Interpolation and sampling on the Fock space

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INTERPOLATION AND SAMPLING ON THE FOCK SPACE

by

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ABSTRACT
INTERPOLATION AND SAMPLING ON THE FOCK SPACE

Daniel Stevenson

The theory interpolating and sampling sequences on spaces of analytic functions is one that has been widely studied and continues to produce open problems. In this dissertation we seek to further the study of these sequences in the Fock space setting.

We begin by extending the results of K. Zhu, found in [18], to the Fock space setting, thus providing a characterization of interpolating and sampling sequences in terms of a certain evaluation operator. Next, we apply some classical results of circle packing and circle covering to provide some easily verifiable geometric conditions for interpolation and sampling.
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Chapter 1

Introduction

Unlike the Bergman and Hardy spaces, which have been extensively studied and standardized, the Fock spaces go by at least two other names: the Bargmann-Fock spaces and the Segal-Bargmann spaces [17]. Since the Fock spaces are defined over the complex plane, the geometry is much nicer than the Bergman and Hardy spaces, which use a curved metric. We begin by defining the Fock space $F^p_\alpha$ and stating some important results concerning duality and point-wise estimates. Next we introduce the topics of atomic decomposition, zero sets and Fock-Carleson measures. Each one of these topics on its own makes for an interesting area of study, and all of the results below are found and expanded upon in [17].
1.1 The Fock Space

The Fock space $F^p_\alpha$, $0 < p < \infty$, $\alpha > 0$, is the space of entire functions $f$ in $\mathbb{C}$ such that

$$||f||_{p,\alpha}^p = \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-p\alpha|z|^2/2} dA(z) < \infty,$$

where $dA$ is the standard area measure.

The space $F^\infty_\alpha$ consists of all entire functions $f$ on $\mathbb{C}$ such that

$$||f||_{\infty,\alpha} = \sup \left\{ |f(z)| e^{-\alpha|z|^2/2} : z \in \mathbb{C} \right\} < \infty.$$

Equipped with the $|| \cdot ||_{p,\alpha}$ norm, the Fock space $F^p_\alpha$ is a Banach space for $1 \leq p \leq \infty$ and a complete metric space for $0 < p < 1$. Furthermore, for $0 < p < \infty$, $F^p_\alpha$ is separable with the set of polynomials forming a dense subset. It is elementary to show that for $0 < p < \infty$ and $\alpha > 0$, $F^p_\alpha$ is a closed subset of the Lebesgue space $L^p_\alpha$, which consists of all measurable functions such that $||f||_{p,\alpha} < \infty$. Hence the space $F^2_\alpha$ is a separable Hilbert space with inner product

$$\langle f, g \rangle_\alpha = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-\alpha|z|^2} dA(z).$$

The following point-wise estimate follows from the sub-mean value theorem.

**Theorem 1.1.1.** For any $f \in F^p_\alpha$, $0 < p \leq \infty$,

$$|f(z)| \leq ||f||_{p,\alpha} e^{p\alpha|z|^2/2}$$

for all $z \in \mathbb{C}$. 

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As a corollary to the above, we find a maximal growth rate for functions in the Fock spaces.

**Corollary 1.1.2.** For any $0 < p \leq \infty$ and $z \in \mathbb{C}$, we have that

$$\sup\{|f(z)| : \|f\|_{p,\alpha} \leq 1\} = e^{p\alpha|z|^2/2}$$

It is clear that since in the Fock space setting we are dealing with the entire complex plane, we lack the membership of bounded functions. However, the above theorem and corollary demonstrate that the growth of functions in the Fock space are capped by exponential growth. This maximal growth rate will also be important when we seek to provide a suitable notion of interpolation on the Fock space.

Consistent with general theory of Hilbert spaces of analytic functions, there is an orthogonal projection,

$$P_\alpha : L^2_\alpha \to F^2_\alpha.$$  

**Theorem 1.1.3.** The orthogonal projection

$$P_\alpha : L^2_\alpha \to F^2_\alpha.$$  

is an integral operator given by

$$P_\alpha f(z) = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(z)e^{\alphazw}e^{-\alpha|w|^2}dA(w)$$

**Proof.** See [17]

From this it follows that the function

$$K_\alpha(z, w) = e^{\alpha zw}$$

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is the reproducing kernel for $F_2^2$. It is easy to show that $K_\alpha(\cdot, w) \in F_2^p$ for all $p > 0$ and $w \in \mathbb{C}$. Furthermore, we denote by

$$k_w(z) = e^{\alpha \pi z - \frac{\alpha}{2}|w|^2}$$

the normalized reproducing kernel at $w$. It is clear that each $k_w$ is a unit vector in $F_2^2$.

We now turn to the issue of duality in the Fock spaces. The first result follows from the usual duality of the $L^p$ spaces.

**Theorem 1.1.4.** Suppose $\beta > 0$, $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual space of $F_\alpha^p$ can be identified with $F_\beta^q$ under the integral pairing

$$\langle f, g \rangle_\gamma = \frac{\gamma}{\pi} \int_\mathbb{C} f(z) \overline{g(z)} e^{-\gamma |z|^2} dA(z),$$

where $\gamma = \sqrt{\alpha \beta}$.

*Proof.* See [17]

We now turn to the case where $0 < p \leq 1$.

**Theorem 1.1.5.** Suppose $\beta > 0$, $0 < p \leq 1$. Then the dual space of $F_\alpha^p$ can be identified with $F_\beta^\infty$ under the integral pairing

$$\langle f, g \rangle_\gamma = \lim_{R \to \infty} \frac{\gamma}{\pi} \int_{|z|<R} f(z) \overline{g(z)} e^{-\gamma |z|^2} dA(z),$$

where $\gamma = \sqrt{\alpha \beta}$. 

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Of course, an easy consequence of this result is obtained by setting $\alpha = \beta$, giving us that the dual space of $F^p_\alpha$ can be identified with $F^q_\alpha$ under the integral pairing $\langle , \rangle_\alpha$ whenever $1/p + 1/q = 1$.

This result on the duality of the Fock space $F^p_\alpha$ for $0 < p \leq 1$ diverges quite a bit from the related theory on the Hardy space and the Bergman space, whose dual spaces are the BMO and Bloch space, respectively.
1.2 Atomic Decomposition

It is easy to show that for $0 < p < \infty$, the Fock spaces $F^p_\alpha$ are separable with the set of polynomials forming a dense subset. In addition, since $F^p_\alpha$ consists of entire functions, for each $f \in F^p_\alpha$ we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for each $z \in \mathbb{C}$, where $a_n$ is the $n^{th}$-Taylor coefficient of $f$.

The goal of atomic decomposition is to express each function as an infinite series of kernel functions. Recall the normalized reproducing kernel at $a$ for $F^2_\alpha$ is the function

$$k_a(z) = e^{\alpha z - \frac{\alpha^2 |a|^2}{2}}.$$

It is clear that $k_a(z)$ is a unit vector in $F^2_\alpha$, but we can actually say a little more:

**Lemma 1.2.1.** The normalized reproducing kernel at $w$, $k_w(z)$, is a unit vector in $F^p_\alpha$ for all $0 < p \leq \infty$ and $w \in \mathbb{C}$.

**Proof:** Proof follows from direct calculation □

For any positive $r$, let $r\mathbb{Z}^2 = \{mr + inr : m, n \in \mathbb{Z}\}$ denote the square lattice of side length $r$. Not concerning the boundary points, we express the fundamental region of this lattice as the half open square

$$S_r = \{z = x + iy : -r/2 \leq x < r/2, -r/2 \leq y < r/2\}.$$
It is clear that the complex plane admits the following decomposition,

\[ \mathbb{C} = \bigcup_{z \in r \mathbb{Z}^2} (S_r + z) \]

Now we arrive at the main result for this section:

**Theorem 1.2.2.** Let \( 0 < p \leq \infty \). There exists a positive constant \( r_0 \) such that for any \( 0 < r < r_0 \), the Fock space \( F^p_\alpha \) consists of exactly the functions

\[ f(z) = \sum_{w \in r \mathbb{Z}^2} c_w k_w(z), \]

where the sequence \( \{c_w : w \in r \mathbb{Z}^2\} \in l^p \).

Here, an interesting feature of the above theorem is the existence of the constant \( r_0 \). We can re-read the above result as follows: given a square lattice of sufficiently small side length \( r \) we can perform atomic decomposition on that lattice. Later in this paper, we will be able to give a precise value for the constant \( r_0 \) such that atomic decomposition for \( F^2_\alpha \) holds on the associated lattice.

Now we say that a sequence \( Z = \{z_n\} \) in \( \mathbb{C} \) and the Fock space \( F^p_\alpha \) form an atomic pair if \( F^p_\alpha \) consists of exactly the functions

\[ f(z) = \sum_{z_n \in Z} c_{z_n} k_{z_n}(z), \]

where the sequence \( \{c_{z_n} : z_n \in Z\} \in l^p \). Later we will develop a necessary and sufficient condition for when the sequence \( Z = \{z_n\} \) and the space \( F^p_\alpha \) form an atomic pair.
1.3 Zero Sequences

We begin with the following definition:

**Definition 1.3.1.** We say a sequence of points \(Z = \{z_n\}\) in \(\mathbb{C}\) is a zero sequence for the space \(F^p_\alpha\) if there exists a function \(f \in F^p_\alpha\), not identically zero, such that \(f\) vanishes exactly on \(Z\).

Note that the definition of a zero sequence in the Fock space setting differs subtly from that in the Bergman or Hardy space setting. This is due to the observation that multiplication by the coordinate function \(z\) is unbounded in the Fock spaces. However, we do have the following observation:

**Lemma 1.3.2.** If \(f \in F^p_\alpha\) for some \(0 < p \leq \infty\) and \(\alpha > 0\), then for any complex number \(a\), the function \(g(z) = (z - a)f(z)\) belongs to \(F^p_\beta\) for all \(\beta > \alpha\).

**Proof:** Suppose \(0 < p \leq \infty\), \(\alpha > 0\) and \(f \in F^p_\alpha\). Now let \(a \in \mathbb{C}\) and consider the entire function \(g(z) = (z - a)f(z)\). Now for any \(\beta > \alpha\) we have that

\[
\int_{\mathbb{C}} |(z - a)f(z)|^p e^{-p\beta|z|^2/2}dA(z) = \int_{\mathbb{C}} |f(z)|^p e^{-p\alpha|z|^2/2}|z - a|^p e^{-p(\beta - \alpha)|z|^2/2}dA(z)
\]

\[
\leq ||f||_{p,\alpha}^p \int_{\mathbb{C}} |z - a|^p e^{-p(\beta - \alpha)|z|^2/2}dA(z)
\]

\[
< \infty
\]

since the Fock space \(F^p_{\beta - \alpha}\) contains the set of polynomials. \(\square\)

To view specific examples of zero sequences on the Fock space, we can invoke the Weierstrass \(\sigma\)-function. Assume that \(\Lambda = \{\omega_{mn}\}\) is a lattice based at the origin.
with $\omega_{mn} = m\omega_1 + n\omega_2$. Define the Weierstrass $\sigma$-function as follows:

$$\sigma(z) = z \prod_{m,n \neq 0} \left[ \left(1 - \frac{z}{\omega_{mn}}\right) \exp\left(\frac{z}{\omega_{mn}} + \frac{z^2}{2\omega_{mn}^2}\right) \right].$$

It can be shown that the function $\sigma$ above is entire and has $\Lambda = \{\omega_{mn}\}$ as its zero sequence.

In addition to the notion of a zero set we also have the following definition:

**Definition 1.3.3.** We say that a subset $Z$ of $\mathbb{C}$ is a uniqueness set for $F_p^\alpha$ if the zero function is the only function in $F_p^\alpha$ that vanishes on $Z$. In other words, if $f \in F_p^\alpha$ vanishes on $Z$, then $f \equiv 0$.

Of course a zero set cannot be a uniqueness set. However, if $Z$ is not a set of uniqueness, that does not imply that $Z$ is a zero set. Zero sets are further studied in [17] and [19].
1.4 Fock-Carleson Measures

There are many rich and insightful results regarding Carleson measures, some of which can be found in [11] and [17].

We begin with the following theorem:

**Theorem 1.4.1.** Suppose $\mu$ is a positive Borel measure on $\mathbb{C}$, $0 < p < \infty$, and $0 < r < \infty$. Then the following conditions are equivalent:

(i) There exists a positive constant $C$ such that

$$
\int_{\mathbb{C}} |f(z)|^p e^{-p\alpha |z|^2/2} d\mu(z) \leq C \int_{\mathbb{C}} |f(z)|^p e^{-p\alpha |z|^2/2} dA(z)
$$

for all $f \in F^p_\alpha$.

(ii) There exists a positive constant $C$ such that

$$
\int_{\mathbb{C}} e^{-\frac{p\alpha}{2} |z-w|^2} d\mu(w) \leq C
$$

for all $z \in \mathbb{C}$. 

(iii) There exists a constant $C > 0$ such that $\mu(B(z,r)) \leq C$ for all $z \in \mathbb{C}$

Any measure that satisfies any of the above equivalent conditions we will call a Fock-Carleson measure.

A detailed proof of the above result can be found in [17]. We will use the above theorem later to find conditions upon which a certain integral operator is bounded on the Fock space.
Chapter 2

Sampling and Interpolating Sequences

In this chapter we outline the origins of interpolating and sampling on analytic function spaces and provide some well-known results for the Hardy space, the Bergman space and finally, the Fock space.
2.1 Introduction

The classical origins of the subject of interpolation on function spaces begin with
the Nevanlinna-Pick problem. It asks the question, given \( z_1, \ldots, z_n \) and \( a_1, \ldots, a_n \) in
the unit disk \( \mathbb{D} \), under what conditions does the interpolation problem \( f(z_j) = a_j, \)
\( j = 1, \ldots, n \) have a solution \( f \), where \( f \) is analytic in \( \mathbb{D} \) and \( |f| \leq 1 \) on \( \mathbb{D} \). Pick’s
theorem states that the interpolation problem has a solution if and only if the
matrix

\[
\begin{pmatrix}
1 - a_j a_k \\
1 - \overline{z_j} z_k
\end{pmatrix}
\]

is positive semi-definite. The solution to the interpolation problem can always be
taken to be a Blaschke product of degree at most \( n \).

Next we turn to Carleson interpolation which extends the Nevanlinna-Pick prob-
lem to infinite sequences of data. We will say that a sequence of points \( Z \) in the
unit disk \( \mathbb{D} \) is uniformly separated if

\[
\inf_j \prod_{n \neq j} \left| \frac{z_j - z_n}{\overline{z_j} - \overline{z_n}} \right| > 0.
\]

Furthermore, we say that a sequence \( Z = \{z_n\} \) of distinct points in \( \mathbb{C} \) is an
interpolating sequence for \( H^\infty(\mathbb{D}) \) if for any bounded sequence \( \{a_n\} \) in \( \mathbb{C} \), there
exists a bounded, analytic function \( f \) on \( \mathbb{D} \) such that \( f(z_j) = a_j \), for each \( j \). We
now state Carleson’s theorem:

**Theorem 2.1.1.** \( Z \) is an interpolation sequence for \( H^\infty(\mathbb{D}) \) if and only if \( Z \) is
uniformly separated.
A similar result for the Hardy space $H^p$ was proved by Shapiro and Shields in 1961:

**Theorem 2.1.2.** Let $Z = \{z_n\}$ be a sequence of distinct points in $\mathbb{D}$. For any sequence $\{v_n\} \in l^2$, there exists a function $f \in H^2$ such that

$$f(z_n)(1 - z_n^2)^{1/p} = v_n$$

if and only if $Z$ is uniformly separated.

Here the Hardy space, $H^p$, consists of analytic functions on the disk such that

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$$

, and we say that Any sequence $Z$ satisfying theorem is an interpolating sequence for $H^p$. The study of the well-defindedness of the mapping, $f \mapsto f(z_n)(1 - |z_n|^2)^{1/2}$ gave natural rise to the study of Carleson measures, of which we have already seen from the Fock space perspective.

When the analysis shifts to the Bergman space $A^p$, $0 < p < \infty$, a slightly different approach must be taken. For $0 < p < \infty$, the Bergman space $A^p$, consists of analytic functions on the unit disk such that,

$$\int_{\mathbb{D}} |f(z)|^p \, dA < \infty.$$  

Here we use an annalogue of the Beurling densities on $\mathbb{C}$ to characterize sequences of interpolation for $A^p$. 
Theorem 2.1.3. Let $Z = \{z_n\}$ be a sequence of distinct points in $\mathbb{D}$. For any sequence $\{v_n\} \in l^p$, there exists a function $f \in A^p$ such that

$$f(z_n)(1 - |z_n|^2)^{2/p} = v_n$$

if and only if $Z$ is uniformly discrete and $D^+(Z) < 1/p$.

We call any sequence $Z$ satisfying the above theorem an interpolating sequence for $A^p$. Here, $D^+$ is the upper uniform density as described in [11]. Consistent among the previous theorems is the notion that sequences must be separated and sufficiently sparse to be interpolating.

The classical origins of sampling on function spaces come from communication theory and in particular, Analog/Digital and Digital/Analog conversion. More information about the origins of sampling and interpolation, along with several results on a wide variety of function spaces, can be found in [11].
2.2 Beurling’s Density

When studying sequences of points in the complex plane, it is useful to characterize these sequences using a geometric notion of density.

Let \( Z = \{z_n\} \) be a sequence of distinct points in \( \mathbb{C} \). For any subset \( S \) of \( \mathbb{C} \) we let \( n(Z, S) = |Z \cup S| \) denote the number of points in the intersection of \( Z \) and \( S \). Now let \( B(\zeta, r) = \{ |z - \zeta| < r \} \) denote the open disk in \( \mathbb{C} \) centered at \( \zeta \) with radius \( r \). We define the upper and lower densities of \( Z \) as follows:

\[
D^+(Z) = \limsup_{r \to \infty} \sup_{\zeta \in \mathbb{C}} \frac{n(Z, B(\zeta, r))}{\pi r^2}.
\]

\[
D^-(Z) = \liminf_{r \to \infty} \inf_{\zeta \in \mathbb{C}} \frac{n(Z, B(\zeta, r))}{\pi r^2}.
\]

The quantity \( \frac{n(Z, B(\zeta, r))}{\pi r^2} \) gives the average number of points in \( Z \) per unit area of the ball. We can generalize this notion of density as follows:

\[
D^+(Z) = \limsup_{r \to \infty} \sup_{\zeta \in \mathbb{C}} \frac{n(Z, \zeta + rI)}{r^2},
\]

\[
D^-(Z) = \liminf_{r \to \infty} \inf_{\zeta \in \mathbb{C}} \frac{n(Z, \zeta + rI)}{r^2},
\]

where \( I \) is any set with Lebesgue area of one. Proof of equality can be found in [12]. These are known as the Beurling densities or the uniform densities on \( \mathbb{C} \).

We would now like to give an example of a set whose density we can calculate explicitly.
Proposition 2.2.1. Let $\Lambda = \{\omega + m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ denote the lattice spanned by $\omega_1$ and $\omega_2$. Then,

$$D^-(\Lambda) = D^+(\Lambda) = \frac{1}{Im(\omega_1\omega_2)}.$$

The previous quantity, $\frac{1}{Im(\omega_1\omega_2)}$, is the area of the parallelogram spanned by the vectors $\omega_1$ and $\omega_2$. As a special case the square lattice $r\mathbb{Z}^2$ has uniform densities given by

$$D^+(r\mathbb{Z}^2) = D^-(r\mathbb{Z}^2) = 1/r^2.$$

Now let $\Lambda_\alpha = \sqrt{\pi/\alpha}\mathbb{Z}^2$. It is clear that $D^+(\Lambda_\alpha) = D^-(\Lambda_\alpha) = \frac{\alpha}{\pi}$.

The following is an interesting observation.

Proposition 2.2.2. If $Z = \{z_{mn}\}$ is a sequence uniformly close to $\Lambda_\alpha = \{w_{mn}\}$, i.e. there exists a positive real number $Q$ (not necessarily small) such that $|z_{mn} - w_{mn}| \leq Q$ for all $m, n \in \mathbb{N}$, then $D^+(Z) = D^-(Z) = \frac{\alpha}{\pi}$.

Proof: Let $Z$ and $\Lambda_\alpha$ be defined as above and let $Q$ be a positive real number such that $|z_{mn} - w_{mn}| \leq Q$ for all $m, n \in \mathbb{N}$. Then it is easy to show that

$$n(Z, B(w, r - Q)) \leq n(\Lambda_\alpha, B(w, r)) \leq n(Z, B(w, r + Q + 1))$$

for all $w \in \mathbb{C}$. Now taking the $inf$ over all $w$ and letting $r \to \infty$ we have that $D^-(Z) \leq D^-(\Lambda_\alpha) \leq D^-(Z)$ and thus $D^-(Z) = D^-(\Lambda_\alpha)$. The proof that $D^+(Z) = D^+(\Lambda_\alpha)$ is similar. $\square$

In addition we have the following result.
Proposition 2.2.3. If $Z$ is a separated sequence with $D^+(Z) = \frac{\beta}{\pi}$, $\beta < \alpha$, then we can expand $Z$ to a separated sequence $Z'$ such that $Z'$ is uniformly close to the lattice $\Lambda_\gamma$, where $\gamma \in (\beta, \alpha)$.

This allows us to generalize our analysis to lattices, and is a key component in the proof of the Seip-Wallsten density theorem stated later in this paper.
2.3 Interpolating and Sampling Sequences for the Fock Space

We begin with the following definition.

Definition 2.3.1. Let $Z = \{z_n\}$ be a sequence of distinct points in $\mathbb{C}$. We say that $Z$ is separated if there exists a real number $\delta$ such that $|z_m - z_n| \geq \delta$ for all $n \neq m$.

In the case where $\delta$ is the largest such constant satisfying the above inequality, we write $\delta = \delta(Z)$ and call $\delta(Z)$ the separation constant of $Z$.

The following result characterizes the values of a function in $F^p_\alpha$ taken over a separated sequence.

Proposition 2.3.2. Let $Z = \{z_n\}$ be a separated sequence and $0 < p < \infty$. Then there exists a positive constant $C$, independent of $f$, such that

$$\sum_{n=1}^{\infty} |f(z_n) e^{-\alpha |z_n|^2/2}|^p \leq C ||f||_{p,\alpha}^p$$

for all $f \in F^p_\alpha$.

Proof: A detailed proof can be found in [17] □

Using the result above we can now appropriately define sequences of interpolation and sampling for $F^p_\alpha$.

Definition 2.3.3. Let $Z = \{z_n\}$ denote a sequence of distinct points in $\mathbb{C}$ and let $0 < p < \infty$. We say that $Z$ is an interpolating sequence for $F^p_\alpha$ if for every sequence
of values \( \{v_n\} \) satisfying
\[
\sum_{n=1}^{\infty} |v_n|^p e^{-p\alpha|z_n|^2/2} < \infty
\]
there exists a function \( f \in F^p_\alpha \) such that \( f(z_n) = v_n \), for all \( n \geq 1 \).

Similarly, we say that a sequence \( Z = \{z_n\} \) of distinct points in \( \mathbb{C} \) is an interpolating sequence for \( F^\infty_\alpha \) if for every sequence of values \( \{v_n\} \) satisfying
\[
\sup_{n \geq 1} |v_n| e^{-\alpha|z_n|^2} < \infty,
\]
there exists a function \( f \in F^\infty_\alpha \) such that \( f(z_n) = v_n \), for all \( n \geq 1 \).

**Definition 2.3.4.** Let \( Z = \{z_n\} \) denote a sequence of distinct points in \( \mathbb{C} \) and let \( 0 < p < \infty \). We say that \( Z \) is a sampling sequence for \( F^p_\alpha \) if there exists a positive constant \( C \) such that
\[
C^{-1} ||f||_{p,\alpha}^p \leq \sum_{n=1}^{\infty} |f(z_n)|^p e^{-p\alpha|z_n|^2/2} \leq C ||f||_{p,\alpha}^p
\]
for all \( f \in F^p_\alpha \).

For sampling in \( F^\infty_\alpha \) we must use a slightly different approach. We say that an arbitrary set \( Z \) in \( \mathbb{C} \) is a sampling set for \( F^\infty_\alpha \) if there exists a constant \( C > 0 \) such that
\[
||f||_{\infty,\alpha} \leq C \sup_{z \in Z} |f(z)| e^{-\frac{\alpha}{2}|z|^2}
\]
for all \( f \in F^\infty_\alpha \).

It is immediately clear that every sampling sequence is necessarily a set of uniqueness, and that a set of uniqueness can not be an interpolating sequence.

We now list some preliminary results.
Lemma 2.3.5. Suppose $0 < p \leq \infty$ and $Z = \{z_n\}$ is an interpolating sequence for $F^p_\alpha$. Then $Z$ must be separated.

Lemma 2.3.6. Suppose $0 < p \leq \infty$ and $Z = \{z_n\}$ is a sampling sequence for $F^p_\alpha$. Then $Z$ contains a separated subsequence $Z'$ that is also a sampling sequence for $F^p_\alpha$.

The latter of these two lemmas allows us to generalize our analysis to separated sequences. This will be particularly helpful later when we study a certain evaluation operator on the Fock space.

We now introduce the following theorems which characterize interpolating and sampling sequences in terms of our notion of density from above.

Theorem 2.3.7. If $Z$ is separated and $0 < p \leq \infty$, then $Z$ is a sampling sequence for $F^p_\alpha$ if and only if $D^-(Z) > \alpha/\pi$.

Theorem 2.3.8. If $Z$ is separated and $0 < p \leq \infty$, then $Z$ is an interpolating sequence for $F^p_\alpha$ if and only if $D^+(Z) < \alpha/\pi$.

The theorems above are attributed to Seip and Wallsten, and their proofs can be found in [17]. It is now clear that examples of interpolating and sampling sequences are not only quite common in the Fock space, but are easy to construct by taking an appropriate lattice.
Chapter 3

The Evaluation Operator on the Fock Space

We now characterize sequences of sampling and interpolation using a certain evaluation operator on the Fock space. The results found here are an extension of those found in [18].
3.1 Introduction and basic properties

Let $Z = \{z_m\}$ be a sequence of distinct points in $\mathbb{C}$. Define an operator $T_Z$ from the Fock space $F^p_\alpha$ into $l^p$ by

$$T_Z(f) = \{f(z_n)e^{-\alpha|z_n|^2/2}\}_{n=1}^\infty.$$ 

We will call $T_Z$ the evaluation operator on the Fock space $F^p_\alpha$. Evaluation operators have already been extensively studied in the Bergman space setting as they pertain to interpolating and sampling sequences. Our goal is to extend these results to the Fock space setting. We begin with some initial observations.

1. The operator $T_Z$ is one-to-one if and only if $\{z_n\}$ is a uniqueness set for $F^p_\alpha$.

2. The operator $T_Z$ maps $F^p_\alpha$ onto $l^p$ iff $\{z_n\}$ is an interpolating sequence for $F^p_\alpha$.

3. The operator $T_Z$ from $F^p_\alpha$ into $l^p$ is bounded from above and below if and only if $\{z_n\}$ is a sampling sequence for $F^p_\alpha$.

We now would like to determine when $T_Z$ maps $F^p_\alpha$ boundedly into $l^p$.

**Theorem 3.1.1.** Suppose $0 < p < \infty$. Then $T_Z$ maps $F^p_\alpha$ boundedly into $l^p$ if and only if $Z$ is a finite union of separated sequences.

**Proof.** Let $f \in F^p_\alpha$. Then

$$||T_z f||^p = \sum |f(z_n)|^p e^{-p\alpha|z_n|^2/2} = \int_{\mathbb{C}} |f(z)|^p e^{-p\alpha|z|^2/2} d\mu(z),$$

where $d\mu(z)$ is the probability measure on $\mathbb{C}$. This bound is achieved if and only if $Z$ is a finite union of separated sequences.
where $\mu = \sum \delta_{z_n}$ with $\delta_{z_n}$ being the point measure at $z_n$. Now we have that $T_Z$ maps $F_\alpha^p$ boundedly into $l^p$ if and only if

$$\int_C |f(z)|^p e^{-p\alpha|z|^2/2} d\mu(z) \leq c \int_C |f(z)|^p e^{-p\alpha|z|^2/2} dA(z)$$

for some $c > 0$, in other words that $\mu$ is a Fock-Carleson measure. By theorem 3.28 of [17], $\mu$ is a Fock-Carleson measure if and only if for $0 < r < \infty$, there exists a constant $C > 0$ such that $\mu(B(z, r)) \leq C$ for all $z \in \mathbb{C}$. This is clearly equivalent to the condition that $Z$ is the union of finitely many separated sequences. □

Now we would like to consider the adjoint of $T_Z$. Let $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and define the operator $S_Z : l^q \to F_\alpha^q$ by

$$S_Z \{a_n\} (z) = \sum a_n k_{z_n}(z).$$

It is easy to show that the above sum converges uniformly on compact subsets of $\mathbb{C}$. Now we consider the following inner product,

$$\langle T_Z f, \{a_n\} \rangle = \langle \{f(z_n)/k_{z_n}(z_n)\}, \{a_n\} \rangle$$

$$= \sum \bar{a}_n \frac{f(z_n)}{k_{z_n}(z_n)}$$

$$= \sum \bar{a}_n \langle f, k_{z_n}(z) \rangle$$

$$= \langle f, S_Z \{a_n\} \rangle$$

This yields the following corollary.

**Corollary 3.1.2.** Let $1 < p, q < \infty$. Then the operator $S_Z : l^q \to F_\alpha^q$, $\frac{1}{p} + \frac{1}{q} = 1$,
defined by $S_Z \{a_n\}(z) = \sum a_n e^{\alpha z - \alpha |z|^2/2}$ is bounded if and only if $Z$ is the union of finitely many separated sequences.

Proof. This follows from duality. $S^*_Z = T_Z$ and $T^*_Z = S_Z$ if either $T_Z$ or $S_Z$ is bounded. $\square$

3.2 Closed Range Property

Recall that a sequence $Z = \{z_n\}$ in $\mathbb{C}$ and the Fock space $F^p_\alpha$ form an atomic pair if $F^p_\alpha$ consists of exactly the functions

$$f(z) = \sum_{z_n \in Z} c_{z_n} k_{z_n}(z),$$

where the sequence $\{c_{z_n} : z_n \in Z\} \in l^p$. We begin with the following results from basic functional analysis.

Lemma 3.2.1. Let $X$ and $Y$ be Banach spaces and suppose that $T : X \to Y$ is bounded. Then $T$ is onto if and only if $\|T^*y^*\| \geq \|y^*\|$ for all $y^* \in Y^*$.

Note that this is equivalent to saying that a bounded operator is onto if and only if its adjoint is bounded from below.

Proof: See Rudin [6].

We now prove the following result which characterizes sampling sequences by way of atomic decomposition.
Theorem 3.2.2. Suppose $1 < p < \infty$ and let $Z$ be a sequence of distinct points in $\mathbb{C}$. Then $Z$ is a sampling sequence for $F^p_\alpha$ if and only if atomic decomposition for $F^q_\alpha$, $\frac{1}{p} + \frac{1}{q} = 1$, holds on $Z$, or in other words, $(F^q_\alpha, Z)$ is an atomic pair.

Proof: We begin by proving necessity. Let $Z$ be a sampling sequence for $F^p_\alpha$. Then by definition there exists a positive constant $C$ such that

$$\sum_{n=1}^{\infty} |f(z_n)|^p e^{-p\alpha|z_n|^2/2} = ||T_Z f||_p^p \leq C ||f||_{p,\alpha}^p$$

for all $f \in F^p_\alpha$. Thus $T_Z$ maps $F^p_\alpha$ boundedly into $l^p$. Now by lemma 3.1 we have that $Z$ is the union of at most finitely many separated sequences and hence, by Corollary 3.2, the adjoint $S_Z$ is bounded as well. Now we have that for any sequence $\{a_n\} \in l^q$ the series

$$\sum a_n e^{\alpha z_n - \alpha |z_n|^2/2}$$

converges normally in $\mathbb{C}$ to a function in $F^p_\alpha$. Furthermore, as $Z$ was sampling we have that $T_Z$ is bounded below and thus by lemma 3.3, $S_Z$ is onto. Therefore, for any $f \in F^q_\alpha$ there exists a sequence $\{b_n\} \in l^q$ such that

$$S_Z \{b_n\} = \sum a_n e^{\alpha z_n - \alpha |z_n|^2/2} = f(z)$$

Hence $(F^q_\alpha, Z)$ form an atomic pair.

Now we prove sufficiency. Assume that atomic decomposition for $F^q_\alpha$ holds on $Z$. Thus the series

$$\sum a_n e^{\alpha z_n - \alpha |z_n|^2/2}$$
converges in $F^p_\alpha$ for each sequence $\{a_n\} \in l^q$. Then by the closed graph theorem the operator $S_Z$ is bounded and hence its adjoint, $T_Z$, is bounded as well. Furthermore, as atomic decomposition for $F^q_\alpha$ holds on $Z$, $S_Z$ is onto and thus $T_Z$ is bounded below. Hence $Z$ is a sampling sequence for $F^p_\alpha$. $\square$

We now turn to the question of when $T_Z$ has closed range. For the remainder of this section we assume that the sequence $Z = \{z_n\}$ is a finite union of separated sequences and that $1 < p < \infty$ so the evaluation operator $T_Z$ is a bounded linear operator.

The easiest example of an evaluation operator with closed range is when $Z$ is an interpolating sequence for $F^p_\alpha$. In this case, the range of $T_Z$ is all of $l^p$.

Recall that a sequence $Z$ is a set of uniqueness for $F^p_\alpha$ if whenever $f \in F^p_\alpha$ is such that $f|_Z \equiv 0$, then $f \equiv 0$. It is easy to show that interpolating sequences for $F^p_\alpha$ are not uniqueness sets for $F^p_\alpha$. It is important to note that as opposed to the Bergman Space $L^p_\alpha(\mathbb{D})$, there exist interpolating sequences for $F^p_\alpha$ that are not zero sequences for $F^p_\alpha$. We now arrive at a necessary and sufficient condition for a set to be sampling for $F^p_\alpha$.

**Theorem 3.2.3.** Suppose $T_Z$ has closed range. Then $Z$ is a sampling sequence for $F^p_\alpha$ if and only if $Z$ is a uniqueness set for $F^p_\alpha$.

**Proof.** It is clear that, if $Z$ is not a uniqueness set then $Z$ can not be a sampling set. Now assume that $Z$ is a uniqueness set for $F^p_\alpha$. By the Open Mapping Theorem
there exists $c > 0$ such that
\[ \|T_z f\|_p^p \geq c \|f\|_{p,\alpha}^p \]
for all $f \in F^p_\alpha/k\ker T_z$. However since $Z$ is a uniqueness set for $F^p_\alpha$ we have that
\[ F^p_\alpha/k\ker T_z = F^p_\alpha \]
and thus $T_z$ is bounded below. Furthermore as $T_z$ is bounded above we have that $Z$ is a sampling sequence for $F^p_\alpha$. □

For the remainder of the section we will assume that the sequence $Z = \{z_n\}$ is not a uniqueness set for $F^p_\alpha$. Since $Z$ is not a uniqueness set, we have that $\ker T_z \neq 0$ and we let $I_Z$ denote the set of functions in $F^p_\alpha$ that vanish on $Z$. It is clear that $I_Z$ is norm-closed in $F^p_\alpha$.

Now let $g \in I_Z$ such that $g \neq 0$. Define a sequence of functions $\{g_k\}$ as follows:
\[ g_k(z) = \frac{cg(z)}{(z - z_k)^n} \]
where $c = (\|g(z_k)\|_{p,\alpha})^{-1}$ and $n$ is the order of the zero of $g$ at $z_k$. It is easy to show that $g_k \in F^p_\alpha$ for all $k \geq 1$, and furthermore we have that:
\[ g_k(z_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \]

For a Banach space $X$ and a subset $M$, define the annihilator $M^\perp$ of $M$ as
\[ M^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0 \ \forall x \in X\} \]
The following is a standard result from Functional analysis.
Lemma 3.2.4. Let $M$ be a closed subspace of a Banach space $X$. Then the spaces $(X/M)^*$ and $M^\perp$ are isometrically isomorphic as Banach spaces.

Proof: See Rudin [6].

Now we arrive at a necessary condition for a separated sequence to be interpolating for $F_p^\alpha$.

Theorem 3.2.5. Suppose that $T_Z$ has closed range and that $Z$ is not a uniqueness set for $F_p^\alpha$. Then $Z$ is an interpolating sequence for $F_p^\alpha$.

Proof: Let $X$ be the range of $T_Z$. Define $T : F_p^\alpha/I_Z \to X$ to be the restriction of $T_Z$ to $F_p^\alpha/I_Z$. It is clear that $T$ is linear, bounded, one-to-one and onto and thus so is its adjoint $T^* : X^* \to (F_p^\alpha/I_Z)^*$. As $\ker T_Z$ is a closed subspace of $F_p^\alpha$, by the above Lemma we write, $T^* : X^* \to I_Z^\perp$.

Now it suffices to show that $X^* = l^q$. Given $\{a_n\} \in l^q$, by Cor. 2 the series $g(z) = \sum a_n k_{z_n}(z)$ converges in $F_p^\alpha$. Furthermore, as

$$\langle \sum a_n k_{z_n}(z), f \rangle = \sum a_n f(z_n)$$

for any $f \in F_p^\alpha$, we have that $g \in I_Z^\perp$. Since $T^*$ is onto, there exists $\{b_n\}$ in $X^*$ such that

$$g = T^* \{b_n\}$$
 Hence
\[ \sum (a_n - b_n)k_{z_n}(z) = 0, \]
Now taking the inner product with \( g_n \) gives us that \( a_n - b_n = 0 \) for all \( n \geq 1 \) and thus
\[ \{a_n\} = \{b_n\} \in X^* \]

Combining the previous two results we arrive at the following corollary.

**Corollary 3.2.6.** Suppose that \( Z = \{z_n\} \) is a separated sequence and \( 1 < p < \infty \). Then \( T_z \) has closed range in \( l^p \) if and only if \( Z \) is either interpolating or sampling for \( F_\alpha^p \).
3.3 Further Remarks/Extensions

Using Theorem 3.1.2 and the Buerling uniform densities, we can restate Theorem 1.2.2 as follows:

**Proposition 3.3.1.** Let $1 < p \leq \infty$. Then for any $0 < r < \frac{\alpha}{\pi}$, the Fock space $F^p_\alpha$ consists of exactly the functions

$$f(z) = \sum_{w \in rZ} c_w k_w(z),$$

where the sequence $\{c_w : w \in rZ\} \in l^p$.

**Proof:** Let $0 < r < \frac{\alpha}{\pi}$ and $Z$ be the square lattice $rZ$. Then by Theorem 2.2.8, $Z$ is a sampling sequence for $F^p_\alpha$. It follows from Theorem 3.1.2 that $(F^q_\alpha, Z)$ form an atomic pair, and thus the above proposition is proved. □

Obviously, this proposition can not handle the case when $0 < p \leq 1$, but it is nonetheless, quite elegant.

The results on boundedness of the evaluation operator and atomic decomposition may be expanded to higher dimensional Fock spaces. For $\Omega \subset \mathbb{C}^n$, the Fock space, $F^2_\alpha(\Omega)$, is a reproducing Hilbert space, and thus Theorem 3.0.9 and Theorem 3.1.2 will hold in the setting as well. Whether the other main theorems from chapter three hold in this setting is left for future study.
Chapter 4

An Application of Circle Packing and Covering

The motivation for the following section comes from the following result obtained by James Tung in [15].

**Theorem 4.0.2.** If \( Z = \{z_n\} \) is a sequence of points in the complex plane satisfying

\[
\inf_{n \neq m} |z_n - z_m| > \frac{2}{\sqrt{\alpha}},
\]

then \( Z \) is an interpolating sequence for \( F^p_\alpha \).

Tung’s result gives us an easily verifiable sufficient condition for a sequence to be interpolating for \( F^p_\alpha \). In this section we show that this constant in Tung’s theorem above can be improved to

\[
\sqrt{\frac{2\pi}{\sqrt{3} \alpha}}.
\]
which is strictly smaller than $2/\sqrt{\alpha}$. We will also obtain a similar result for $F_{\alpha}^p$ sampling sequences. To accomplish this we will use some classical results from circle packing and covering of the plane.
4.1 Introduction to Circle Packing

A circle packing of the plane is a countable collection of non-overlapping circles in $\mathbb{C}$. For many years it had remained a curiosity to mathematicians as to what arrangement of circles of a fixed radius would cover the largest proportion of the plane.

It was already known to Joseph Louis Lagrange in 1773 that, among lattice arrangements of circles, the highest density is achieved by the hexagonal lattice of the bee’s honeycomb, in which the centers of the circles form a hexagonal lattice, with each circle surrounded by 6 others. The density of such a packing is given by

$$\frac{\pi}{\sqrt{12}} = 0.9069\cdots.$$

In 1890, Axel Thue showed that this density was actually maximal among all possible circle packings (not necessarily lattice packings). But his proof was considered to be incomplete by some mathematicians, and a more rigorous proof was finally found by László Fejes Tóth in 1940.

Let $S(z, r) = \partial B(z, r)$ denote the circle centered at $z$ with radius $r$. If $S = \{S(z_n, r_0)\}$ is a circle packing in the plane, its packing density is defined as

$$\Delta(S) = \limsup_{r \to \infty} \sup_{\zeta \in \mathbb{C}} \frac{1}{\pi r^2} \sum_{z_n} \{\pi r_0^2 : B(z_n, r_0) \cap B(\zeta, r) \neq \emptyset\}.$$

See page 22 of [5]. Therefore, the historical result about circle packing in the plane can be stated as follows. See page 1 of [5] for example.
**Theorem 4.1.1.** For any circle packing $S$ we always have

$$\Delta(S) \leq \frac{\pi}{\sqrt{12}} = \frac{\pi}{2\sqrt{3}} < 1.$$ 

Furthermore, equality is achieved by the hexagonal packing.

There is also a corresponding notion of circle covering. More specifically, we say that a countable collection of circles $C = \{S(z_n, r_0)\}$ is a circle covering of the plane if the union of $\{B(z_n, r_0)\}$ covers the whole plane $\mathbb{C}$. The number

$$\delta(C) = \lim_{r \to \infty} \inf_{\zeta \in \mathbb{C}} \frac{1}{\pi r^2} \sum \{ \pi r_0^2 : B(z_n, r_0) \subset B(\zeta, r) \}$$

will be called the covering density of $C$. See page 22 of [5]. The following theorem is a classical result from circle covering in the plane. See page 16 of [5] for example.

**Theorem 4.1.2.** For any circle covering $C$ we always have

$$\delta(C) \geq \frac{2\pi}{3\sqrt{3}} > 1.$$ 

Furthermore, equality is achieved by circles centered at any hexagonal lattice with the same radius chosen to be the minimum so that these circles cover the plane.
4.2 Statement and Proofs of Main Results

We now apply the classical results about circle packing and circle covering to obtain sufficient conditions for interpolating and sampling sequences for Fock spaces.

**Theorem 4.2.1.** If \( Z = \{z_n\} \) is a sequence of points in the complex plane and
\[
\inf_{n \neq m} |z_n - z_m| > \sqrt{\frac{2\pi}{\sqrt{3\alpha}}},
\]
then \( Z \) is an interpolating sequence for \( F^p_\alpha \).

**Proof:** Suppose \( \sigma > 0 \) and \( |z_n - z_m| \geq \sigma \) for all \( n \neq m \). It is then clear that \( S = \{S(z_n, \sigma/2)\} \) is a circle packing in the complex plane. By Theorem 4.2, \( \Delta(S) \leq \pi/\sqrt{12} \). It follows from the definition of packing density that for any \( \varepsilon > 0 \) there exists some positive number \( R \) such that for all \( r > R \) and all \( \zeta \in C \) we have
\[
\frac{1}{\pi r^2} \sum \left\{ \frac{\pi \sigma^2}{4} : B(z_n, \sigma/2) \cap B(\zeta, r) \neq \emptyset \right\} < \pi \sqrt{\frac{12}{\varepsilon}}.
\]
Since \( z_n \in B(\zeta, r) \) implies
\[
B(z_n, \sigma/2) \cap B(\zeta, r) \neq \emptyset,
\]
we must also have
\[
\frac{1}{\pi r^2} \sum \left\{ \frac{\pi \sigma^2}{4} : z_n \in B(\zeta, r) \right\} < \frac{\pi}{\sqrt{12}} + \varepsilon.
\]
Rewrite this as
\[
\frac{\sigma^2}{4\pi^2} n(Z \cap B(\zeta, r)) < \frac{\pi}{\sqrt{12}} + \varepsilon,
\]
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or equivalently,

\[
\frac{n(Z \cap B(\zeta, r))}{\pi r^2} < \frac{4}{\sqrt{12} \sigma^2} + \frac{4\varepsilon}{\sigma^2 \pi}.
\]

Take the supremum over \(\zeta \in \mathbb{C}\) and let \(r \to \infty\). We obtain

\[
D^+(Z) \leq \frac{2}{\sqrt{3} \sigma^2} + \frac{4\varepsilon}{\sigma^2 \pi}.
\]

Since \(\varepsilon\) is arbitrary, we must have \(D^+(Z) \leq 2/(\sqrt{3} \sigma^2)\).

Now if

\[
\sigma > \sqrt{\frac{2\pi}{\sqrt{3} \alpha}},
\]

then

\[
\frac{2}{\sqrt{3} \sigma^2} < \frac{\alpha}{\pi},
\]

so that \(D^+(Z) < \alpha/\pi\). Combining this with Theorem 4.1.1, we conclude that the condition

\[
\inf_{n \neq m} |z_n - z_m| > \sqrt{\frac{2\pi}{\sqrt{3} \alpha}}
\]

implies that \(Z = \{z_n\}\) is a sequence of interpolation for \(F^{p}_{\alpha}\). \(\Box\)

Suppose \(Z = \{z_n\}\) is a hexagonal lattice and \(\sigma\) is the distance from any point in \(Z\) to its nearest neighbor. When \(r\) is very large, the difference between the number of points \(z_n\) satisfying \(B(z_n, \sigma/2) \cap B(\zeta, r) \neq \emptyset\) and the number of points \(z_n\) satisfying \(z_n \in B(\zeta, r)\) is insignificant. Since the hexagonal circle packing has the largest packing density, a careful examination of the proof above shows that the constant \(\sqrt{(2\pi)/(\sqrt{3} \alpha)}\) in Theorem 4.2.1 is best possible.

A companion result for sampling sequences is the following.
Theorem 4.2.2. Let \( Z = \{z_n\} \) be a sequence of distinct points in the complex plane. If there exists a positive number
\[
\sigma < \sqrt{\frac{2\pi}{3\sqrt{3}\alpha}}
\]
such that \( C = \{S(z_n, \sigma)\} \) is a circle covering for \( C \), then \( Z \) is a sampling sequence for \( F_{\alpha}^p \).

Proof: Suppose that \( C = \{S(z_n, \sigma)\} \) is a circle covering in the complex plane. By Theorem 4.1.2, we have \( \delta(C) \geq (2\pi)/(3\sqrt{3}) \). It follows from the definition of covering density that for any \( \varepsilon > 0 \) there exists a positive number \( R \) such that for all \( r > R \) and all \( \zeta \in \mathbb{C} \) we have
\[
\frac{1}{\pi r^2} \sum \{\pi \sigma^2 : B(z_n, \sigma) \subset B(\zeta, r)\} \geq \frac{2\pi}{3\sqrt{3}} - \varepsilon.
\]
Since \( B(z_n, \sigma) \subset B(\zeta, r) \) implies that \( z_n \in B(\zeta, r) \), we must also have
\[
\frac{1}{\pi r^2} \sum \{\pi \sigma^2 : z_n \in B(\zeta, r)\} \geq \frac{2\pi}{3\sqrt{3}} - \varepsilon.
\]
Rewrite this as
\[
\frac{\pi \sigma^2}{\pi r^2} n(Z \cap B(\zeta, r)) \geq \frac{2\pi}{3\sqrt{3}} - \varepsilon,
\]
or equivalently,
\[
\frac{n(Z \cap B(\zeta, r))}{\pi r^2} \geq \frac{2}{3\sqrt{3} \sigma^2} - \frac{\varepsilon}{\pi \sigma^2}.
\]
Take the infimum over \( \zeta \) and let \( r \to \infty \). We obtain
\[
D^{-}(Z) \geq \frac{2}{3\sqrt{3} \sigma^2} - \frac{\varepsilon}{\pi \sigma^2}.
\]
Since $\varepsilon$ is arbitrary, we must have

$$D^{-}(Z) \geq \frac{2}{3\sqrt{3}\alpha^2}.$$ 

It is then easy to see that the condition

$$\sigma < \sqrt{\frac{2\pi}{3\sqrt{3}\alpha}}$$

implies $D^{-}(Z) > \alpha/\pi$. This along with Theorem 2.2.8 shows that $Z = \{z_n\}$ is a sampling sequence for $F_{\alpha}^{\sigma}$. □

Again, if $\mathcal{C} = \{S(z_n, \sigma)\}$ is an optimal hexagonal circle covering of the complex plane, then for very large $r$, the difference between the number of points $z_n$ satisfying $B(z_n, \sigma) \subset B(\zeta, r)$ and the number of points $z_n$ satisfying $z_n \in B(\zeta, r)$ is negligible. Therefore, the constant $\sqrt{(2\pi)/(3\sqrt{3}\alpha)}$ in Theorem 4.2.2 is best possible.
Bibliography


