Bayesian model testing of models for ellipsoidal variation on stars due to hot Jupiters

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Bayesian Model Testing of Models for Ellipsoidal Variation on Stars Due to Hot Jupiters

by

Anthony D. Gai

A Thesis

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To my parents, Mia and Dave, and sister, Anna, you make this all possible. To Nonna, Jay, and Debbie your support keeps the cold Albany nights at bay. Thank you.
Abstract

A massive planet closely orbiting its host star creates tidal forces that distort the typically spherical stellar surface. These distortions, known as ellipsoidal variations, result in variations in the photometric flux emitted by the star, which can be detected by the Kepler Space Telescope. Currently, there exist several models describing such variations and their effect on the photometric flux [1] [2] [3] [4]. By using Bayesian model testing in conjunction with the Bayesian-based exoplanet characterization software package EXON-EST [4] [5] [6], the most probable representation for ellipsoidal variations was determined for synthetic data and two systems with confirmed hot Jupiter exoplanets: HAT-P-7 and Kepler-13. The models were indistinguishable for the HAT-P-7 system likely due to noise within the dataset washing out the differences between the models. The most preferred model for ellipsoidal variations was determined to be EVIL-MC. The Modified Kane & Gelino model [4] provided the best representation of ellipsoidal variations, of the trigonometric models, for the Kepler-13 system and may serve as a fast alternative to the more computationally intensive EVIL-MC [3]. The computational feasibility of directly modeling the ellipsoidal variations of a star are examined and future work is outlined. Providing a more accurate model of ellipsoidal variations is expected to result in better estimations of planetary properties.
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Chapter 1

Introduction

This is a pivotal time in the existence of humanity. It is now that we, as a species, have the ability to search our neighboring systems for other worlds. Other worlds are not just speculation anymore.

As more planets are discovered we begin to paint a more detailed picture of the composition of our galaxy. A quote from a 2013 press release speaking on the Kepler Mission sums the situation up perfectly, “My God, it’s full of planets! They should have sent a poet” [7].

Now is the time to address the question of “what is out there?” To describe the contents of our galaxy we must first sharpen the tools of our analysis. Bayesian inference provides the whetstone for the tools to characterize the planets.

1.1 History of Exoplanet Detection

Many techniques have been developed to probe the stars for the faint traces of planets orbiting other stars. These “extra-solar” planets have been termed exoplanets.

The first successful detection of an exoplanet was in 1992 by precise timing of pulsar pulses from PSR1257+12. The system was determined to contained multiple planets which
Figure 1.1: Exoplanet discoveries by year, color coded by method [8]. Original discoveries were primarily made using radial velocity measurements. The launch of the Kepler Space Telescope in 2009 provided a huge boost to the field, doubling the number of discoveries in just one year of observation. As of 27 October 2016 there have been 3402 confirmed exoplanets with 4696 exoplanet candidates.

were around three times more massive than Earth orbiting between 0.33-0.5 AU [9].

The first discovery of an exoplanet around a main sequence star had to wait until 1995 for the team of Mayor and Queloz [10]. 51-Pegasai is a Sun-like star with a companion Jupiter-sized planet. The planet was discovered by measuring the radial velocity of the star. The star and planet orbit the barycenter, or center of mass, of the system. Measuring the line of sight velocity determined the minimum mass is about half of Jupiter’s mass. The reported value is a minimum mass because the mass is coupled with the inclination of the orbit. A small planet in an edge-on orbit may provide the same line-of-sight velocity as a large planet in an inclined orbit. This detection method is biased toward larger planets.
with smaller inclined orbits. The bias is because a system with a larger planet and a nearly edge-on orbit will provide the largest Doppler effect.

The most prolific method of exoplanet discovery and characterization is the transit method. If a star-planet system is aligned so that the planet passes between the observer (Earth) and the star, the total flux of light from the system will dip as the planet passes in front of the star blocking out starlight emitted by the stellar disk. This “eclipse” is known as a transit. In addition to the transit a companion will induce other effects to the total flux observed. These photometric variations are small in comparison to the transit and the total flux of the system but still detectable in some cases. Like the radial velocity method, transits are biased towards larger planets with shorter orbital periods. The probability of a transit occurring is much larger for short period planets and a larger planet will correspond to larger photometric variations. The largest contributor to the discoveries made by the transit method to date is the Kepler Space Telescope.

Other methods which have been applied toward exoplanet detection include microlensing and direct imaging. Microlensing looks for the slight change in flux due to gravitational lensing by a planetary system passing in front of a background star. Direct imaging is the “brute force” method which uses a mask to block out the light from the star and allow for observations of the exoplanet by itself. The Wide Field Infrared Survey Telescope (WFIRST) will be a 2.4 m space telescope capable of detecting exoplanets using both microlensing and direct imaging techniques. An accompanying coronagraph will allow for direct imaging and spectroscopy measurements of exoplanets and debris disks [11].

Exoplanet atmospheres have been characterized using spectral recordings from the Spitzer Space Telescope including the detection of water vapor [12]. Future characterization missions of exoplanetary atmospheres will use the James Webb Space Telescope [13].
1.2 The Kepler Space Telescope

The Kepler Space Telescope monitored the flux of over 100,000 individual stars during its primary mission from March 2009 to May 2013. Exoplanets may be detected by looking for small dips in the total flux from the light of a system by an eclipsing planet and observing the small oscillations within the total flux from a star due to a companion planet. The Bayesian based exoplanet hunting algorithm, EXONEST, developed by Placek, Knuth, and Angerhausen [4] [5] [6], uses Multinest [14] [15] [16] to perform Bayesian model testing and focuses on modeling transits and the four most prominent photometric effects found in Kepler lightcurves: Reflected Light, Thermal Emission, Doppler Boosting, and Ellipsoidal Variation.

The effect due to reflected light assumes the star radiates isotropically and the planet reflects as a Lambertian sphere. The planet appears to cycle through phases just as the Moon or Mercury in our Solar System. The total reflected light flux is computed by integrating over the illuminated portions of the planet observable from the line of sight. This effect has a period equal to the orbital period of the planet.

Thermal emission, which also has a period equal to the orbital period, represents the contribution of the blackbody radiation from the planet. In circular orbits, reflected light and thermal emission vary identically and cannot be disentangled. Combining observations from multiple bandpasses (e.g. Kepler and TESS), reflected light and thermal emission can be distinguished and provide improved constraints on geometric albedo, day-side temperatures, and night-side temperatures [17]. A sufficient eccentricity in the orbit breaks the degeneracies and allows one to independently identify reflected light and thermal emissions [6].

Doppler boosting is a relativistic effect caused by an increase in observed stellar flux as the star moves toward the observer as the star orbits the system center of mass and a decrease in observed flux as the star recedes. This approximation is the combination of
many smaller effects: the transformation of the energy-momentum four-vector between the frame of the star and the observer, increase/decrease of apparent angular size of the star as the star approaches/recedes, increased/decreased travel time of photons from the star as the star recedes/approaches, and the shift of the stellar emission flux due to the Doppler effect within the Kepler bandpass. The first three effects will all create a decrease in observed flux when the star is receding from the observer and an increase when the star approaches the observer. The fourth effect is more complicated and depends on the location of peak emission. Jackson et al. [3] report that HAT-P-7, a bluer star, will exhibit an increase in stellar flux as the star approaches and a decrease in stellar flux when receding. Doppler boosting has a period equal to the orbital period but has a 90° phase shift compared to reflected light or thermal emission.

Ellipsoidal variation is caused by distortion of the star due to gravitational effects of a companion exoplanet. This distortion results in photometric variations consisting of two maxima per orbit, each located when the greatest stellar surface area is being observed. The photosphere of a star has a large temperature gradient. Cooler plasma will appear dimmer compared to hotter plasma. Limb-darkening is the apparent dimmer edge on the star, as compared to the brighter center of the disk, due to observing cooler plasma closer to the surface. Similarly, gravity-darkening is an effect of observing the cooler plasma in the tidal bulge. Stellar rotation will cause the star to have a larger equator than meridian which will cause a dimming to occur around the equator and a brightening at the poles.

Future missions will improve the photometric variation precision into the low ppm range. Project goals for CHaracterising ExOPlanet Satellite (CHEOPS) is to reach <10 ppm precision in flux detection. CHEOPS is scheduled for launch in 2017 and will target systems with previously confirmed exoplanets. This mission moves beyond the goal of detection into more detailed characterization and identification. The bandpass will span between 0.4 and 1.1 microns which ranges from blue wavelengths into the infrared.

The PLAnetary Transits and Oscillations (PLATO) is a European Space Agency run
mission which will have the photometric precision to look for earthlike planets orbiting
Sun-like stars. In addition, the project will study the effects of stellar oscillations and
will provide more information about star mass, size, and age. The target launch date is
2024 [18].
Chapter 2

System Description

In order to describe the photometric effects caused by a companion exoplanet we must first be able to define how the star and planet system evolves in time. This section details the progression from Lagrange’s equations of motion into more convenient variables for the purposes of describing photometric variations.

2.1 Equations of Motion

First consider a star and planet system where \( \vec{r}_1 \) and \( \vec{r}_2 \) point to the star and planet respectively. The Lagrangian can then be written as

\[
L = T - V = \frac{1}{2} m_{\text{Tot}} \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 - V(\vec{r})
\]  

(2.1)

where we have performed a change of coordinates to \( \vec{r} = \vec{r}_1 - \vec{r}_2 \), the reduced mass \( \mu = \frac{m_1 m_2}{m_1 + m_2} \), \( \dot{R} \) which describes the center of mass motion of the system, the total mass of the system \( m_{\text{Tot}} \), and the term for the gravitational interaction potential, \( V(\vec{r}) \).

For simplicity we can set the relative motion of the system equal to zero. Additionally,
we write the Lagrangian in spherical coordinates. If $\dot{R} = 0$

$$L = \frac{1}{2} \mu \dot{r}^2 - V(\vec{r})$$

$$= \frac{1}{2} \mu (r^2 + r^2 \dot{\theta}^2) - V(\vec{r}).$$

The two generalized coordinates provide two equations of motion. Since the angle $\theta$ does not appear in the Lagrangian there is a conserved quantity known as angular momentum $\ell$

$$\ell = \mu r^2 \dot{\theta}.$$  \hfill (2.3)

The other equation of motion reflects the radial term

$$\mu \ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial V(r)}{\partial r}.$$ \hfill (2.4)

The solution of this equation gives the separation distance between the star and planet as a function of orbital angle $\theta$. Substitution of the angular momentum allows the format to closely resemble Newton’s Second Law. Here the left-hand side resembles mass multiplied by acceleration. The right-hand side is the net force written with two contributors: the first representing a ‘pseudo-force’, the centrifugal force, caused by the non-inertial reference frame of an orbit, and the second contributing the gravitational force between the two objects. For exoplanet motion it is reasonable to use Newton’s gravitational potential written in the form $V(r) = \frac{k}{r}$. Here the multiplicative factor containing the gravitational constant and masses is contained in $k$. The current form is

$$\mu \ddot{r} = \frac{\ell^2}{\mu r^3} + \frac{k}{r^2}.$$ \hfill (2.5)

It is convenient to perform a change of variables of $s = \frac{1}{r}$. The resulting second-order differential equation can be solved using separation of variables to obtain the separation
distance as a function of the orbit.\(^1\)

\[
\frac{\ell^2}{\mu^2} \left( \frac{d^2 s}{d\theta^2} + s \right) = -\frac{k}{\mu}.
\]  

(2.6)

The most general solution for this differential equation is

\[
s = \frac{k\mu}{\ell^2} \left( 1 + e \cos(\theta - \theta_o) \right)
\]  

(2.7)

Substituting in for \(s = 1/r\) the solution becomes

\[
r(\theta) = \frac{a(1 - e)}{1 + e \cos(\theta - \theta_o)}.
\]  

(2.8)

In the above equation \(a\) is the orbital semi-major axis, \(e\) is the orbital eccentricity, and \(\theta_o\) is the starting orbital position, usually set to be zero. The angle \(\theta\) is known as the true anomaly and hereafter will be denoted as \(\nu\). The true anomaly will be used to compute the separation distance as a function of time.

### 2.2 Orbital Anomalies

The mean anomaly, eccentric anomaly, and true anomaly are useful tools when determining the mechanics of a planet-star system. The mean anomaly, \(M(t)\), describes the mean motion of the system. It is given by the angular distance between the planet’s location and the planet’s point of closest approach to the star (periastron). This may be written as

\[
M(t) = M_o + \frac{2\pi}{T} t.
\]

(2.9)

\[
= \frac{2\pi}{T} (t - t_p).
\]

Above \(M_o\) is the initial mean anomaly, \(T\) is the orbital period, and \(t_p\) is the time since

\(^1\)Use \(\frac{ds}{dt} = \frac{ds}{d\theta} \frac{d\theta}{dt}\)
the planet passed through periastron. For eccentric orbits the eccentric anomaly is defined as

\[ E(t) = M(t) + e \sin (E(t)). \] (2.10)

The eccentric anomaly represents the angular separation between periastron and the projection of the planet onto an axillary circle which circumscribes the orbit. Numerical methods must be used in order to solve this transcendental equation.

![Diagram](image)

Figure 2.1: Diagram displaying the coordinate system used to describe the location of a planet within an eccentric orbit. The red plane represents the orbital plane while gray plane is the reference plane, or plane of the sky, for an observer located in the +z-direction. The inclination of the orbit is described by \( i \).

Last the true anomaly, \( \nu(t) \), is given by

\[ \tan \left( \frac{\nu(t)}{2} \right) = \sqrt{\frac{1 + e}{1 - e}} \tan \left( \frac{E(t)}{2} \right). \] (2.11)
Combining the eccentric and true anomalies, the separation distance between the star and planet can be determined

\[ r(t) = a(1 - e \cos E(t)) \]
\[ = \frac{a(1 - e^2)}{1 + e \cot \nu(t)}. \]  
\[ (2.12) \]

### 2.3 Euler Angles

The combination of three angles can be used to determine the position of a body in three dimensions. The three angles \( (\theta, \psi, \phi) \) represent rotations about each of the Cartesian coordinate axes: X, Y, and Z respectively. Each of these angles are defined using orbital parameters and the true anomalies, \( (\theta, \psi, \phi) = (i, \omega + \nu, \Omega) \). Here \( i \) is the inclination, defined as the angle between the reference plane (the plane of the sky) and the orbital plane. A face on orbit has an inclination of \( i = 0^\circ \). An edge on orbit, the orientation most beneficial to exoplanet searches using the transit technique, has an inclination of \( i = 90^\circ \). The variable \( \omega \) is the argument of periastron and represents the angle between the point of closest approach (periastron) and the intersection between the orbital plane and the reference plane. The argument of periastron is also the rotation of the orbit about the axis perpendicular to the orbital plane and is only definable in eccentric orbits. The longitude of the ascending node, \( \Omega \), rotates the orbit in the reference plane about the line-of-sight, typically taken to be along the z-axis.

The position of the planet as a function of time can therefore be defined as

\[ \mathbf{r}(t) = R(\psi)R(\theta)R(\phi)\mathbf{r}'(t). \]  
\[ (2.13) \]

This can be translated into Cartesian coordinates with
\[
\begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix} = 
\begin{pmatrix}
\cos \Omega \cos(\omega + \nu(t)) - \sin \Omega \sin(\omega + \nu(t)) \cos i \\
\sin \Omega \cos(\omega + \nu(t)) + \cos \Omega \sin(\omega + \nu(t)) \cos i \\
\sin(\omega + \nu(t)) \sin i \\
\end{pmatrix} \cdot r(t).
\]

\[\text{(2.14)}\]

For transits we set \(\Omega = 0\) since rotation about the line of sight will not affect the observed lightcurve. The resulting equation is

\[
\begin{pmatrix}
x \\
y \\
z \\
\end{pmatrix} = 
\begin{pmatrix}
\cos(\omega + \nu(t)) \\
\sin(\omega + \nu(t)) \cos i \\
\sin(\omega + \nu(t)) \sin i \\
\end{pmatrix} \cdot r(t).
\]

\[\text{(2.15)}\]

The position of a planet in Cartesian coordinates is written as

\[
\vec{r}(t) = x(t) \hat{x} + y(t) \hat{y} + z(t) \hat{z}.
\]

Therefore the unit vector is \(\hat{r} = \frac{\vec{r}}{r(t)}\). The line-of-sight, \(\hat{r}'\), is taken to be along the \(z\)-axis, \(\hat{r}' = \hat{z}\).

The phase angle, which is useful when describing motion around the orbit, is defined as

\[
\theta(t) = \arccos(\hat{r}(t) \cdot \hat{r}')
\]

\[
= \arccos\left(\frac{z(t)}{r(t)}\right)
\]

\[\text{(2.16)}\]

\[
= \arccos\left(\sin(\omega + \nu(t)) \sin(i)\right).
\]

This phase angle is used when describing the photometric effects due to a planet.
Ellipsoidal Variation Models

Ellipsoidal stars were first examined in non-eclipsing binary stars. The effect was extended to describe the interaction between a star and companion exoplanet. The amplitude of the effect was described by Loeb and Gaudi [19]

\[
\text{Amplitude} = \beta \frac{M_p}{M_s} \left( \frac{R_s}{r(t)} \right)^3
\]  

(3.1)

where \( M_p \) is the planet mass, \( M_{\text{star}} \) is the star’s mass, \( R_{\text{star}} \) is the stellar radius, \( r(t) \) is the star-to-planet distance, and \( \beta \) accounts for gravity and linear limb-darkening. There are two ways to compute \( \beta \), one involving the approximations from stellar parameters (Mass, Radius, and Temperature) and one involving the gravity-darkening coefficient and the linear limb-darkening coefficient. Typically, gravity and limb-darkening coefficients are used.

The method using coefficients is [20]

\[
\beta = 0.15 \frac{(15 + u)(1 + g)}{(3 - u)}
\]  

(3.2)

where \( u \) and \( g \) are the limb-darkening and gravity-darkening coefficients respectively. Limb-darkening and gravity-darkening coefficients can be extracted by modeling metallicity and
effective temperature [21] [22]. The metallicity of the star is given by the fraction of mass which is not hydrogen or helium. In stellar physics any element heavier than helium is classified as a metal. Metallicity increases as the star ages and more elements are fused.

Alternatively, the stellar parameters of effective stellar temperature $T_{\text{eff}}$, star mass $M_S$, and stellar radius $R_S$, can be used to determine $\beta$

\[
\beta = \frac{\log\left(\frac{G M_S}{R_S^2}\right)}{\log(T_{\text{eff}})}.
\]  

(3.3)

3.1 BEER

BEaming, Ellipsoid, and Reflection (BEER) was developed by [23] to describe the periodic deviations in the observed flux of a star due to a short period planet. The model describes the effects introduced in Section 1.2 as trigonometric functions. The following equation is the BEER model for the flux observed from ellipsoidal variations. BEER modifies the amplitude from Equation 3.1 by accounting for the effect due to the inclination of the orbit ($i$) with respect to the observer. Since BEER approximates the effect as trigonometric functions the algorithm runs quickly. However, the actual effect is more complicated than a simple sinusoid. BEER accounts for the phase angle with the orbital period term

\[
\frac{F_{\text{ellip}}(t)}{F_S} \approx -\beta \frac{M_P}{M_S}\left(\frac{R_S}{a}\right)^3 \sin^2(i) \cos\left(\frac{2\pi P_{\text{orb}}/2}{P_{\text{orb}/2}}t\right),
\]

(3.4)

where $F_S$ is the normalization factor equal to the stellar flux, $M_P$ is the planet’s mass, $M_S$ is the star’s mass, $R_S$ is the star’s radius, $a$ is the star-planet distance, $i$ is the inclination, and $P_{\text{orb}}$ is the orbital period of the planet. For circular orbits, the phase angle, $\theta$, and $\frac{2\pi}{P_{\text{orb}}}$ are equal. The relation may not hold true in eccentric orbits except in the limiting case where $e \to 0$. Replacing the factor in BEER $\frac{2\pi}{P_{\text{orb}}}t$, with orbital phase, the parameter
useful for compatibility for EXONEST, the effect becomes

\[
\frac{F_{\text{Ellip}}(t)}{F_S} \approx -\beta \frac{M_p}{M_S} \left( \frac{R_S}{a} \right)^3 \sin^2(i) \cos(2\theta). \tag{3.5}
\]

### 3.2 Kane and Gelino (2012)

Another model for ellipsoidal variation was proposed by Kane and Gelino [2]. The amplitude of the effect is the same as BEER (Loeb and Gaudi) however differs from BEER by accounting for the change in the observed flux by using the phase angle (\(\theta\)). Like the BEER algorithm, this model only contains a trigonometric function and therefore will run relatively quickly.

The model in terms of Euler angles is given by

\[
\frac{F_{\text{Ellip}}(t)}{F_S} \approx \beta \frac{M_p}{M_S} \left( \frac{R_S}{a} \right)^3 \left( \cos^2(\omega + f) + \sin^2(\omega + f) \cos^2 i \right)^{\frac{1}{2}}. \tag{3.6}
\]

Simplification by substitution using the phase angle and Cartesian coordinates provides

\[
\frac{F_{\text{Ellip}}(t)}{F_S} \approx \beta \frac{M_p}{M_S} \left( \frac{R_S}{a} \right)^3 \sin(\theta), \tag{3.7}
\]

where \(\theta\) is the phase angle.

Unlike the BEER representation for ellipsoidal variations the Kane & Gelino model has a minimum value of zero. Additionally, the model contains a discontinuity in the first derivative at the locations of minimum ellipsoidal variations. It is not clear where such a discontinuity should arise. Further analysis using the projected area from an arbitrarily oriented ellipsoid, derived by Vickers [24], will provide further insight.
3.3 Kane and Gelino (Modified)

The modified version of the Kane and Gelino (2012) model, proposed by Placek, Knuth, and Angerhausen [4], introduces a square above the sinusoid which removes the discontinuity in the first derivative of the original model (See Figure 3.4)

\[ \frac{F_{\text{ellip}}(t)}{F_S} \approx \beta \frac{M_P}{M_S} \left( \frac{R_S}{a} \right)^3 \sin^2(\theta). \]  

(3.8)

3.4 EVIL-MC

Ellipsoidal Variations Induced by a Low-Mass Companion (EVIL-MC) was developed by Jackson [3] in IDL to more accurately model the ellipsoidal variation and has been translated for compatibility with the EXONEST algorithm which was written in MATLAB. This model determines the shape of the star by projecting a grid onto a sphere and determining the deviation from sphericity caused by the planet for each projected stellar grid point. Jackson has applied this model to circular orbits [3].

3.4.1 Derivation

The gravitational potential for the stellar surface is described by

\[ U = \frac{GM_S}{R_S} + \frac{GM_P}{(A^2 - 2R_SA \cos \psi + R_S^2)^{1/2}} - \frac{GM_P}{A^2} R_S \cos \psi + \frac{1}{2} \omega_S^2 R_S^2 (1 - \cos^2 \lambda), \]  

(3.9)

where \( G \) is Newton’s gravitational constant, \( M_S \) is the stellar mass, \( R_S \) is the distance from the center of the star to a point on the photosphere, \( M_P \) is the planet’s mass, \( A \) is the distance between the star and the planet, \( \omega_S \) is the stellar rotation rate, \( \cos \psi = \hat{R}_S \cdot \hat{A} \), and \( \cos \lambda = \hat{R}_S \cdot \hat{\omega}_S \). For this paper, the stellar rotation was set to zero, hence \( \omega_S = 0 \).

The first and second terms in Equation 3.9 represent the contributions to the potential
by the star and planet respectively. If the second term were expanded in a Taylor series in $R_S/A$ the first term would be proportional to $\cos \psi$. This term represents the force which keeps the star in orbit around the system’s barycenter. Since this force is constant it does not contribute to the tidal interaction and therefore the third term removes this contribution. The fourth term represents the contribution to the potential due to centrifugal accelerations from rotational motion.

To set up the expansion $U$ is normalized to produce $\Phi$

$$\Phi \equiv U \left( \frac{R_o}{GM_S} \right) = \frac{1}{R} + \frac{q}{(a^2 - 2a\cos \psi + R^2)^{1/2}} - \frac{q}{a^2} R \cos \psi. \quad (3.10)$$

with $R = R_S/R_o$, $q = M_P/M_S$, and $a = A/R_o$. Additionally, define $|R_S| \equiv R_o$ when $\vec{R}_S \perp \vec{A}$. From here we can rewrite $R_o$, so that the stellar radius can be written as $R = (1 + \delta R)R_o$. The isopotential contour, $\Phi = \text{const}$, defines the surface of the star. For simplicity the potential $\Phi_o$ is the potential at $\vec{R}_o$ (when $\cos \psi = 0$)

$$\Phi_o = 1 + \frac{q}{(a^2 + 1)^{1/2}}. \quad (3.11)$$

If the departure from sphericity, $\delta R$, is small, then we can perform an expansion

$$\Phi = \frac{1}{1 + \delta R} + \frac{q}{(a^2 - 2a(1 + \delta R) \cos \psi + (1 + \delta R)^2)^{1/2}} - \frac{q}{a^2} (1 + \delta R) \cos \psi$$

$$\approx (1 - \delta R) + \frac{q}{(a^2 - 2a \cos \psi + 1)^{1/2}} - \frac{q}{a^2} \cos \psi. \quad (3.12)$$

Second-order and higher terms were dropped. Setting Equation 3.11 equal to Equation 3.12 and solving for $\delta R$

$$\delta R = q \left( [a^2 - 2a \cos \psi + 1]^{-1/2} - [a^2 + 1]^{-1/2} - \frac{\cos \psi}{a^2} \right) - \frac{\omega^2}{2a^3} \cos^2 \lambda. \quad (3.13)$$
To evaluate the algorithm in the original code a 31x31 grid is projected onto a sphere. The algorithm is computed for each orbital phase angle. The proportion of the area observed is computed by taking the dot product between the normal area vector and the normal vector to the observer ($\hat{z}$). The observed flux within the Kepler bandpass is computed using Equation 2 from [19].

Limb-darkening is caused by the observation of plasma at different temperatures when observing different locations on the star. When looking at the center of the star the observations penetrate into the hotter layers closer to the stellar core. Observations on the limb of the star are limited only to the cooler plasma closer to the surface. This causes the edges of the star to appear darker than the center. In the original EVIL-MC algorithm a quadratic limb-darkening mask is applied to account for this effect.

Similarly, plasma on the star which is pulled further away from the center will be cooler. This will occur primarily at the bulge due to the gravitational interaction between the planet and star. Since the plasma is cooler it will appear darker with respect to the other locations on the star. This is known as gravity-darkening.

Stellar rotation will cause the star to become an oblate spheroid. An oblate spheroid is obtained by rotating an ellipse about a symmetry axis and appears as a “squashed” sphere. If we take the symmetry axis to be the vertical axis (z-axis), the radius along the x and y directions are the same with a smaller radius in the z direction. Since the poles of the spheroid are closer to the hot center the plasma at the poles will be hotter than the equator. Therefore the poles will be brighter than the equator giving rise to gravity brightening.

The magnitude of this effect may result in a temperature change of a few 0.1 K on points on the stellar surface. This effect is modeled in the original EVIL-MC by looking at changes to the gravity vector on the stellar surface. This effect has not been implemented into the version contained in EVIL-MC due to computational limitations. Gravity-darkening will provide the smallest contribution to the lightcurve and therefore is lower priority.

The ellipsoidal variation observed is computed with
\[ \Phi_{\text{Ellipse}} = 1 - \frac{\phi_{\text{Sphere}}}{\phi_{\text{Star}}}. \]  \tag{3.14}

Here, \( \phi_{\text{Sphere}} \) is the flux from a spherical star and \( \phi_{\text{Star}} \) is the flux from the ellipsoidal star.

While computationally intensive, this model should improve upon the accuracy provided by both BEER and the Kane and Gelino models by directly accounting for limb-darkening and gravity-darkening. Additionally, none of the trigonometric models are able to model the photometric effects due to stellar rotation.

### 3.4.2 Implementation in EXONEST

EVIL-MC was originally written in IDL. For use in EXONEST, EVIL-MC had to be translated into MATLAB. The new function first constructs a geodesic dome, with \( f=8 \), to represent the star. The frequency, \( f \), determines the number of times each base triangle is subdivided. The geodesic dome is useful because it is constructed through a nearly uniformly distributed number of points along the sphere which form easy to work with triangles. Each triangle is formed with three grid-points. Additionally, since only half of the star is possible to observe at any given time, only half of the star needs to be constructed. This improves the efficiency of the computation over the original EVIL-MC code. A frequency of 8 will give 640 triangles across the half-dome which is a denser distribution of points compared to the 31x31 grid over a full sphere used in the original EVIL-MC and may be generated in less time.

The deviations from spherical are computed for each grid-point on the geodesic sphere using Equation 3.13. These deviations are added to the locations of the grid-point effectively adjusting the location of the surface of the star. Limb-darkening profiles can be applied by accounting for the location of the center of each triangle. Next, if stellar rotation is used, the Doppler shift of the blackbody function can be computed for each triangle.
Figure 3.1: A scale model looking edge on to the Kepler-13 system using EVIL-MC for EXONEST. A geodesic sphere of each point in orange represents a location where the deviations to the stellar surface have been computed. Kepler-13b is a massive planet, estimated to be around $9 \ M_J \ [25]$, with an orbital period of 1.763 days. A large planet orbiting closely to the star should provide significant ellipsoidal variation. However, while the effect may be detectable to Kepler, the distortions on the star are not easily observable to the eye.

The area of a single polygon is computed by taking the cross product between two vectors connecting three different corners of triangles. The observed flux within the Kepler bandpass is observed using Equation 2 from [19]. To compute the amount of flux we must take the dot product between the light of sight vector and the flux vector. In practice this is simply the $z$-component of the surface normal to the center of the triangle. The total flux is summed for the ellipsoidal star and a spherical reference star. The resulting effect observed in the Kepler data is modeled by Equation 3.14.

### 3.4.3 Limb-Darkening Models

Limb-darkening is accounted for by applying a mask following a limb-darkening model. The original EVIL-MC by Jackson uses a quadratic limb-darkening model. The new im-
plementation in EXONEST has the ability to choose different models some of which can be seen in Table 3.1. It is not currently possible to directly model all the stellar details directly. Instead, ad hoc “laws” or models are applied. The models use free parameters to match with stellar atmosphere models. The limb-darkening models can then be applied to obtain the lightcurve data. To describe a location on the surface of the star the parameter $\mu = \cos \theta$ is used. Currently the quadratic model is used in EXONEST for transits.

<table>
<thead>
<tr>
<th>Model Name</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear [26]</td>
<td>$\frac{I(\mu)}{I(1)} = 1 - a_2(1 - \mu)$</td>
</tr>
<tr>
<td>Quadratic [27]</td>
<td>$\frac{I(\mu)}{I(1)} = 1 - a_2(1 - \mu) - a_4(1 - \mu)^2$</td>
</tr>
<tr>
<td>4 Parameter Non-Linear [28]</td>
<td>$\frac{I(\mu)}{I(1)} = 1 - \sum_{n=1}^{4} a_n (1 - \mu^{n/2})$</td>
</tr>
<tr>
<td>3 Parameter Non-Linear [22]</td>
<td>$\frac{I(\mu)}{I(1)} = 1 - \sum_{n=1}^{3} a_n (1 - \mu^{(n+1)/2})$</td>
</tr>
<tr>
<td>Square-Root [29]</td>
<td>$\frac{I(\mu)}{I(1)} = 1 - a_2(1 - \mu) - a_1(1 - \sqrt{\mu})$</td>
</tr>
<tr>
<td>Logarithmic [30]</td>
<td>$\frac{I(\mu)}{I(1)} = 1 - a_2(1 - \mu) - a_5 \mu \ln \mu$</td>
</tr>
<tr>
<td>Exponential [31]</td>
<td>$\frac{I(\mu)}{I(1)} = 1 - a_2(1 - \mu) - \frac{a_6}{1 - e^\mu}$</td>
</tr>
</tbody>
</table>

Table 3.1: Various limb-darkening models.

Figure 3.2: Shown is the linear limb-darkening profile model.
Figure 3.3: The quadratic model is a commonly used representation of limb-darkening.

3.5 All Models Comparison

Each model exhibits the general characteristics of having two maxima per orbit, located around quarter and three-quarters orbital phase and has a maximum amplitude roughly described by Loeb and Gaudi [19]. All models can be seen plotted together in Figure 3.4. Included are two versions of EVIL-MC: one shown in magenta which models only the star shape and the second which also includes a limb-darkening mask. Also plotted are the trigonometric models. The modified version of Kane & Gelino seems to approximate the version of EVIL-MC which only models the stellar shape.
Figure 3.4: Each of the ellipsoidal models plotted over one orbital period. The 1 M$_J$ mass planet is in a circular orbit around a 1 M$_S$ star with an orbital period of 1 day.
Chapter 4

Bayesian Model Testing

Often, different descriptions are used to represent the same situation. These differences can stem from relying on different parameters or numbers of parameters. For example, a model may account for an extra effect using more parameters, say using a 5th degree polynomial instead of a 4th degree polynomial. One criteria for model selection should be the representation of the data. Models which poorly describe the data should not be favored. Traditionally, comparison between model and data has relied on the Chi-Squared method of analysis. Models which poorly describe the data will perform poorly in a Chi-Squared test. Generally, a model with more parameters will fit to the data closer than a simpler model will. However, this can’t be the only criteria. If we are given 10 data points one could easily fit a 10-dimensional polynomial which will exactly hit every data point but we may more easily describe the data with a simple line. This is related to the problem of over-fitting. Therefore another condition should be simplicity or number of parameters. If two models, under equal conditions, describe the same data equally well then the simpler model should be preferred. Additionally, for data in one-dimension, a sinusoid with a ultra fine-tuning on the frequency and amplitude can nearly hit every point exactly. Requirements on the tuning of the parameters must also be taken into account. Bayesian model testing allows us to extend beyond the traditional and outdated Chi-Squared method
Bayes’ Rule can be written as

\[
P(\theta_M | D, I) = \frac{P(\theta_M | I) \ P(D | \theta_M, I)}{P(D | I)}.
\] (4.1)

Here M is chosen to label a particular model, D is the data to be considered, I is the prior information, and \( \theta_M \) represents parameter values assigned to the particular model. The quantity \( P(\theta_M | D, I) \) is the posterior probability and represents the probability distribution of a set of parameters after considering the data. \( P(\theta_M | I) \) is the prior probability and is the distribution of the set of parameters considering only the previous knowledge and constraints and without consideration of the data. \( P(D | \theta_M, I) \) is known as the likelihood function which represents the probability that the selected parameter values and model could have reproduced the data. Lastly, \( P(D | I) \) is the evidence which represents the probability that the data could have been generated by the model without considering particular model values. This term is used to keep the posterior probability normalized and in model selection.

### 4.1 Model Selection and the Evidence

If we are just interested in parameter selection then the posterior probability is the most important quantity. Evaluating the posterior will allow us to determine the most likely parameter values to represent the data.

In model selection the most important quantity is the evidence. Model selection is done by comparing the evidences of each model. The evidence is determined by integrating over all parameter values which essentially acts as a prior-weighted average for the likelihood. Since this process involves marginalizing over all parameter values the evidence is commonly called the marginalized likelihood. The evidence can be written as
\[ P(D|I) = Z = \int P(\theta_M|M, I)P(D|\theta_M, I)d\theta_M. \]  \hspace{1cm} (4.2)

Models with more parameters will have a lower prior probability because the unit probability is spread over a larger volume. If all models have equal a priori probabilities (there is no reason for one model to be biased over another), then the Bayesian evidence will naturally favor the simpler model that describes the data. If two models describe the same situation equally then the simpler model should prevail.

Models can be compared by taking the ratio of the evidences, called the Bayes’ factor, which provides a numerical value for model preference

\[ \text{Bayes’ Factor} = \frac{\text{Evidence of Model } A}{\text{Evidence of Model } B}. \]  \hspace{1cm} (4.3)

If the Bayes’ Factor from Equation 4.3 is much larger than unity then model \( A \) will be significantly preferred over model \( B \). Similarly, if the Bayes’ Factor is small, close to zero, then model \( B \) is preferred. If the Bayes’ Factor returns of order unity then neither model is significantly preferred over the other. Model preference is inherently subjective. The cutoff defining when one model should be preferred is difficult to define. Generally, we say there is a significant distinction between the models when the difference in the evidence (Bayes’ Factor) is larger than 150 [32] [33].

To ease the computation of the Bayes’ Factor and comparison of the evidences it is useful to work with the natural logarithm of the evidence, \( \log Z \). General properties of logarithms apply. That is the Bayes’ Factor simplifies to

\[ \log(\text{Bayes’ Factor}) = \log \left( \frac{Z_A}{Z_B} \right) = \log(Z_A) - \log(Z_B). \]  \hspace{1cm} (4.4)

We can therefore use simple subtraction rather than dividing large numbers. A Bayes’ Factor of 150 occurs when the difference in the \( \log Z \)’s is around 5 since \( e^5 \approx 150 \).
4.2 Priors

Selection of a prior must be done carefully and take into account any prior information or constraints but not assume more than what is known. We assign uniform probabilities to each parameter within the range defined by the constraints or our search area which can be seen in Table 4.1. An example parameter is orbital inclination. Currently there is no information to indicate a connection between a planet’s orbital inclination and the orientation of the observer. Therefore, to select a uniform prior on the region, we select from a uniform distribution over a sphere. Impossibilities are also accounted within the priors. Examples include a planet of negative mass which is nonphysical and orbital eccentricities which are greater than one and would not create a periodic orbit for which we could observe. These can be excluded in the search by a proper selection of the prior. The priors for planetary radius and planetary mass have been chosen to span the range from below the detectability threshold and into the brown dwarf region. A posterior containing many results with masses larger than the boundary between planet and brown dwarf of around $13 \, M_J$ will lead to the classification of a brown dwarf star rather than an exoplanet.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Interval</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbital Period (Days)</td>
<td>$T$</td>
<td>$[0.01, 15]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>$e$</td>
<td>$[0, 1]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Stellar Mass</td>
<td>$M_S$</td>
<td>—</td>
<td>Known</td>
</tr>
<tr>
<td>Mean Anomaly</td>
<td>$M_0$</td>
<td>$[0, 2\pi]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Arg. of Periastron</td>
<td>$\omega$</td>
<td>$[0, 2\pi]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Orbital Inclination</td>
<td>$i$</td>
<td>$[0, \frac{\pi}{2}]$</td>
<td>Uniform on a Sphere</td>
</tr>
<tr>
<td>Planetary Radius ($R_J$)</td>
<td>$R_P$</td>
<td>$[10^{-4}, 20]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Geometric Albedo</td>
<td>$A_g$</td>
<td>$[0, 1]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Stellar Radius</td>
<td>$R_S$</td>
<td>—</td>
<td>Known</td>
</tr>
<tr>
<td>Planetary Mass ($M_J$)</td>
<td>$M_P$</td>
<td>$[0.1, 20]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Dayside Temperature (K)</td>
<td>$T_d$</td>
<td>$[0.5000]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Nightside Temperature (K)</td>
<td>$T_n$</td>
<td>$[0.5000]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Standard Deviation of Noise (ppm)</td>
<td>$\sigma$</td>
<td>$[10^{-6}, 10^{-4}]$</td>
<td>Uniform</td>
</tr>
<tr>
<td>Limb-Darkening Coefficients</td>
<td>$\mu_1$, $\mu_2$</td>
<td>—</td>
<td>Known</td>
</tr>
</tbody>
</table>

Table 4.1: A list of the priors assigned to each parameter. Distributions may be adjusted beyond uniform given additional information. Parameters associated with quantities of prior information are designated as ‘known’ and assigned the known value.
Chapter 5

Nested Sampling

Model selection is concerned with both the posterior distributions as well as the evidence. If we were just concerned with the posterior we wouldn’t need to spend the extra computation time on the evidence. An alternative method would be pure Markov Chain Monte Carlo (MCMC) simulations.

Nested Sampling, developed by John Skilling [34] [35], uses MCMC methods to explore probability space and iterate toward a posterior distribution. Writing Bayes’ Rule relating prior $\pi$, likelihood $L$, evidence $Z$, and posterior $P(x)$ we obtain [36]

$$L(x)\pi(x) = ZP(x). \quad (5.1)$$

This provides the direction of nested sampling. We input a likelihood and a prior and receive an output of the evidence and a posterior. We require that normalization holds true for the prior and posterior. Marginalization over all parameters returns unity

$$\int \int \ldots \int \pi(x)dx = 1 \quad \text{and} \quad \int \int \ldots \int P(x)dx = 1. \quad (5.2)$$

With a posterior defined using the above normalization conditions and Bayes’ Rule in the form
\[ P(x) = \frac{\mathcal{L}(x) \pi(x)}{Z}, \]  
(5.3)

we can solve for the evidence as

\[ Z = \int \int ... \int \mathcal{L}(x) \pi(x) dx. \]  
(5.4)

Nested sampling provides a useful method for exploring the parameter space while focusing on areas of parameter space that have the greatest contribution to the posterior and evidence. This is especially useful in systems with many parameters. For example let one parameter have a total of 10 possible values. Doubling the number of parameters increases the number of possible states by another factor of 10. An \( N \)-dimensional system will have \( 10^N \) possible states. Even in simple cases the number of possible states is so numerous that directly evaluating every possible situation is impossible.

The process is to use \( n \) objects within parameter space \( x \), randomly distributed within the prior \( \pi \), and iterate using the constraint \( \mathcal{L}(x) > \mathcal{L}^* \). A new object is randomly selected from the remaining probability space that satisfies the constraint on the likelihood. The new object replaces the worst object in the previous iteration which, due to the constraint, will correspond to an improved likelihood \( \mathcal{L}(x) > \mathcal{L}^* \). This will naturally contract towards regions with higher likelihood. The algorithm collapses the space at a rate of roughly equal to \( e^{1/n} \). Increasing the number of samples explores parameter space more densely but increases computation time.

### 5.1 Multinest

Multinest [14] [15] [16] is a nested sampling algorithm with capabilities to efficiently sample multimodal and highly complex distributions. Multinest follows the nested sampling philosophy and provides a posterior distribution and an associated evidence summation. The
algorithm shows significant acceptance rate improvements for multimodal and degenerate distributions over traditional nested sampling. Efficient sampling is required for timely computation. The algorithm works by creating ellipsoidal bounds around the samples and choosing new samples from within the bounds. This reduces the time spent trying uninteresting samples in low probability regions. Separate ellipsoidal regions can be created to account for multimodal distributions and degeneracies. Multinest has drawbacks. In general, the clustering of samples into ellipsoids is only computationally feasible in problems where the dimensionality is on the order of 10 parameters. Also the elliptical regions may not encompass the high likelihood regions [37].

5.1.1 Computation of the Evidence

The evidence can be viewed as the normalization factor or average of the likelihood over the prior. This is the integral of the likelihood as a function of the proportion of the prior satisfying the nested sampling constraint, that is the region contained within the iso-likelihood contour. The evidence can be written as

\[ Z = \int_0^1 \mathcal{L}(x_i) dx_i \]  

(5.5)

where \( x_i \) represents the volume of the prior after the \( i^{th} \) iteration.

This integral can be approximated numerically using a weighted sum and the trapezoid rule

\[ Z = \sum_{i=1}^{M} \mathcal{L}_i \frac{1}{2}(x_{i-1} - x_{i+1}). \]  

(5.6)

This process begins by randomly selecting \( N \) points from the full prior, \( \pi(\theta) \), which will serve as the initial set of active samples. The initial prior volume is \( x_o = 1 \). Samples are sorted by their likelihood values. The sample with the lowest likelihood is dropped and deemed inactive. A new sample is randomly selected from the remaining prior satisfying
the constraint that the new likelihood value will be larger than the previous worst sample, that is \( L_i > L_{i-1} \). The new volume of the prior which is contained within an iso-likelihood contour is defined as \( x_1 = t_1 x_o \) where \( t_1 \) is a randomly chosen variable from the distribution \( Pr(t) = N t^{N-1} \) which is obtained using order statistics. The volume will contract at an average at a rate of \( \log x_i \approx -(i \pm \sqrt{i}/N) \).

The stopping criteria is chosen by computing the evidence to a specified precision. Initially the likelihood increases faster than the width of the prior decreases when high probability regions are being discovered. Eventually the likelihood plateaus and the decreasing width of the prior overshadows the increase of the likelihood. At this point the evidence will not change much and the computation can be terminated. This termination criteria is subjective however for this paper a tolerance of 0.01 in Multinest was used as a stopping criteria. This means that the evidence has been roughly computed to within \( e^{0.01} \) according to the model.
Chapter 6

Sources of Noise

Ellipsoidal variations and the other photometric effects are small modulations on the total amplitude of the photometric flux from the system. Noise arises from instrument precision, image drift, thermal variations on the detector, pixel variances, and stellar activity. While many of these effects are noticeable or well-known and can be removed to some extent in post-processing, sometimes this compiles together to dominate any photometric variations present in the data other than the transit. In this case, noise will mask any photometric effects due to ellipsoidal variations of the host star.

6.1 Stellar Activity and Starspots

Noise caused by stellar activity, which includes flares, prominences/filaments, coronal mass ejections, and starspots. In particular, starspots have been a topic of much modern research [38] [39] [40] [41]. Starspots are essentially the same as sunspots, where large magnetic fields, usually associated with active regions, produce significantly cooler plasma. The cool plasma appears dark compared to the rest of the photosphere. Starspots, depending on size, can mimic the appearance of a transit. Additionally, the change in flux due to starspots and other stellar activity can dominate or mask other photometric effects. Since the rotational
period of the star is unlikely to be the same as the period of the photometric effects, phase folding over many orbits will wash out any effect other than the transit. In the following plot of flux from Kepler-8, Figure 6.1, data from one Kepler quarter \(^1\) is shown phase folded over the planet’s orbital period.

Figure 6.1: A phase folded plot of the flux from Kepler-8. The background noise and effects wash out photometric variations other than the primary transit. The secondary eclipse is also completely washed out by the background effects. Choosing a shorter dataset, one spanning only a couple orbits, may reduce the amount of longterm noise associated with stellar variability, starspots, and pixel drift. Longterm variations to the lightcurve may be able to be subtracted out. Additionally choosing a system which doesn’t display as much longterm variation like HAT-P-7 and Kepler-13 may not require post-processing.

\(^1\)A Kepler quarter is the 93 day (\(\approx 3\) months) set of observations. Kepler has published 18 quarters worth of data thus far.
6.2 Stellar Rotation

A rotating star will exhibit broader emission lines due to a Doppler shift. Portions on the star which are moving toward the observer will be shifted to “bluer” shorter wavelengths while receding material will be “red” shifted toward longer wavelengths. The Kepler band-pass is not uniform and so the observed spectra is hard to predict without direct computation. This may be accomplished using a model like EVIL-MC however, in order to model rotation, the parameter space would need to be expanded by three parameters (one for each Cartesian axis).

Additionally, stellar rotation will cause the star to compress along the axis parallel to the rotation. The star becomes a slight oblate spheroid. A more tangible example is the Earth which has a larger circumference along the Equator than the Prime Meridian. Rotation changes the shape of the star and makes gravity-darkening/brightening a more important factor. As stated in 3.4.1, gravity-darkening occurs in plasma which is pulled further from the hotter plasma closer to the center of the star. This will be most important at locations on the star perpendicular to the axis of rotation. Additionally, points on the star near the poles will appear brighter because they have been compressed toward the center.

Currently no trigonometric model has the ability to represent stellar rotation. Direct modeling like EVIL-MC theoretically has the ability to account for stellar rotation but the increased parameter space is currently computationally infeasible. Morris et al. [42] used a least-squares method to show that gravity-darkening induced by a planetary companion may be detectable in the HAT-P-7 system. The group modeled the gravity-darkening as a spot of cooler temperature on the star. Future analysis should address possible implementations of gravity-darkening more fully and using the Bayesian framework. Starspots complicate the transit lightcurve but may allow for examination of the rotation rates of the star. The Sun contains a latitudinally dependent rotational rate. More direct modeling
of a star with starspots and differential rotation may be able to better parameterize the system [43].
Chapter 7

Processing of the Data

Some parameters like the limb-darkening and gravity-darkening coefficients were taken as prior knowledge. These were determined by using prior knowledge of the star class and temperature. Additionally, the data was prepared to allow for efficient computation within the EXONEST algorithm.

7.1 Limb-darkening Coefficients

Limb-darkening coefficients are interpolated using data from Claret and Bloemen [44]. They computed limb-darkening coefficients using different models and at a range of effective stellar temperatures ($2000 K \leq T_{\text{eff}} \leq 50000 K$), metallicities ($10^{-5}$ to $10^{1}$ Solar abundances), surface gravities ($0.0 \leq \log (g) \leq 5.0$), and microturbulent velocities ($V_\xi = 0, 1, 2, 4, 8 \text{ km s}^{-1}$). The coefficients were generated by performing a least-squares fit to stellar atmosphere models ATLAS and PHOENIX for within the Kepler, Spitzer, and CoRoT bandpasses. Further analysis was conducted by Howarth [45] on comparing limb-darkening coefficients from models photometry. Howarth discovered a significant range of quality of fits even on slightly different stellar parameters.
7.2 Gravity-darkening coefficient

The coefficient used to account for gravity-darkening has a wavelength dependence. A general expression for the gravity-darkening coefficient as a function of wavelength was given by Bloemen et al. [46]

\[
\beta(\lambda, T_{\text{eff}} \log g, V_\xi) = \frac{d \ln I}{d \ln g} = \left( \frac{\partial \ln I(\lambda)}{\partial \ln g} \right)_{T_{\text{eff}}} + \left( \frac{d \ln T_{\text{eff}}}{d \ln g} \right) \left( \frac{\partial \ln I(\lambda)}{\partial \ln T_{\text{eff}}} \right) g. \tag{7.1}
\]

A change in the temperature of a body will have a large change in the blackbody emission from an object. The luminosity from an object with area A is given by the Stefan-Boltzmann equation

\[
L = A \sigma T^4. \tag{7.2}
\]

Therefore even a small change in the temperature could have a detectable effect on the observed photometric flux. Gravity-darkening is a small but not negligible effect [44]. The stellar atmosphere models PHOENIX and ATLAS disagree on the significance of the effect to some degree. In EXONEST the gravity-darkening coefficient is assumed to be constant over the Kepler bandpass which extends from 420 nm to 900 nm.

7.3 Phase Folding

The lightcurve data was downloaded from the Exoplanet Archive hosted by IPAC - CalTech [8]. The data was normalized with the division method to create a continuous dataset. Essentially this divides the data by the average flux of the star [8]. The models of the photometric effects have been defined as ratios of the photometric flux divided by the stellar flux. Additionally this removed the discontinuities from observations in different quarters.
The normalized data was refined to the desired length, typically only one to a few quarters. The period of the planet can be determined by using a Lomb-Scargle Periodogram [47] [48] which folds the data on many different periods. The Lomb-Scargle Periodogram plots the spectral power as a function of period. A transiting planet (transits are present) will have the maximum power at the orbital period of the planet. The data was then folded on the planet’s orbital period to create the phase curve covering one complete orbit of the planet. This is advantageous because the forward model only needs to be computed for one orbit which saves on computational time.
Chapter 8

Application

8.1 Synthetic Data

Before applying the Bayesian framework to Kepler data, the analysis was conducted in a more controlled environment. The models can be compared by taking a simple residual subtraction and comparing the result to the estimated noise level within the Kepler telescope. Additionally synthetic data can be constructed by running the model for a given set of parameters and adding Gaussian noise to the resultant lightcurve. This provides a controlled environment where the answer is known and can be compared to the posteriors generated by EXONEST.

8.1.1 Residuals

Examination of the residuals between the models may reveal the extent to which the models may be distinguished. The residual was taken to be the largest differences between the models. A larger difference between the models should occur when ellipsoidal variations have a greater amplitude. This occurs at closer orbits (shorter orbital periods) and larger planet masses. In Figures 8.1 and 8.2, the estimated precision of Kepler is shown as the green plane and the estimated precision of CHEOPS is shown as a black plane. While the
differences between the models may be within the noise of the telescope, measurements over multiple orbits may still contain enough information to distinguish the models. The Kepler telescope has an estimated precision of 30 ppm (green plane on Figures 8.1 and 8.2) for a 10 mag star. The CHEOPS mission (black plane) will reduce the precision level down around 10 ppm. For CHEOPS, the ellipsoidal models should be clearly distinguishable in most hot Jupiter systems. The largest difference between the models occurs between the BEER and EVIL-MC (with limb-darkening) models which can be seen in Figure 3.4. Because the BEER model estimates the variation using a sinusoid the model predicts a “negative” flux when the smallest portion of the star is being observed. EVIL-MC (with limb-darkening) estimates the flux to be non-zero at this point. This occurs at an orbital phase angle of 0.25 and 0.75 on Figure 3.4. This is the largest difference between all of the models at any point along the orbit.

![Residuals of BEER and Kane & Gelino (2012)](image)

Figure 8.1: The maximum residuals between the BEER and Kane & Gelino models. The planet mass range extends between [0,15] M_J and the orbital period [2,8] days.
Figure 8.2: The maximum residuals between the BEER and EVIL-MC model containing limb-darkening are plotted at different planet masses and orbital periods. The planet mass range extends between $[0,15]$ M$_J$ and the orbital period $[2,8]$ days.

8.1.2 Synthetic Kepler Data

The ellipsoidal models were compared using synthetic data generated by adding Gaussian noise to data generated by using one of the models. Models with noise at levels 0 ppm, 10 ppm, and 30 ppm were compared, which includes the estimated uncertainty of the future mission CHEOPS and current mission Kepler.

Synthetic data with Gaussian noise added up to 30 ppm was able to determine the original model for ellipsoidal variation. The data was generated using parameters described in Table 8.1.

In every case, except for data with no noise and no added variation, the preferred model was determined correctly. The posterior distributions were correctly determined and matched the parameters from Table 8.1. The Kane & Gelino (2012) model and the Kane & Gelino (Modified) model were difficult to distinguish, especially with increasing noise level. The Kane & Gelino (2012) model and the Kane & Gelino (Modified) models are similar, that is their residuals are small, which can be seen in Figure 3.4. Generating
<table>
<thead>
<tr>
<th>Parameter (Units)</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stellar Mass ($M_\odot$)</td>
<td>$M_S$</td>
<td>1</td>
</tr>
<tr>
<td>Stellar Effective Temperature (K)</td>
<td>$T_{Eff}$</td>
<td>6500</td>
</tr>
<tr>
<td>Orbital Period (Days)</td>
<td>$T$</td>
<td>2</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>$e$</td>
<td>0</td>
</tr>
<tr>
<td>Inclination</td>
<td>$i$</td>
<td>85°</td>
</tr>
<tr>
<td>Planetary Radius ($R_J$)</td>
<td>$R_p$</td>
<td>2</td>
</tr>
<tr>
<td>Planetary Mass ($M_J$)</td>
<td>$M_p$</td>
<td>6</td>
</tr>
<tr>
<td>Dayside Temperature (K)</td>
<td>$T_d$</td>
<td>3500</td>
</tr>
<tr>
<td>Nightside Temperature (K)</td>
<td>$T_n$</td>
<td>3000</td>
</tr>
<tr>
<td>Geometric Albedo</td>
<td>$A_g$</td>
<td>0</td>
</tr>
<tr>
<td>Limb-Darkening Coefficient</td>
<td>$u_1$</td>
<td>0.3</td>
</tr>
<tr>
<td>Limb-Darkening Coefficient</td>
<td>$u_2$</td>
<td>0.3</td>
</tr>
<tr>
<td>Gravity-Darkening Coefficient</td>
<td>$g$</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table 8.1: Table containing the values used to construct the synthetic data. This represents a super-Jupiter type planet orbiting closely to a Sun-like star in a nearly edge-on circular orbit. The geometric albedo is set to zero to eliminate reflected light.

Figure 8.3: Synthetic lightcurve data generated using the BEER model for ellipsoidal variations. Gaussian noise has been added with a maximum amplitude of 30 ppm. A plot showing the photometric variations has also been included by removing data points within the primary and secondary transits. The effect from thermal emission dominates the photometric variations. The thermal variations are of order $10^{-3}$ while the ellipsoidal variations and Doppler boosting effects are on the order of $10^{-5}$. 
Table 8.2: Ellipsoidal variation model testing for synthetic data with no noise added. The row represents the evidence value for the model. The column indicates which model was used to generate the synthetic data.

<table>
<thead>
<tr>
<th>Model</th>
<th>Simulated BEER</th>
<th>Simulated Kane &amp; Gelino (2012)</th>
<th>Simulated Kane &amp; Gelino (Mod)</th>
<th>Simulated No Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>BEER</td>
<td>1223.25 ± 1.00</td>
<td>1020.84 ± 0.89</td>
<td>1044.64 ± 0.90</td>
<td>979.25 ± 0.86</td>
</tr>
<tr>
<td>Kane &amp; Gelino (2012)</td>
<td>907.93 ± 0.86</td>
<td>1223.46 ± 1.00</td>
<td>1103.32 ± 0.93</td>
<td>1012.92 ± 0.89</td>
</tr>
<tr>
<td>Kane &amp; Gelino (Mod)</td>
<td>1001.64 ± 0.78</td>
<td>1114.35 ± 0.94</td>
<td>1223.92 ± 1.00</td>
<td>920.86 ± 0.87</td>
</tr>
<tr>
<td>No Variation</td>
<td>903.51 ± 0.81</td>
<td>991.64 ± 0.87</td>
<td>968.11 ± 0.83</td>
<td>993.01 ± 0.94</td>
</tr>
</tbody>
</table>

Table 8.3: Ellipsoidal variation model testing for synthetic data with 10 ppm Gaussian noise added. The row represents the evidence value for the model. The column indicates which model was used to generate the data.

<table>
<thead>
<tr>
<th>Model</th>
<th>Simulated BEER</th>
<th>Simulated Kane &amp; Gelino (2012)</th>
<th>Simulated Kane &amp; Gelino (Mod)</th>
<th>Simulated No Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>BEER</td>
<td>1011.74 ± 0.88</td>
<td>991.77 ± 0.87</td>
<td>999.40 ± 0.87</td>
<td>959.99 ± 0.85</td>
</tr>
<tr>
<td>Kane &amp; Gelino (2012)</td>
<td>954.63 ± 0.84</td>
<td>1015.77 ± 0.88</td>
<td>1008.63 ± 0.93</td>
<td>983.49 ± 0.86</td>
</tr>
<tr>
<td>Kane &amp; Gelino (Mod)</td>
<td>973.71 ± 0.85</td>
<td>1012.92 ± 0.54</td>
<td>1020.56 ± 0.88</td>
<td>974.48 ± 0.85</td>
</tr>
<tr>
<td>No Variation</td>
<td>899.74 ± 0.81</td>
<td>970.00 ± 0.84</td>
<td>952.96 ± 0.83</td>
<td>1015.02 ± 0.88</td>
</tr>
</tbody>
</table>

Table 8.4: Ellipsoidal variation model testing for synthetic data with 30 ppm Gaussian noise added. Each row represents the evidence value for the particular model. The column indicates which model was used to generate the data.

<table>
<thead>
<tr>
<th>Model</th>
<th>Simulated BEER</th>
<th>Simulated Kane &amp; Gelino (2012)</th>
<th>Simulated Kane &amp; Gelino (Mod)</th>
<th>Simulated No Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>BEER</td>
<td>911.96 ± 0.82</td>
<td>914.04 ± 0.82</td>
<td>908.78 ± 0.82</td>
<td>915.65 ± 0.81</td>
</tr>
<tr>
<td>Kane &amp; Gelino (2012)</td>
<td>896.94 ± 0.80</td>
<td>918.30 ± 0.81</td>
<td>913.13 ± 0.81</td>
<td>920.21 ± 0.82</td>
</tr>
<tr>
<td>Kane &amp; Gelino (Mod)</td>
<td>902.57 ± 0.81</td>
<td>918.88 ± 0.81</td>
<td>912.94 ± 0.81</td>
<td>918.69 ± 0.82</td>
</tr>
<tr>
<td>No Variation</td>
<td>875.28 ± 0.78</td>
<td>907.58 ± 0.81</td>
<td>895.50 ± 0.79</td>
<td>925.10 ± 0.81</td>
</tr>
</tbody>
</table>

8.2 Application to Kepler Data

Targeted systems were chosen because the host stars are quiet and large planets with short orbital periods have been confirmed using both radial velocity and transit methods. This provides the best conditions to observe ellipsoidal variations. Both planets have an apparent magnitude in the Kepler bandpass of around 10 mag. The estimated precision of
Kepler long cadence data is around 29 ppm. Effects smaller than \(\approx 29\) ppm are increasingly difficult to detect.

<table>
<thead>
<tr>
<th>(K_p)</th>
<th>P(ppm) Short Cadence</th>
<th>P(ppm) Long Cadence</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.0</td>
<td>39</td>
<td>7.1</td>
</tr>
<tr>
<td>8.0</td>
<td>62</td>
<td>11.3</td>
</tr>
<tr>
<td>9.0</td>
<td>99</td>
<td>18.1</td>
</tr>
<tr>
<td>10.0</td>
<td>159</td>
<td>29.0</td>
</tr>
<tr>
<td>11.0</td>
<td>259</td>
<td>47.4</td>
</tr>
<tr>
<td>12.0</td>
<td>441</td>
<td>80.5</td>
</tr>
<tr>
<td>13.0</td>
<td>804</td>
<td>147</td>
</tr>
</tbody>
</table>

Table 8.5: A table containing the estimated precision of the Kepler telescope from calibrations for stars of different magnitudes (\(K_p\)). Targeted systems that appear brighter in the Kepler field are preferred for model testing ellipsoidal variations. For the full table see [49]. Short cadence data was recorded at 1 minute intervals while long cadence data was recorded at 30 minute intervals.

EXONEST was applied to the Kepler-13 and HAT-P-7 systems. For stopping criteria the tolerance was set to 0.01. These results will be discussed in Sections 8.4 and 8.5.

### 8.3 Other Photometric Effects Modeled

In addition to ellipsoidal variations the other effects modeled are transits, thermal emission, Doppler boosting, and reflected light\(^1\).

#### 8.3.1 Transits

In EXONEST the primary transit (when the planet blocks out light from the star) is modeled using the cross-sectional areas of the planet and the star. The ratios simplify down to

\[
\delta F_P = \left(\frac{R_P}{R_S}\right)^2
\]

\(^1\)Reflected light was only modeled in eccentric orbits
where $R_P$ is the radius of the planet and $R_S$ is the radius of the star.

A secondary eclipse can occur when light from the planet is obscured by the star. This secondary transit is modeled using

$$\delta F_P = \frac{T_d}{T_{\text{eff}}} \left( \frac{R_P}{R_S} \right)^2,$$

(8.2)

where $T_d$ and $T_{\text{eff}}$ are the day-side temperature of the planet and effective temperature of the star, respectively. Mandel and Agol [50] provided analytic solutions to transit lightcurves for quadratic limb-darkening law and a non-linear limb-darkening law. Currently EXONEST implements the quadratic limb-darkening law.

### 8.3.2 Thermal Emission

Planets which are close in to their stars will likely be intensely hot, potentially in the thousands of Kelvin. The Kepler Space Telescope observes from around 420 nm to 900 nm. As such the blackbody radiation from the planet is detectable within the bandpass of Kepler. The Kepler response function shows how Kepler’s CCD sensors respond to light.

The thermal contribution is broken up between day-side and night-side components. The two components are shifted by half an orbit from each other assuming that exactly half of the planet is illuminated by the star.. The day-side contribution is given by:

$$\frac{F_{\text{Th,d}}(t)}{F_S} = \frac{1}{2} \left( 1 + \cos \theta(t) \right) \left( \frac{R_P}{R_S} \right)^2 \int B(T_d) K(\lambda) d\lambda \int B(T_{\text{eff}}) K(\lambda) d\lambda$$

(8.3)

where $B(T_d)$ is the blackbody radiation at the day-side temperature of the planet and effective temperature of the star and $K(\lambda)$ represents the wavelength-dependent Kepler response function. Similarly the night-side contribution is given by:

$$\frac{F_{\text{Th,n}}(t)}{F_S} = \frac{1}{2} \left( 1 + \cos \theta(t) - \pi \right) \left( \frac{R_P}{R_S} \right)^2 \int B(T_n) K(\lambda) d\lambda \int B(T_{\text{eff}}) K(\lambda) d\lambda$$

(8.4)
where $T_n$ is the night-side temperature of the planet.

### 8.3.3 Reflected Light

The normalized reflected light from the planet over the stellar emission is given by

$$\frac{F_P(t)}{F_S} = A_g \frac{R_P^2}{r(t)^2} \left(1 + \cos \theta(t)\right)$$

(8.5)

where $A_g$ is the geometric albedo of the planet. Typically for hot Jupiters most of the reflective material has been burnt away. As such, the geometric albedo for these planets tends to be small.
For circular orbits the reflected light and thermal light cannot be distinguished. Both effects have a period equal to the orbital period of the planet and are maximum at the new phase. Trying to model both effects simultaneously leads to degeneracies in the data. To prevent unnecessary degeneracies reflected light was not included for circular orbits. The planetary emission component of the lightcurve is modeled entirely as thermal emission. Placek showed that thermal emission is more significant than reflected light in the Kepler-13 lightcurve [4] [6].

8.3.4 Doppler Boosting

The star and planet(s) orbit the barycenter of the system. This leads to a photometric variation rising from a relativistic effect. As the star moves toward the observer there is an increase in observed flux. As the star recedes from the observer the observed flux is reduced.

The radial velocity (along $\hat{z}$) may be represented using parameters defined in Section 2.2

$$V_z(t) = K \left( \cos (\nu(t) + \omega) + e \cos \omega \right)$$  \hspace{1cm} (8.6)

where $K$ is defined as the radial velocity semi-amplitude

$$K = \left( \frac{2\pi G}{T} \right)^{1/3} \frac{m_P \sin i}{m_S^{2/3} \sqrt{1-e^2}}$$  \hspace{1cm} (8.7)

The normalized line of sight velocity, $\beta_r(t)$, for the star is

$$\beta_r(t) = \frac{V_z(t)}{c}.$$  \hspace{1cm} (8.8)

The effect is modeled using the following
\[
\frac{F_{\text{boost}}(t)}{F_S} = 4\beta_r(t). \tag{8.9}
\]

## 8.4 Circular Orbits

### 8.4.1 Kepler-13

The Kepler-13 system contains a hot Jupiter-like exoplanet orbiting an A-type star. The host star is about 2.4 M\(_\odot\) and about three times the radius of the Sun. Within the Kepler field the star has an apparent magnitude in the Kepler bandpass of 9.958 mag. The companion planet has an orbital period of 1.76 days and was detected by Shporer et al. [25] through only photometric variations using the BEER algorithm.

<table>
<thead>
<tr>
<th>Parameter (Units)</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stellar Mass (M(_\odot))</td>
<td>M(_\odot)</td>
<td>2.466(^{+0.047}_{-0.025})</td>
</tr>
<tr>
<td>Stellar Effective Temperature (K)</td>
<td>(T_{\text{Eff}})</td>
<td>9107(^{+25}_{-42})</td>
</tr>
<tr>
<td>Orbital Period (Days)</td>
<td>T</td>
<td>1.763587569 ±3.2E-08</td>
</tr>
<tr>
<td>Metalicity (dex)</td>
<td>—</td>
<td>0.070(^{+0.140}_{-0.065})</td>
</tr>
<tr>
<td>Limb-Darkening Coefficient</td>
<td>(u_1)</td>
<td>0.2278</td>
</tr>
<tr>
<td>Limb-Darkening Coefficient</td>
<td>(u_2)</td>
<td>0.2884</td>
</tr>
<tr>
<td>Gravity-Darkening Coefficient</td>
<td>g</td>
<td>0.5319</td>
</tr>
</tbody>
</table>

Table 8.6: Table containing the values assumed to be known in the trials for the Kepler-13 system. Values are recorded from the NASA Exoplanet Archive. The limb-darkening coefficients were determined by interpolation from coefficients computed by Claret and Bloemen [44] (see 7.1).

EXONEST was run using each model for ellipsoidal variations using 75 samples within the nested sampling algorithm. For the Kepler-13 system EVIL-MC provided the best evidence with a LogZ = 14.257.46 ± 0.74. The trigonometric models BEER and Kane & Gelino (Modified) models were nearly indistinguishable to EVIL-MC with Bayes’ factors \(e^{1.58}\) and \(e^{1.16}\). All three models were slightly more preferred over the Kane & Gelino (2012) model with a Bayes’ Factor of around \(e^{3.1\text{-}e^{4.7}}\). This is not significant enough to conclude
that one model should be preferred over any other given the noise within the dataset. Postprocessing may remove longterm variations in the data. This could be done using a polynomial fit [25] or the PyKE detrending algorithm [52]. All models for ellipsoidal variations were significantly preferred over a model containing no variation. Therefore ellipsoidal variations are detectable within the Kepler-13 system.

All the models agreed on orbital parameters other than the mass of the planet. The BEER model estimated the planet mass at around 1 M$_J$. This is significantly less than the masses estimated by the other models (see Table 8.5.1). The mean value of the variation under the BEER model is 0. For both Kane & Gelino models the effect is always greater than or equal to zero. This means that the BEER model only requires a planet of around half the mass of the Kane & Gelino models to represent the same amplitude (See Figure 3.4). This is confirmed in the posteriors. Additionally the planetary masses are underestimated by roughly a factor of two. By normalizing the observed flux by the average stellar flux of two A-type stars we are estimating only about half of the total ellipsoidal variations present. Accounting for this factor produces similar planetary mass estimates of 4.94-8.09 M$_J$ determined by Shporer et al. [53] but less than the 14.8 M$_J$ or 9.4 M$_J$ estimated by Saterne et al. [54] (depending on which star the planet is orbiting).

The modified Kane & Gelino model provided a posterior distribution nearly identical to the distribution generated using EVIL-MC. The posteriors for each model may be seen in Figures 8.4.1-8.4.1. The corner plot shows the distribution of the posterior and will reveal any anomalies or correlations between parameters. The posteriors are generally well-behaved for Kepler-13 in all of the models. The planetary mass has proportionately larger uncertainties than the other parameters. This is likely due to the noise within the dataset blurring some of the photometric variations. An additional factor to be considered in model selection is the computational speed of the model. The modified Kane & Gelino model (and the other trigonometric models) will complete a full nested sampling run on the order of minutes to an hour. EVIL-MC took 7.80 days to complete. The Modified Kane &
Gelino model is numerically most similar to EVIL-MC. As a result, the Modified Kane & Gelino model may provide a computationally fast approximation for EVIL-MC.

Figure 8.5: Plot displaying the evidence values for five runs of each of the trigonometric models. A control model where the ellipsoidal variation was set to zero is also displayed. EVIL-MC performed the best and is slightly more preferred over the modified Kane & Gelino model and BEER model. The modified Kane & Gelino model performed the best of the trigonometric models but was nearly indistinguishable from the BEER model. Both were slightly preferred over Kane & Gelino (2012). All the ellipsoidal variation models were significantly preferred over no variation which indicates ellipsoidal variations are present in Kepler-13 data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BEER</th>
<th>Kane &amp; Gelino (2012)</th>
<th>Kane &amp; Gelino (Modified)</th>
<th>EVIL-MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>LogZ</td>
<td>14,256.29 ± 0.74</td>
<td>14,253.66 ± 0.72</td>
<td>14,256.71 ± 0.73</td>
<td>14,257.46 ± 0.74</td>
</tr>
<tr>
<td>cos I</td>
<td>0.30386 ± 1.6E-4</td>
<td>0.30390 ± 1.6E-4</td>
<td>0.30386 ± 1.7E-4</td>
<td>0.30384 ± 1.6E-4</td>
</tr>
<tr>
<td>Mean Anomaly</td>
<td>3.54352 ± 2.0E-4</td>
<td>3.54351 ± 2.0E-4</td>
<td>3.54348 ± 2.0E-4</td>
<td>3.54351 ± 2.0E-4</td>
</tr>
<tr>
<td>Planet Radius (R_J)</td>
<td>2.1111 ± 2.2E-3</td>
<td>2.1100 ± 2.1E-3</td>
<td>2.1111 ± 2.1E-3</td>
<td>2.1113 ± 2.0E-3</td>
</tr>
<tr>
<td>Planet Mass (M_J)</td>
<td>0.91 ± 0.11</td>
<td>2.32 ± 0.31</td>
<td>1.66 ± 0.23</td>
<td>1.67 ± 0.20</td>
</tr>
<tr>
<td>Dayside Temp. (K)</td>
<td>3696 ± 61</td>
<td>3661 ± 64</td>
<td>3695 ± 58</td>
<td>3712 ± 51</td>
</tr>
<tr>
<td>Nightside Temp (K)</td>
<td>2973 ± 337</td>
<td>2710 ± 64</td>
<td>2969 ± 393</td>
<td>3042 ± 286</td>
</tr>
</tbody>
</table>

Table 8.7: Table displaying posterior values of the Kepler-13 system in a circular orbit for the parameters assigned by each of the ellipsoidal variation models using runs providing the best evidence.
8.4.2 HAT-P-7/Kepler-2

HAT-P-7b, also known as Kepler-2b, was discovered in 2008 [55], early in the Kepler mission. The star has an apparent magnitude of 10.463 mag within the Kepler bandpass.
Figure 8.8: Corner plot displaying the posterior of Kepler-13 in a circular orbit using the Kane & Gelino (Modified) model for ellipsoidal variations.

Figure 8.9: Corner plot displaying the posterior of Kepler-13 in a circular orbit using the EVIL-MC model for ellipsoidal variations.

The system includes a Jupiter-sized planet, in a near circular orbit, transiting with a period of 2.205 days. The F-type host star has a mass of 1.52 M$_\odot$ and a radius of 1.991 R$_\odot$. HAT-P-7b is a well studied system and may provide an additional system to determine the preferred model.
<table>
<thead>
<tr>
<th>Parameter (Units)</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stellar Mass ($M_\odot$)</td>
<td>$M_S$</td>
<td>$1.52 \pm 0.036$</td>
</tr>
<tr>
<td>Stellar Effective Temperature (K)</td>
<td>$T_{\text{Eff}}$</td>
<td>$6350 \pm 80$</td>
</tr>
<tr>
<td>Orbital Period (Days)</td>
<td>$T$</td>
<td>$2.204735365 \pm 3.8E-08$</td>
</tr>
<tr>
<td>Metallicity (dex)</td>
<td>——</td>
<td>$0.260 \pm 0.080$</td>
</tr>
<tr>
<td>Limb-Darkening Coefficient</td>
<td>$u_1$</td>
<td>$0.3181$</td>
</tr>
<tr>
<td>Limb-Darkening Coefficient</td>
<td>$u_2$</td>
<td>$0.3120$</td>
</tr>
<tr>
<td>Gravity-Darkening Coefficient</td>
<td>$g$</td>
<td>$0.5510$</td>
</tr>
</tbody>
</table>

Table 8.8: Table containing the values assumed to be known in the trials for HAT-P-7b. Values are recorded from the NASA Exoplanet Archive. The limb-darkening and gravity-darkening coefficients were determined by interpolation from coefficients computed by Claret and Bloemen [44] (see 7.1).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BEER</th>
<th>Kane &amp; Gelino (2012)</th>
<th>Kane &amp; Gelino (Modified)</th>
<th>EVIL-MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>LogZ</td>
<td>$31.73178 \pm 0.72$</td>
<td>$31.73114 \pm 0.71$</td>
<td>$31.73267 \pm 0.71$</td>
<td>$31.73231 \pm 0.72$</td>
</tr>
<tr>
<td>cosI</td>
<td>$0.12578 \pm 2.2E-4$</td>
<td>$0.12577 \pm 2.4E-4$</td>
<td>$0.12579 \pm 2.3E-4$</td>
<td>$0.12578 \pm 2.4E-4$</td>
</tr>
<tr>
<td>Mean Anomaly</td>
<td>$4.97164 \pm 1.2E-4$</td>
<td>$4.97163 \pm 1.4E-4$</td>
<td>$4.97161 \pm 1.4E-4$</td>
<td>$4.97162 \pm 1.3E-4$</td>
</tr>
<tr>
<td>Planet Radius ($R_J$)</td>
<td>$1.54051.2E-3$</td>
<td>$1.5405 \pm 1.2E-3$</td>
<td>$1.5405 \pm 1.2E-3$</td>
<td>$1.5404 \pm 1.1E-3$</td>
</tr>
<tr>
<td>Planet Mass ($M_J$)</td>
<td>$0.50 \pm 0.26$</td>
<td>$0.73 \pm 0.46$</td>
<td>$0.91 \pm 0.47$</td>
<td>$0.89 \pm 0.41$</td>
</tr>
<tr>
<td>Dayside Temp. (K)</td>
<td>$2897 \pm 36$</td>
<td>$2901 \pm 34$</td>
<td>$2898 \pm 39.0$</td>
<td>$2894 \pm 36$</td>
</tr>
<tr>
<td>Nightside Temp (K)</td>
<td>$1385 \pm 746$</td>
<td>$1363 \pm 722$</td>
<td>$1382 \pm 759$</td>
<td>$1328 \pm 728$</td>
</tr>
</tbody>
</table>

Table 8.9: Table displaying the posterior values of the HAT-P-7b system in a circular orbit for the parameters assigned by each of the ellipsoidal variation models. The posterior values were selected from the run with the highest evidence in the set. This represents the best parameters as selected by a particular model.

For HAT-P-7 none of the models for ellipsoidal variations are preferred. Since the Bayes’ Factor for comparison between any of the models would be nearly unity the models are completely indistinguishable. The evidences for each run are plotted in Figure 8.10 and listed in Table 8.9. The models are indistinguishable likely due to HAT-P-7b being a smaller planet in a larger orbit than Kepler-13b. The noise level within the dataset will have a larger effect on the total photometric variation when compared to Kepler-13 and is enough to wash out any distinguishing characteristics between the models. This may also be seen in the posterior distributions. For each model the uncertainty in the planet’s mass was up to 50% of the value. The ellipsoidal variations were not a significant portion of the photometric data and therefore were not able to provide a proper estimation of the planet
mass. The posteriors for the best trials of each model are provided in corner plot form in Figures 8.11 - 8.14. The nightside temperatures and the planetary masses were not well constrained. The ranges in the values for these parameters in the posteriors clearly show this in the corner plots.

Figure 8.10: Plot displaying the evidence values for five runs of each of the trigonometric models. The Modified Kane & Gelino model is slightly preferred over the other models however the differences are within the uncertainties. The noise within this dataset does not permit a preferred model to be determined. Noise reduction techniques will need to be addressed before a preferred model can be determined.

8.5 Eccentric Orbits

While tidal interactions may cause most eccentric orbits to slowly become circularized, there is nothing precluding the observation of eccentric orbits. The model testing was extended to include the possibility of eccentric orbits. By considering eccentric orbits, we may break the degeneracy between reflected light and thermal emission. For eccentric orbits the prior is extended to include the argument of periastron ($\omega$), eccentricity ($e$), and the geometric albedo.
8.5.1 Kepler-13

The Kepler-13 system may have an eccentric orbit [4] [6]. The evidence for eccentric orbits is significantly more preferred over circular orbits by a Bayes’ Factor of around $e^{73}$. For
eccentric orbits the Modified Kane & Gelino model was the most preferred over BEER and Kane & Gelino (2012) by a factor of around $e^3$. All trigonometric models are significantly more preferred over no ellipsoidal variations by a Bayes’ Factor of $\approx e^{25}$. The temperatures of the planet’s dayside and nightside are not well estimated by any of the models.
The nightside of the planet should not be hotter than the dayside. For the case of the Kane & Gelino (2012) model the dayside and nightside are roughly equal. This could be explained through methods of heat transfer. A near equilibrium between the two sides of a tidally-locked planet could be explained by strong planetary winds creating a method for heat transfer. Winds between 2-5 km/s have been observed on planet HD 189733b [56]. It is likely that with the small eccentricity found the reflected light and thermal emissions are not distinguishable. Placek estimates that an eccentricity of around 0.3 would be required to distinguish reflected light and thermal emission [6]. The geometric albedo generated by each of the models is approximately equal to that of the moon with a geometric albedo of 0.12 [57].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BEER</th>
<th>Kane &amp; Gelino (2012)</th>
<th>Kane &amp; Gelino (Modified)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LogZ</td>
<td>14.326.15 ± 0.99</td>
<td>14,325.90 ± 0.90</td>
<td><strong>14,329.50 ± 0.95</strong></td>
</tr>
<tr>
<td>cosI</td>
<td>0.31280 ± 6.1E-4</td>
<td>0.31286 ± 5.6E-4</td>
<td>0.31275 ± 5.7E-4</td>
</tr>
<tr>
<td>Mean Anomaly</td>
<td>5.07567 ± 0.03656</td>
<td>5.08609 ± 0.03145</td>
<td>5.07487 ± 0.04110</td>
</tr>
<tr>
<td>ω</td>
<td>4.75566 ± 0.04076</td>
<td>4.74404 ± 0.03509</td>
<td>4.75646 ± 0.04578</td>
</tr>
<tr>
<td>e</td>
<td>0.05148 ± 0.003501</td>
<td>0.05167 ± 0.00316</td>
<td>0.05112 ± 0.00321</td>
</tr>
<tr>
<td>Planet Radius (R_J)</td>
<td>2.08760 ± 0.00229</td>
<td>2.08597 ± 0.00316</td>
<td>2.08795 ± 0.00220</td>
</tr>
<tr>
<td>Planet Mass (M_J)</td>
<td>0.81 ± 0.10</td>
<td>2.19 ± 0.29</td>
<td>1.46 ± 0.18</td>
</tr>
<tr>
<td>Dayside Temp. (K)</td>
<td>1303 ± 797</td>
<td>1561 ± 861</td>
<td>1653 ± 970</td>
</tr>
<tr>
<td>Nightside Temp. (K)</td>
<td>3139 ± 110</td>
<td>2180 ± 880</td>
<td>3018 ± 168</td>
</tr>
<tr>
<td>Albedo</td>
<td>0.183 ± 0.013</td>
<td>0.167 ± 0.018</td>
<td>0.175 ± 0.024</td>
</tr>
</tbody>
</table>

Table 8.10: Table displaying the posterior values of the Kepler-13 system considering an eccentric orbit. The columns represent each of the ellipsoidal variation models. The posterior values were selected from the run with the highest evidence in the set. This represents the mean parameters as selected by a particular model.
Chapter 9

Future Work

9.1 Additional Models

Ellipsoidal variations are modeled in binary star systems. One of the traditionally used models is the Wilson-Devinney (WD) code [58] which is currently in use with PHysics Of Eclipsing BinariEs (PHOEBE). The WD model is written in FORTRAN so some modifications may be required for compatibility with EXONEST. The model has had relatively frequent updates since 1971, the most recent occurring in 2007. WD works by following the classical Roche model of binaries in synchronous rotation. This should produce a result similar to EVIL-MC but may be more accurate for low-mass stars. Similarly to EVIL-MC, the WD code can account for stellar rotation and directly model limb-darkening and gravity darkening.

For circular orbits the flux from reflected light and thermal emission can not be disentangled. In general, for hot Jupiters, the thermal emission is usually the dominant component however this need not be the case. Reflected light could be modeled instead of thermal emission. Additionally both the reflected light and thermal emission may be combined together into one effect which represents the total flux from the planet. The will prevent prediction of temperature and albedo for the planet but still model both effects. It
is likely better to model a single effect (either reflected light or thermal emission) and use the estimated value as a maximum estimated value. The true value is likely to be less.

9.2 Model Averaging

A potential alternative to selecting a specific preferred model is to use model averaging. Classification of the system is the primary end goal. Determination of the posterior is dependent on the model. Instead of relying on a particular model to determine the posteriors many models may be used with a weighted average.

Suppose that there are $K$ models ($M_1, ..., M_K$) each with their respective prior probabilities $P(M_k | I)$ [59]. Additionally suppose $P(\theta_k | M_k, I)$ is given. The posterior for a desired quantity $\omega$ given data $d$ and prior information $I$ is given by

$$P(\omega | d, I) = \sum_{k=1}^{K} P(\omega | d, M_k, I) P(M_k | d, I).$$

(9.1)

The posterior for $\omega$ using the $k^{th}$ model is $P(\omega | d, M_k, I)$. The posterior model probability is accounted for using the weighted average

$$P(M_k | d, I) = \frac{P(d | M_k, I) P(M_k | I)}{\sum_k P(d | M_k, I) P(M_k | I)}$$

(9.2)

where

$$p(d | M_k, I) = \int P(d | \theta, M_k, I) P(\theta | M_k, I) d\theta.$$  

(9.3)

The result is a posterior probability which may be dependent on many different models. If one particular model is significantly more preferred over the others than the posterior will be strongly based on the values output by that model. If instead no model is preferred then the resultant posterior will be more evenly determined by contributions from all of the models.
The extra computation required for this method will probably not provide a significantly improved posterior estimation to warrant the extra effort. In the case of three possible trigonometric models to describe ellipsoidal variation the amount of computation required to determine the posterior and evidences would be roughly tripled. Additionally the model averaging would be better suited to cases where the models are more direct representations of reality. In the case with ellipsoidal variation the only model which has been closely connected to physical phenomena is EVIL-MC. The trigonometric models are simple representations of the variations and are designed by closely matching the expectation value of the effect at the location of maximum ellipsoidal variation. The effect between maxima is ad hoc and therefore averaging over the models would not be providing a better estimation.

9.3 Limb-darkening and Gravity-Darkening Coefficient

Priors

The limb-darkening and gravity coefficients in this paper are assumed to be known. They have been extrapolated from quantities with uncertainties like stellar effective temperature, stellar radius, and the surface gravity. These quantities are inherently uncertain. The prior may be extended to account for other limb-darkening and gravity-darkening coefficients. The could be implemented by inserting a search radius around the coefficients determined by the stellar parameters and can be accounted for within the uncertainty. Additionally this extension will allow for comparison of limb-darkening models containing different numbers of parameters (coefficients). A comparison of the different limb-darkening models may provide insight on a preferred representation for limb-darkening. More parameters may not be more preferred. Spreading out the prior over more parameters will create the need for the model to describe the data significantly better than models with fewer parameters in order to be preferred. The evidence in the Bayesian framework already takes this into
EVIL-MC is compatible with any desired limb-darkening model. Model testing can be extended to limb-darkening models which must include modeling both ellipsoidal variations (with EVIL-MC) and transits. A study by Heyrovský [21] showed that the 4-parameter model fit the PHOENIX atmosphere model with a maximum uncertainty of about half those from the other models [21]. However, the difference may not be detectable in Kepler data or in the proposed future missions.

9.4 Other Effects to Model

9.4.1 Bulge Lag

The tidal distortion on the star may not point directly at the planet. In the trigonometric models the maximum amplitude of the ellipsoidal variations occurs when the planet would be observed in quarter or three-quarters orbital phase. Additionally, EVIL-MC treats the system as a series of static snapshots. There is no reference to a previous state. Instead, the “bulge” could lag behind the planet due to resistances to change on the stellar surface. This may be accounted for with an initially ad hoc parameter known as the phase shift. This extra parameter could explored using the nested sampling algorithm. Additional resources would need to be invested to connect the magnitude of the phase offset to stellar intrinsic parameters. A phase shift parameter could be added to the trigonometric models. For EVIL-MC, a rotation of the star on an axis perpendicular to the orbital plane could provide the desired effect.

9.4.2 Stars Are Not Blackbodies

To first order the flux from a star appears as a blackbody however their true nature is more complicated. Absorption lines, wavelengths where the stellar atmosphere is opaque, is a
function of density, composition, and temperature [60]. A simple comparison of different spectra can be seen in Figure 9.4.2.

Figure 9.1: “Digitized spectra of main sequence classes O5-F0 displayed in terms of relative flux as a function of wavelength.” [60] Data is from [61]

Absorption at specific wavelengths are mostly due to four interactions between photons and baryons: bound-bound transitions, bound-free absorptions, free-free absorption, and
electron scattering.

Bound-bound transitions occur when an electron jumps from one energy level to another. Absorption occurs when a photon bumps an electron from a lower energy level into a higher energy level. Transmission may also occur if an electron jumps from an excited state into a lower energy state. An absorbed photon will cause the electron to jump from a lower to higher energy level. If the electron returns back to its original energy level then the same wavelength photon will be emitted in a random direction. The result will be a scattered photon. If the electron returns to a different energy level then a photon with a different wavelength will be emitted in a random direction. Additionally, if the atom collides with another atom collisional de-exitation may occur. Some energy will be transferred into thermal variations in the gas. Either situation will cause a net reduction in average energy of the photon.

Bound-free absorption, or photoionization, occurs when a photon of sufficient energy is absorbed to strip an electron from the atom and creating an ion. Recombination can occur creating one or more photons which causes a net reduction in average photon energy.

Free-free absorption, also known as bremsstrahlung or “breaking radiation,” occurs when a free electron within the vicinity of an ion absorbs an incoming photon.

Electron scattering is most significant when the medium contains a high density of free electrons. This is the most significant source of absorption lines in hot stars [60]. Thompson scattering is when the electron oscillates with the electromagnetic field of the photon. A similar effect occurs when a photon is scattered by an electron which is loosely bounded by the nucleus of an atom. If the wavelength is much smaller than the atom the process is called Compton scattering. If the wavelength is much larger then the interaction is called Rayleigh scattering. Rayleigh scattering is assumed to be elastic scattering while Compton scattering is inelastic scattering.

Modeling every possible interaction or scattering process would be computationally too complicated and intensive but identifying the most important processes for the visible spec-
trum may provide a simpler solution. Additionally, the Rosseland mean opacity prioritizes
the most significant forms of opacity in the spectrum. The weighing function depends on
the Planck function or change in the blackbody function with respect to temperature [60].
The Rosseland mean opacity is given by

\[
\bar{\kappa} = \frac{\int_0^\infty \frac{1}{\kappa} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(t)}{\partial T} d\nu},
\]

(9.4)

where \( \kappa \) represents the opacity and \( B(T) \) is the blackbody function at temperature \( T \).

There are too many interactions to simply express the absorption due to bound-bound
transitions however approximations have been developed to describe bound-free and free-
free interactions,

\[
\kappa_{bf} = 4.34 \times 10^{21} \frac{g_{bf}}{t} Z(1 + X) \rho \frac{m^2 kg^{-1}}{T^{3.5}}
\]

(9.5)

and

\[
\kappa_{ff} = 3.68 \times 10^{18} g_{ff}(1 - Z)(1 + X) \rho \frac{m^2 kg^{-1}}{T^{3.5}}
\]

(9.6)

where \( \rho \) is the density, \( T \) is temperature, \( X \) and \( Z \) are the mass fractional abundances
of hydrogen and metals respectively. The terms \( g_{bf} \) and \( g_{ff} \) are called Gaunt factors. They
are quantum mechanical correction factors however for the visible and ultraviolet spectrum
are approximately equal to one. The quantity \( t \) in the bound-free equation corresponds to
a “guillotine factor.” “[This factor] describes the cutoff of an atom’s contribution to the
opacity after it has been ionized. Typical values of \( T \) lie between 1 and 100.” [60]

The stellar spectra is further complicated with effects that broaden spectral lines.
Natural Broadening

Natural broadening is due to the Heisenberg uncertainty principle. When an electron occupies an orbital level it does so for a finite amount of time. Therefore there is some uncertainty in the value of the energy level which is connected to the estimated time spent in the level. The uncertainty principle states

\[ \Delta E \approx \frac{\hbar}{\Delta t} \]  

(9.7)

where \( \hbar \) is the reduced Planck constant. This can be converted into a wavelength and expanded to estimate the uncertainty in the wavelength of the photons

\[ \Delta \lambda \approx \frac{\lambda^2}{2\pi c} \left( \frac{1}{\Delta t_i} + \frac{1}{\Delta t_f} \right). \]  

(9.8)

The estimated time until transition of an electron in first and second excited states of hydrogen is about \( \Delta t = 10^{-8} \) s. Therefore the estimated width of the Hydrogen-alpha (H-\( \alpha \)) line due to natural broadening is about 4.57E-5 nm. Natural broadening will likely not have much of an effect on the total flux from the star.

Pressure Broadening

Pressure broadening or collisional broadening is caused by the perturbations of energy levels due to collisions with atoms (collisional) or interactions with the electric field of ions (pressure). In general, both effects are encompassed under pressure broadening. Because of all the possible types of interactions between different atoms and ions directly modeling the effect is nearly impossible. The order of magnitude of the effect can be estimated by estimating the duration between collisions. Usually the estimation is done by dividing the mean free path of the atoms by the post probable speed from a Boltzmann distribution. Therefore the estimated time between collisions is given by
\[ \delta t_o \approx \frac{\ell}{\nu} = \frac{1}{n\sigma \sqrt{2kT/m}} \] (9.9)

which depends on the collision cross-section, \( \sigma \), and the number density of the atoms, \( n \). The estimated width of the spectral line is then given by

\[ \Delta \lambda = \frac{\lambda^2}{c} \frac{1}{\pi \Delta t_o} \approx \frac{\lambda^2}{c} \frac{n\sigma}{\pi} \sqrt{\frac{2kT}{m}}. \] (9.10)

This effect is most significant in smaller stars where the number density is larger. The number density of hydrogen in the Sun is approximately 1.5E23 \( m^{-3} \). This corresponds to a width in the H-\( \alpha \) line of approximately 2.36E-5 nm. The effect can be up to an order of magnitude larger in stars with denser atmospheres [60]. This effect will be most significant in small dense stars like K or M-dwarfs.

**Doppler Broadening**

Doppler broadening is likely the most significant source of spectral line broadening in stars. The atoms in the gas move in random directions with speeds described by the Maxwell-Boltzmann distribution

\[ n\nu d\nu = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-m\nu^2/2kT} 4\pi \nu^2 d\nu \] (9.11)

where the most probable velocity is given by

\[ \nu_{mp} = \sqrt{\frac{2kT}{m}}. \] (9.12)

Combining the Doppler shift and the most probable speed of the atom, the expected shift in wavelength of the spectral lines is roughly

\[ \frac{\Delta \lambda}{\lambda} \approx \pm |\nu/c| \rightarrow \Delta \lambda \approx \frac{2\lambda}{c} \sqrt{\frac{2kT}{m}}. \] (9.13)

67
For the Sun with an effective temperature of 5777 K, the Hydrogen-alpha (H-α) line at 656.3 nm has an estimated width of around 0.0427 nm. For the Sodium (Na - D₁) line the estimated width is 0.00804 nm. This approximation assumes the system is in thermal equilibrium however stellar atmospheres are constantly changing. To account for the turbulent velocities the full width half max of the line can be approximated with

\[ (\Delta \lambda)_{1/2} \approx \frac{2\lambda}{c} \sqrt{\left(\frac{2kT}{m} + \nu_{\text{turb}}\right) \ln 2}. \]  

(9.14)

The quantity \( \nu_{\text{turb}} \) represents the most probable turbulent velocity for the atoms on top of the thermal effects [60].

The magnitude of this effect can be approximated using

\[ \text{Spectral Line Flux} = (F_{\text{BB}}(\lambda_{\text{SL}}))(K(\lambda))(\delta\lambda) \]  

(9.15)

where \( F_{\text{BB}}(\lambda_{\text{SL}}) \) is the blackbody flux at the wavelength of the spectral line, \( K \) is the value of the Kepler Response Function at the wavelength of the spectral line, and \( \delta\lambda \) is the width of the spectral line as computed in Equations 9.10 or 9.13.

To determine how significant the spectral line is compared to the total flux we use

\[ \text{Fraction Removed} = \frac{\text{Spectral Line Flux}}{\text{Total Flux}}. \]  

(9.16)

For a Sun-like star the Doppler broadening of the H-α line corresponds to a 0.01% difference in the observed flux. In proportion to the total flux this is a 1.2E-4 difference which is an order of magnitude greater than the estimated Kepler noise level. For the Sodium-D₁ line the change in observed flux is about 2.9E-5. This is the same order of magnitude as the estimated Kepler noise level. This effect may not be necessary to include in transit data because the spectral lines will not change significantly during the orbital motion. The effect may be approximated by adding a constant to the total estimated flux.
from the star, simply translating the net flux from the star up or down. However, the EVIL-MC model may be used to generate the shape of a star under tidal interactions with a companion exoplanet. Adding spectral lines to the outward flux from the star will be able to generate a realistic model of ellipsoidal stars. This may be beneficial if examining the spectra of a transiting exoplanet’s atmosphere using the James Webb Space Telescope [13].

9.5 Computational Efficiency

Model testing in the Bayesian framework using the evidence does not take into account the effort required to complete a model. Currently the trigonometric models are quick to compute, each taking only milliseconds to run. EVIL-MC is a much more computational intense model and takes at least a few thousand times longer to run (on the order of seconds). Over many tens of thousands to hundreds of thousands of iterations the difference in computation time becomes significant, taking weeks instead of hours to complete. This corresponds to significant difficulty when using a search algorithm like nested sampling. When new samples are generated they are often not an improvement over the lowest likelihood sample from the previous group. The new sample is rejected and the computation time spent on that sample has been wasted. It is not uncommon for 2-5 samples to be rejected before a new sample is found and sometimes takes significantly many more attempts to produce a new sample. A run of EXONEST using one of the trigonometric models for a circular orbit can take a few minutes to an hour to run on most datasets depending on the amount of data and number of samples used to explore the space. The larger the datafile and the more samples used, the longer it takes to run the algorithm. For HAT-P-7 the EVIL-MC model took 17.25 days to explore the parameter space, determine the posterior, and calculate the evidence. EVIL-MC may not produce a significantly better estimation of system parameters to warrant the extra effort.

In order to create a more usable model EVIL-MC must be sped up by at least an order
of magnitude. Computation time may be significantly improved by translating EXONEST and EVIL-MC into a faster machine-native coding language. Python and C++ are potential options with the former being a strong candidate for the universal use within the scientific community and having MULTINEST already available in the language. Additionally, it is possible to compile the EVIL-MC code into C and calling the function through MATLAB. This will benefit from the structure of MATLAB while having the speed of a simpler language.

Many of the functions, primarily the computation of the flux emitted from a section of the star, will benefit from parallelization. Initial tests did not show a significant improvement on speed but the optimization can be improved.

Additionally more efficient search algorithms should be explored. Efficient sampling will significantly reduce computation time.
Chapter 10

CONCLUSIONS

Precise modeling of photometric variations within transit data will become increasingly more important with the launch of future missions like CHEOPS and PLATO. One of the photometric variations which is significant for large, close-in exoplanets is ellipsoidal variations. Gravitational interactions between the stellar atmosphere and the planet cause the star to distort from a uniform spherical shape. Rotation will also cause the star to deviate from a spherical shape. In general ellipsoidal variations will have two maxima per orbit of the planet, each located when the largest cross-sectional area of the star is being observed. This will occur roughly when the planet is at half and three-quarters phase. Ellipsoidal variations in photometric data are described using many different models [1] [2] [3] [4]. Model testing in the Bayesian framework uses the evidence as a representation of how well the data is represented by the model. The evidence is computed by marginalizing over all model parameters. Taking the ratio of the evidences, or subtracting the logarithm of the evidences, provides the Bayes’ Factor or preference of one model over the other.

The Bayesian-based exoplanet detecting and characterizing algorithm EXONEST was used to evaluate the evidence and determine posterior distributions using each model for ellipsoidal variations. The modified nested sampling search algorithm MULTINEST was used as a framework within EXONEST [14] [15] [16]. MULTINEST provides a search
algorithm which may efficiently sample high dimensional and multi-modal spaces while computing the evidence.

The models were compared using residuals by determining the full photometric effect of ellipsoidal variations for each model for planet masses between [0-15] M\textsubscript{J} and orbital periods between [2 and 8] days. The difference between each model was computed at each orbital phase. The largest difference between the models was determined to occur between the BEER and EVIL-MC models. The differences between BEER and EVIL-MC should be detectable in exoplanets with orbital periods shorter than 2 days within Kepler data. The differences should be noticeable below orbital periods of \( \approx 6 \) days. The differences between BEER and the other trigonometric models is more significant for larger planets. The models may be detectable in optimum Kepler data with large planets (>5 M\textsubscript{J}) with short orbital periods (<3 days).

Synthetic data was created using each of the different models for ellipsoidal variations, including a null model where the ellipsoidal variations were set to zero. The data was generated at levels of Gaussian noise ranging from 0 ppm to 30 ppm. In nearly every case the model used to generate the data was identified by a Bayes' Factor much larger than 1. The Kane & Gelino models were indistinguishable at the 30 ppm noise level but both models were significantly preferred over the BEER model and the model with no ellipsoidal variations. The noise in the Kepler dataset is estimated to be around 29 ppm for stars with apparent magnitudes of around 10 mag.

The model testing was extended to two confirmed hot Jupiter exoplanets, HAT-P-7 (Kepler-2) and Kepler-13. EXONEST was run using each of the models and using circular and eccentric orbits. Beyond ellipsoidal variations, EXONEST also models transits, reflected light and thermal emission from the planet, and Doppler boosting on the star.

The noise level in the HAT-P-7 dataset did not permit distinguishing the models in either circular or eccentric orbits. The evidence computed for all five models were within the uncertainty of each other.
EXONEST determined that an eccentric orbit, albeit with a small eccentricity of around $e \approx 0.05$, was preferred over a circular orbit for Kepler-13 with a Bayes Factor of around $e^{73}$. The Modified Kane & Gelino model was the most preferred trigonometric model by a factor of around $\approx e^3$. All trigonometric models were preferred over no ellipsoidal variations by a factor of $\approx e^{25}$.

Similarly for circular orbits for Kepler-13, EXONEST determined that EVIL-MC was the most preferred model by a factor of around $e^{1.1}$ and $e^{1.6}$ over the Modified Kane & Gelino model and BEER model respectively. The Modified Kane & Gelino model was preferred over the Kane & Gelino (2012) model with a Bayes Factor $\approx e^{3.4}$. Also, all three trigonometric models were significantly preferred over no variations by a Bayes’ factor of around $e^{25}$. This shows that ellipsoidal variations are a significant component of the observed flux from the Kepler-13 system.

The most significant difference between the models is the effect the planet mass has on the amplitude. The BEER model would only require a planet roughly half the mass as either the Kane & Gelino (2012) or Modified Kane & Gelino models to provide the same amplitude of the effect. Selection of the best model is critical in determining the planetary mass.

The synthetic data showed that the model used to construct the data was identifiable using the Bayesian evidence but had difficulty distinguishing the Kane & Gelino (2012) model and the Modified Kane & Gelino model at higher noise levels. This combined with the results from Kepler-13 showing that the BEER and Modified Kane & Gelino models were both slightly better represented than Kane & Gelino (2012) in the lightcurve shows that neither of the trigonometric models exactly represent the effect. It is much more likely that the effect is more similar to EVIL-MC which is indicated in the slightly higher evidence for EVIL-MC in the Kepler-13 circular orbit dataset. It is likely that the differences introduced by BEER are compensated by slightly varying the parameters associated with the Doppler boosting and reflected light photometric variations.
Numerically, the Modified Kane & Gelino model is the closest approximation to, and may serve as a quick alternative to, the more complicated EVIL-MC model. EVIL-MC takes on the order of 1,000-10,000 times longer to compute than the simple trigonometric models. For HAT-P-7 the algorithm took 17.25 days and for Kepler-13 the algorithm took 7.80 days to complete. For this reason the computationally intensive EVIL-MC is not suitable to be used in model testing. One would likely need to test all the possible combinations of effects to determine which are present within the dataset. Performing dozens of combinations when a single run takes multiple weeks to compute is not efficient. Instead, the simpler trigonometric models may serve as a general representation of the effect for use in model testing and the full version may be used when performing exoplanet characterization.
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