\) so we need only consider the case where the two attacking boxes are in consecutive rows. Let \(R''\) and \(R'\) be two rows in \(f(v, T)\) such that \(i' = i - 1\) and let \(u''\) and \(u'\) be the permutations corresponding to these rows. Suppose \(a < b \leq \lambda_i' \leq \lambda_i' + 1\). Any other attacking pair must take on one of the following forms:

*Type 1.* If \(v(a) < v(b)\) then we have the following attacking pair:

\[
\begin{array}{c|c}
R''(a) & R''(b) \\
\hline
v(a) & v(b)
\end{array}
\]

*Type 2.* If \(v(a) > v(b)\) then we have the following attacking pair:

\[
\begin{array}{c|c}
R'(b) & R'(a) \\
\hline
v(a) & v(b)
\end{array}
\]

We will now show that, under the filling map \(f\), all Type 1 and Type 2 attacking pairs are filled with distinct numbers from \([n]\).

*Type 1.* We have the following arrangement, with \(v(a) < v(b)\):

\[
\begin{array}{c|c}
u''(a) & u''(b) \\
\hline
v(a) & v(b)
\end{array}
\]
where

\[ u^i = u'^i T^i \]
\[ = u'^i t_1^i t_2^i \ldots t_m^i. \]

Since \( v(a) < v(b) \), the transpositions \( t_j^i \) first make switches at position \( a \) before making switches at position \( b \).

**Case 1.** Suppose \( b \leq \lambda_i' \). Then the \( t_j^i \) first make switches at position \( a \) with positions to the right of \( \lambda_i' \) before eventually making switches at position \( b \). This assures us that \( u^i(a) \neq u'^i(b) \).

**Case 2.** Suppose \( b = \lambda_i' + 1 \). Then the \( t_j^i \) will first make switches at position \( a \) with positions to the right of \( \lambda_i' + 1 \) before eventually making switches at position \( b \). This pattern again assures us that \( u^i(a) \neq u'^i(b) \).

These two cases cover all possible values of \( b \) so all Type 1 attacking pairs in \( f(v, T) \) are filled with distinct numbers from \([n]\). We now turn our attention to Type 2 attacking pairs.

**Type 2.** We have the following arrangement, with \( v(a) > v(b) \):

```
   \[
   \begin{array}{c}
   \hline
   \text{u}(b) \\
   \text{u}'(a) \\
   v(a) \\
   v(b) \\
   \hline
   \end{array}
   \]
```

where

\[ u^i = u'^i T^i \]
\[ = u'^i t_1^i t_2^i \ldots t_m^i. \]

Since \( v(a) > v(b) \), the \( t_j^i \) will first make switches at position \( b \) with positions to the right of \( \lambda_i' \) before eventually making switches at position \( a \). This pattern assures us that \( u^i(b) \neq u'^i(a) \). Thus, all Type 2 attacking pairs in \( f(v, T) \) are filled with distinct numbers from \([n]\).

We have shown that in \( f(v, T) \), the boxes in each Type 0, Type 1, and Type 2 attacking pair are filled with distinct numbers from \([n]\). Thus, \( f(v, T) \) is non-attacking.

Lemma 4.6. For an arbitrary \((v, T) \in A_v(\Gamma'')\), every triple in \( f(v, T) \) is an inversion triple.
Proof. Let $R^{i'}$ and $R^i$ be two rows in $f(v, T)$ such that $i' = i - 1$ and let $u^{i'}$ and $u^i$ be the permutations corresponding to these rows. Suppose $a < b \leq \lambda'_{i'} \leq \lambda'_i + 1$. Any triple in $f(v, T)$ must take on one of the following forms:

**Type 1.** If $v(a) < v(b)$ then we have the following triple

\[
\begin{array}{|c|c|}
\hline
R^i(a) \\
R^{i'}(a) \\
R^{i'}(b) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
v(a) \\
v(b) \\
\hline
\end{array}
\]

**Type 2.** If $v(a) > v(b)$ then we have the following triple

\[
\begin{array}{|c|c|}
\hline
R^i(a) \\
R^{i'}(a) \\
R^{i'}(b) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
v(a) \\
v(b) \\
\hline
\end{array}
\]

We will now show that, under the filling map $f$, all Type 1 and Type 2 triples must be inversion triples.

**Type 1.** We have the following arrangement, with $v(a) < v(b)$.

\[
\begin{array}{|c|c|}
\hline
u^i(a) \\
u^{i'}(a) \\
u^{i'}(b) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
v(a) \\
v(b) \\
\hline
\end{array}
\]

where

\[
u^i = u^{i'} T^i = u^{i'} t^i_1 t^i_2 \ldots t^i_{m_i}.
\]

Assume, for the sake of contradiction, that this is a coinversion triple. Then

\[u^i(a) < u^{i'}(b) < u^{i'}(a)\]
where \( \hat{u}^i(a) \neq \hat{u}^i(b) \) because they are a Type 0 attacking pair and \( u^i(a) \neq u^i(b) \) because they are a Type 1 attacking pair.

**Case 1.** Suppose \( b \leq \lambda'_i \). Then the \( t^i_j \) will first make switches at position \( a \) with positions to the right of \( \lambda'_i \) before eventually making switches at position \( b \). Since \( u^i(a) < \hat{u}^i(b) < u^i(a) \), we must make a down step of more than 1 to get the Type 1 triple pictured above. This is a contradiction.

**Case 2.** If \( b = \lambda'_i + 1 \), then the \( t^i_j \) will first make switches at position \( a \) with positions to the right of \( \lambda'_i + 1 \) before eventually making switches at position \( b \). Since \( u^i(a) < \hat{u}^i(b) < u^i(a) \), we must make a down step of more than 1 to get the Type 1 triple pictured above. This is a contradiction.

Thus, all Type 1 triples must be inversion triples.

**Type 2.** We have the following arrangement, with \( v(a) > v(b) \).

\[
\begin{array}{cc}
\hat{u}^i(a) & u^i(b) \\
u^i(a) & \\
v(a) & v(b)
\end{array}
\]

where

\[
\hat{u}^i = u^i T^i
\]
\[
= u^i t_1^i t_2^i \ldots t_{m_i}^i.
\]

Assume for the sake of contradiction that this is a coinversion triple. Then

\[
u^i(a) < u^i(b) < u^i(a)
\]

where \( u^i(a) \neq u^i(b) \) because they are a Type 0 attacking pair and \( u^i(a) \neq u^i(b) \) because they are a Type 2 attacking pair.

The \( t^i_j \) will first make switches at position \( b \) with positions to the right of \( \lambda'_i \) before eventually making switches at position \( a \). Thus, by the time we begin making switches at position \( a \), the entry in position \( b \) must already be set. Since \( u^i(a) < u^i(b) < u^i(a) \), to get the arrangement pictured we must make a down step of more than 1, a contradiction. Thus, all Type 2 triples must be inversion triples.
We have now shown that all Type 1 and Type 2 triples in $f(v, T)$ must be inversion triples.

**Lemma 4.7.** Let $R^i$ and $R^{i'}$ be consecutive rows in our filling such that $i' = i - 1$. Let $u^i$ be the permutation corresponding to $R^i$ and $u^{i'}$ be the permutation corresponding to $R^{i'}$. Now define

$$u := u^{i'} t_1^i t_2^i \cdots t_k^i$$

where $t_j^i \in T^i$ and $u(j) = R^i(j)$ for all $j$ such that $v(j) < v(a)$. If $u(a) \neq R(a)$, then there exists a unique sequence $(m_1, \ldots, m_p)$ such that

$$u(a) > u(m_1) > \cdots > u(m_p) = R(a)$$

where each transposition $(a, m_i)$ gives a down step of exactly 1.

**Proof.** **Strategy:** In our chain of transpositions $\Gamma''$, we make every down by 1 switch that brings us closer to our target row. We will show that:

- **Step 1.** We can always make a switch that brings us closer to our target.
- **Step 2.** Each switch we make decreases the length of our word by exactly 1.
- **Step 3.** The sequence of transpositions produced is unique.

We proceed with Step 1:

**Step 1.** We can always reach our target.

Suppose that our target is in position $m$. We know that we can always switch position $a$ with positions $m > \lambda_i^i + 1$, so suppose $m \leq \lambda_i^i + 1$.

If $m < a$, we have the arrangement below.

\[
\begin{array}{c|c}
R^i(m) & R^i(a) \\
\hline
u(m) & u(a) \\
R^{i'}(m) & R^{i'}(a) \\
\hline
v(m) & v(a)
\end{array}
\]

We proceed by cases.
Case 1. If $v(m) < v(a)$, then $u(m) = R^i(m) \neq R^i(a)$ because they are a Type 0 attacking pair. Thus, our target cannot be in such a position.

Case 2. If $v(m) > v(a)$, then $u(m) = R^{i'}(m) \neq R^i(a)$ because they are a Type 2 attacking pair. Thus, our target cannot be in such a position.

Because our target cannot be in a position $m < a$, we must have $a < m \leq \lambda_i' + 1$. This gives us the following arrangement.

\[
\begin{array}{|c|c|}
\hline
R^i(a) & R^i(m) \\
\hline
u(a)   & u(m)   \\
\hline
R^{i'}(a) & R^{i'}(m) \\
\hline
v(a)   & v(m)   \\
\hline
\end{array}
\]

We proceed by cases.

Case 1. If $v(m) > v(a)$ then $u(m) = R^{i'}(m) \neq R^i(a)$ since they are a Type 1 attacking pair. Thus, our target cannot be in such a position.

Case 2. If $v(m) > v(a)$ then we must consider two subcases.

Subcase A. If $m \leq \lambda_i'$, then $u(m) = R^{i}(m) \neq R^i(a)$ as they are a Type 0 attacking pair. Thus, our target cannot be in such a position.

Subcase B. If $m = \lambda_i' + 1$, then we can make the switch $(a,m)$.

The above argument shows that the switches we need to make to get us to our target must always be in positions where we can make them.

Step 2. Each switch we make decreases the length of our word by exactly 1.

Let $(a,m)$ to be the next switch made under the algorithm. Assume for the sake of contradiction that there exists a $b$ such that $a < b < m$ with $u(m) < u(b) < u(a)$. We proceed by cases.

Case 1. Suppose $b > \lambda_i'$. Since we are only making switches that decrease the length of our word by 1, we will not choose to make such a switch.
Case 2. Suppose $b \leq \lambda'_i$. Then we have the following arrangement.

\[
\begin{array}{ccc}
R^i(a) & R^i(b) & u(a) \\
u(b) & u(m) & \\
R'^i(a) & R'^i(b) & \\
v(a) & v(b) & v(m)
\end{array}
\]

We have the following subcases.

**Subcase A.** Suppose $v(b) < v(a)$. Then $u(b) = R^i(b)$ and we have

\[
R^i(a) \leq u(m) < u(b) = R^i(b) < u(a) \leq R'^i(a)
\]

which makes

\[
\begin{array}{ccc}
R^i(a) & R^i(b) & \\
R'^i(a) & \\
v(a) & v(b)
\end{array}
\]

a Type 2 coinversion triple, a contradiction. Thus, such a $b$ cannot exist.

**Subcase B.** Suppose $v(b) > v(a)$. Then $u(b) = R'^i(b)$ and we have

\[
R^i(a) \leq u(m) < u(b) = R'^i(b) < u(a) \leq R'^i(a)
\]

which makes

\[
\begin{array}{ccc}
R^i(a) & \\
R'^i(a) & R'^i(b) & \\
v(a) & v(b)
\end{array}
\]

a Type 1 coinversion triple, a contradiction. Thus, such a $b$ cannot exist.

This concludes Step 2. Steps 1 and 2 show us that we can produce a sequence like the one mentioned in the statement of the lemma. Now we must show that such a sequence is unique.
Step 3. The sequence of transpositions produced is unique.

Suppose another sequence $\lambda_i' < m_i' < \cdots < m_r'$ exists. Suppose $m_1' < m_1$. Then applying the transposition $(a, m_1')$ either gives an up step or takes us past our target, a contradiction.

Now suppose $m_1' > m_1$. If $(a, m_1')$ is a transposition satisfying the down by 1 condition then $u(m_1') > u(m_1)$. Since $R(i) < u(m_1)$ then there is a smallest $\ell > 1$ such that $u(m_\ell') < u(m_1)$. Assume now that we apply the transpositions $(a, m_1'), \ldots, (a, m_{\ell-1}')$. Applying $(a, m_\ell')$ now violates the down by 1 condition because

$$u(m_\ell') < u(m_1) < u(m_{\ell-1}).$$

Therefore, $m_1 = m_1'$. We proceed in this fashion to show that the two sequences are the same.

\[\square\]

**Lemma 4.8.** For every $\hat{\sigma} \in \text{PBSF}(\mu)$, there is a unique $f^{-1}(\hat{\sigma}) \in A_\geq(\Gamma'')$.

**Proof.** If we consider $\hat{\sigma}$ as a concatenation of rows $R^0R^1\ldots R^{\lambda_1}$, then by repeated applications of Lemma 4.7, we see that every filling has a unique inverse. \[\square\]

We can now prove Theorem 4.3.

**Proof.** From Lemmas 4.4, 4.5, and 4.6, we see that $f(A(\Gamma'')) \subset \text{PBSF}(\mu)$. From Lemma 4.8, we see that every element of PBSF($\mu$) has a unique inverse, making $f$ a bijection. \[\square\]

**Conjecture 4.9.** The map $f$ is a weight-preserving bijection between $A_\geq(\Gamma'')$ and PBSF($\mu$).

**Remark 4.10.** In Conjecture 4.9, “weight-preserving” means that for $J \in A_\geq(\Gamma'')$

$$\text{wt}(J) = \text{content}(f(v, T))$$

where the correspondence between $J$ and $(v, T)$ is as described previously.

**Example 4.11.** Let $\mu = (3, 1, 4, 0)$. Then $\lambda = (4, 3, 1, 0)$, $\lambda' = (3, 2, 2, 1)$, $v = 3124$, and

$$\Gamma'' = ( \mid (2, 4), (1, 3), (1, 4), | (2, 3), (2, 4), (1, 3), (1, 4) | (1, 2), (1, 3), (1, 4)).$$
1. 
\[(v, T) = (3124, (| (1, 3) || (1, 2))) \iff \hat{\sigma} = \begin{pmatrix} 
1 \\
2 & 1 \\
3 & 1 & 2 \\
3 & 1 & 2 & 4 
\end{pmatrix} .\]

2. 
\[(v, T) = (3124, (| (1, 3) | | )) \iff \hat{\sigma} = \begin{pmatrix} 
2 \\
2 & 1 \\
3 & 1 & 2 \\
3 & 1 & 2 & 4 
\end{pmatrix} .\]

3. 
\[(v, T) = (3124, (| | (1, 3) | (1, 2))) \iff \hat{\sigma} = \begin{pmatrix} 
1 \\
2 & 1 \\
3 & 1 \\
3 & 1 & 2 & 4 
\end{pmatrix} .\]

4. 
\[(v, T) = (3124, (| | (1, 3) | | )) \iff \hat{\sigma} = \begin{pmatrix} 
2 \\
2 & 1 \\
3 & 1 \\
3 & 1 & 2 & 4 
\end{pmatrix} .\]
4.3 The Symmetric Case

In this section, we show how the bijection $f$ demonstrates a well-known relationship between the symmetric and non-symmetric Macdonald polynomials.

**Theorem 4.12 ([LNS+14]).** For $\lambda \in \Lambda^+$,

$$E_{w_3(\lambda)}(x;0,0) = P_\lambda(x;0,0).$$

**Remark 4.13.** In fact, for $\lambda \in \Lambda^+$, we have

$$E_{w_3(\lambda)}(x;q,0) = P_\lambda(x;q,0)$$

This is also a result of [LNS+14].
For the remainder of this section, we will consider weights $w_\circ(\lambda)$ for $\lambda \in \Lambda^+$. 

**Proposition 4.14.** A filling $\hat{\sigma} \in \text{PBSF}(w_\circ(\lambda))$ must have weakly decreasing columns (considered from bottom to top) and strictly decreasing rows (considered from left to right).

**Proof.** We know that $\hat{\sigma}$ has no descents so it must have weakly increasing columns. We also know that we cannot have equality within rows because our filling is non-attacking. Now assume that there is an increase within one of the rows of $\hat{\sigma}$. In other words, assume we have two boxes $a = (i, j)$ and $b = (i', j)$ with $i < i'$ such that $\hat{\sigma}(a) < \hat{\sigma}(b)$. This gives us the following arrangement:

\[
\begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ a
\end{array} \\
\begin{array}{c}
w_\circ(i)
\end{array} \\
\begin{array}{c}
\ \ \ \ \ \ b
\end{array} \\
\begin{array}{c}
w_\circ(i')
\end{array}
\end{array}
\]

Now consider the boxes $c = (i, j - 1)$ and $d = (i', j - 1)$. This gives the following arrangement:

\[
\begin{array}{c}
\begin{array}{c}
\ \ \ \ \ c
\end{array} \\
\begin{array}{c}
w_\circ(i)
\end{array} \\
\begin{array}{c}
\ \ \ \ \ b
\end{array} \\
\begin{array}{c}
w_\circ(i')
\end{array} \\
\begin{array}{c}
\ \ \ \ \ d
\end{array}
\end{array}
\]

Since $(a, b, c)$ is a Type 2 triple, it must be an inversion triple. We are assuming that $\hat{\sigma}(a) < \hat{\sigma}(b)$, so $\hat{\sigma}(c) < \hat{\sigma}(b)$. Since our filling cannot have any descents, we get the inequality

\[
\hat{\sigma}(c) < \hat{\sigma}(b) \leq \hat{\sigma}(d).
\]

Now we have that $\hat{\sigma}(c) < \hat{\sigma}(d)$. By continuing our argument in this fashion, we must eventually conclude that $w_\circ(i) < w_\circ(i')$, a contradiction. Thus, our filling must have strictly decreasing rows. \hfill \square

Define a semi-standard Young tableau of shape $\lambda$ to be a filling $\tau$ of $\text{dg}(\lambda)$ with strictly increasing rows (considered from left to right) and weakly increasing columns (considered from bottom to top).

**Remark 4.15.** This definition of semi-standard Young tableaux differs from the traditional definition as it has been adapted to fit the orientation of $\text{dg}(\lambda)$ described in Section 2.6.
Let SSYT(\(\lambda\)) be the set of all semi-standard Young tableaux of shape \(\lambda\).

**Theorem 4.16.** We have the following well-known expression for the Schur polynomials

\[
P_{\lambda}(x; 0, 0) = \sum_{\tau \in \text{SSYT}(\lambda)} x^{\tau}.
\]

**Remark 4.17.** Note that left multiplication by \(w_0\) defines a bijection between SSYT(\(\lambda\)) and PBSF(\(w_0(\lambda)\)) (ignoring the basement). This bijection does not preserve the content of the fillings.

Now recall from (10) that for \(J \in \mathcal{A}_c(\Gamma'')\) we have the following sequence of alcoves starting at \(v = w_0\) in which each step decreases the length of the word by 1:

\[
w_0 \xrightarrow{-\beta_{j_1}} w_0 r''_{j_1} \xrightarrow{-\beta_{j_2}} \cdots \xrightarrow{-\beta_{j_s}} w_0 r''_{j_1} \cdots r''_{j_s}.
\]

Via left multiplication by \(w_0\), we get the correspondence

\[
w_0 \xrightarrow{-\beta_{j_1}} w_0 r''_{j_1} \xrightarrow{-\beta_{j_2}} \cdots \xrightarrow{-\beta_{j_s}} w_0 r''_{j_1} \cdots r''_{j_s}
\]

where the chain in the bottom row is an up by 1 path in the Bruhat graph starting at the identity.

Notice that since \(w_0(n) < w_0(n-1) < \cdots < w_0(1)\), for any row of length \(k\), the corresponding chain is

\[
\begin{pmatrix}
(k, k + 1), & (k, k + 2), & \ldots, & (k, n), \\
(k - 1, k + 1), & (k - 1, k + 2), & \ldots, & (k - 1, n), \\
\ldots & \ldots & \ldots & \\
\ldots & \ldots & \ldots & \\
(1, k + 1), & (1, k + 2), & \ldots, & (1, n)
\end{pmatrix}
\]

(38)

by both (35) and (36). By concatenating these chains for \(k\) corresponding to parts of the conjugate partition \(\lambda'\), we get \(\Gamma''\).

Let \(\mathcal{A}_c(\Gamma'')\) be the set of all up by 1 paths in \(\Gamma''\) starting at the identity.

**Theorem 4.18** ([LP07]).

\[
P_{\lambda}(x; 0, 0) = \sum_{J \in \mathcal{A}_c(\Gamma'')} x^{\text{wt}(J)}.
\]
Remark 4.19. Theorem 4.18 is the same as Theorem 2.3 since

\[ P_\lambda(x;0,0) = \text{ch}(V(\lambda)) \]

and \( \Gamma'' \) is a reduced alcove path from \( A_0 \) to \( A_0 - \lambda \). Note that \( \Gamma'' \) starts at \( A_0 \) in this special case because

\[ -w_0(A_0) = -(-A_0) = A_0. \]

To connect the previous two expressions for \( P_\lambda(x;0,0) \), we cite the following theorem.

Theorem 4.20 ([Len12]). There is a weight-preserving bijection \( g \) between \( A_{\geq}(\Gamma'') \) and semi-standard Young tableaux \( \text{SSYT}(\lambda) \).

We will now show that the bijection \( f \) from Theorem 4.3 recovers the previously established connection

\[ E_{w_0(\lambda)}(x;0,0) = P_\lambda(x;0,0) \]

as cited in Theorem 4.12.

Proof. We know:

1. There is a bijection \( f : A_{\geq}(\Gamma'') \rightarrow \text{PBSF}(w_0(\lambda)) \) from Theorem 4.3.

2. There is a weight-preserving bijection \( g : A_{\leq}(\Gamma'') \rightarrow \text{SSYT}(\lambda) \) from Theorem 4.20.

3. There is a bijection between \( A_{\geq}(\Gamma'') \) and \( A_{\leq}(\Gamma'') \) via left multiplication by \( w_0 \) (see Equation (4.3)).

4. There is a bijection between \( \text{PBSF}(w_0(\lambda)) \) and \( \text{SSYT}(\lambda) \) via left multiplication by \( w_0 \) (see Remark 4.17).

We have the following diagram:

\[
\begin{array}{ccc}
A_{\geq}(\Gamma'') & \xrightarrow{f} & \text{PBSF}(w_0(\lambda)) \\
\downarrow{w_0} & & \downarrow{w_0} \\
A_{\leq}(\Gamma'') & \xrightarrow{g} & \text{SSYT}(\lambda).
\end{array}
\]
The definition of the map \( g \) from [Len12] matches the definition of the filling map \( f \) so this diagram commutes and \( f \) is a weight-preserving bijection in this special case. We now see that

\[
E_{w,\otimes}(\lambda)(x; 0, 0) = w \circ P_\lambda(x; 0, 0)
\]

\[
= P_\lambda(x; 0, 0)
\]

since \( P_\lambda \) is symmetric. \square
Bibliography


