On Rudin's pathological submodules over bidisk

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On Rudin’s Pathological Submodules over Bidisk

by

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Abstract

Operator theory has been greatly enriched after the introduction of Hilbert spaces of analytic functions. On the one hand, problems in operator theory can be reformulated and solved via the analytic function space techniques. For example, the (vector-valued) Hardy space over the unit disk is the basic model for the Nagy-Foias’s dilation theory; on the other hand, the studies of analytic function spaces employ tools from algebra, geometry, topology and operator theory, which were interpreted from a new viewpoint.

It is well known that the structure of a submodule of the Hardy space over the unit disk can be determined completely by an inner function. However, inner functions fail to characterize the structure of submodule of the Hardy space over the polydisc. This thesis mainly focuses on Hardy submodules over the bidisk and related operator theory.

First, we study two types of submodules via Beurling-Lax-Halmos theorem— the inner-sequence-based submodules and submodule generated by two inner functions. Two necessary and sufficient conditions about operator-valued inner function \( \Theta \) are obtained. Then we use those results and a spectral formula given by Sz.Nagy and Foias to compute the spectra of \( S_z \) and \( S_w \) on these types of submodules.

Next, we focus on zero-based submodule, which is a submodule generated by a zero set of a \( H^2(\mathbb{D}^2) \) function. We show the Hilber-Schmidtness of submodule \( M \) when the zero set is away from distinguished boundary of \( \mathbb{D}^2 \). In addition, we study a pathological submodule given by W. Rudin that contains no nontrivial bounded function.
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CHAPTER 1

Introduction

The 1949 paper of Beurling built a bridge between operator theory and function spaces, leading to a spate of research in operator theory, $H^p$ theory and other areas. Many problems in operator theory can be reformulated and solved via the analytic function space techniques, while the studies of analytic function spaces employed tools from algebra, geometry, topology and operator theory. Here I mainly focus on the study of a concrete function space $H^2(D^2)$, Hardy space over the bidisk.

Although operator theory lasted for more than one hundred years, multi-variable operator theory is relatively new. One approach is through the study of Hilbert module, a Hilbert space on which an algebra acts. It was firstly introduced by R. Douglas, and then be studied systematically by R. Douglas, V. Paulsen\cite{6} and K.Yan\cite{17}. This module language emphasizes some key problems in the multivariate operator theory from a module theoretical viewpoint and makes clear its connections with geometry and commutative algebra.

A natural example of such Hilbert modules would be the reproducing kernel Hilbert spaces of holomorphic functions on a domain $\Omega$ that is closed under multiplication by function algebra $A$, for example Hardy spaces and Bergman spaces over polynomial algebra. For Hardy space $H^2(D)$, the submodules have a very clear characterization by Beurling’s theorem. However, one immediately finds that $H^2(D^2)$ does not have such a structure. Its function theory was first studied by W. Rudin. As he showed in his book\cite{32}, many techniques from classic analysis do not work efficiently in
several variable case. Lack of effective tools and not knowing what to ex-
pect make even basic problems in multivariable, such as structure of sub-
modules and related operator tuples, hard to solve. Hoping to build a model
for this study, R. Douglas and R. Yang\[29, 30, 19\] as well as many other
researchers\[4, 9, 10, 1, 2, 3, 4, 5, 6, 7, 8\] started a project of building a
systematic operator theory in $H^2(D^2)$. This thesis is a part of it.

The structure of the thesis is as follows:

Chapter 2. We introduce the background, basic conceptions, main tools and
results used in the thesis.

Chapter 3. We study two types of submodules based on the Beurling-Lax-
Halmos theorem-inner-sequence based submodules(first considered by Rudin
in the 1950s) and the submodules generated by two inner functions. Two
necessary and sufficient conditions are obtained. Moreover, we use those
two theorems and a spectral formula given by Sz.Nagy and Foias to com-
pute the spectra of $S_z$ and $S_w$ for these two types of submodules.

Chapter 4. Here we focus on zero-based submodule, that is a submodule
generated by a zero set of a $H^2$ function. We mainly discuss on the Hilber-
Schmidtness of the submodule via the evaluation operator $L(\lambda)|_{M \ominus zM}$. In
particular, when a zero set is away from the distinguish boundary $T^2$, Rudin
showed that there exists a nontrivial $H^\infty(D^2)$ function with the zero set.
And we show that this type of submodule is Hilbert-Schmidt. Here we will
also take a careful look at Rudin’s second pathological submodule, in which
no bounded function exists. We show that it is Hilbert-Schmidt as well.
CHAPTER 2
Background and Preliminary

2.1. Hardy spaces

**Hardy space over the polydisk** We let $\mathbb{C}^n$ denote the Cartesian product of $n$ copies of the complex field. The points of $\mathbb{C}^n$ are thus ordered $n$-tuples $z = (z_1, z_2, \ldots, z_n)$. $\mathbb{D}^n$ will be the unit polydisk in $\mathbb{C}^n$ with distinguished boundary $\mathbb{T}^n$, where $\mathbb{T}$ is the unit circle. The closure of polynomials over $\mathbb{D}^n$ under the supremum norm will be denoted by $A(\mathbb{D}^n)$ and called the polydisk algebra. Let $dm$ be the normalized Lebesgue measure on $\mathbb{T}^n$. The Hardy space $H^2(\mathbb{D}^n)$ is the collection of holomorphic functions over $\mathbb{D}^n$ such that

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(rz)|^2 dm < \infty$$

with norm

$$\|f\|_2 = \sup_{0 \leq r < 1} \left( \int_{\mathbb{T}^n} |f(rz)|^2 dm \right)^{1/2}$$

$H^2(\mathbb{D}^n)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{T}^n} f(z) \overline{g(z)} dm$$

The space $H^\infty(\mathbb{D}^n)$ is defined as the vector space of bounded holomorphic functions on the disk, with the norm

$$\|f\|_\infty = \sup |f(z)|, \ z \in \mathbb{D}^n.$$

It is easily seen that $H^\infty(\mathbb{D}^n)$ is a Banach algebra with pointwise multiplication and addition. By Fatou’s Lemma, given $f \in H^p$, with $p \geq 0$, the radial limit $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ exists for almost every $\theta$. The function $f^*$ belongs to the space $L^p$ on the unit circle, and one has that $\|f^*\|_{L^p} = \|f\|_{H^p}$. 


Therefore, we may identify \( f \) with \( f^* \), and can thus regard \( H^2(\mathbb{D}^n) \) as a closed subspace of \( L^2(\mathbb{T}^n) \), i.e., \( H^2(\mathbb{D}^n) \) and \( H^2(\mathbb{T}^n) \) denote the same space.

**Shift operator** Let \( \varphi \in L^\infty(\mathbb{T}^n) \), the Toeplitz operator with symbol \( \varphi \) is defined by

\[
T_{\varphi}f = P\varphi f, \quad f \in H^2(\mathbb{D}^n)
\]

where \( P \) is the orthogonal projection from \( L^2(\mathbb{T}^n) \) to \( H^2(\mathbb{D}^n) \). In particular, when \( \varphi = z_i \) for some \( 1 \leq i \leq n \), \( T_{z_i} \) is called unilateral shift operator on \( H^2(\mathbb{D}^n) \).

**Inner and outer functions in** \( H^2(\mathbb{D}) \) When \( n = 1 \), the classic Hardy space \( H^2(\mathbb{D}) \) and the shift operator \( T_z \) was studied by many scholars. In 1949, A. Beurling defined the notion of inner function and outer function in \( H^2(\mathbb{D}) \).

**Definition 2.1.1.** One says \( \theta \in H^2(\mathbb{D}) \) is an inner function if \( |\theta(z)| = 1 \) a.e. on \( \mathbb{T} \). A function \( u \in H^2(\mathbb{D}) \) is an outer function if it takes the form

\[
\begin{align*}
u(z) &= c \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\varphi(e^{i\theta})) d\theta \right)
\end{align*}
\]

for some complex number \( c \), and some positive measurable function \( \varphi \) such that \( \log \varphi \) is integrable on \( \mathbb{T} \).

Moreover, it has been shown that \( u \in H^2(\mathbb{T}) \) is an outer function if

\[
\log|u(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|u(e^{i\theta})| d\theta
\]

There are two types of inner functions. Blaschke product is defined as

\[
B(z) = \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}, \quad \text{where} \quad \sum_n (1 - |a_n|) < \infty \text{ holds to ensure the convergence of the product.}
\]

Another type of inner function is the one that has no zeros, called singular inner function. For every singular inner function \( s(z) \), there is a unique positive measure \( \mu \), singular with respect to Lebesgue
measure $d\theta$, and a constant $\alpha$ of modulus 1, such that

$$s(z) = \alpha e^{\exp \left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta} + z \overline{e^{i\theta} - z} d\mu(\theta)\right)}.$$ 

It is well known that every non-zero function $f$ in $H^2(\mathbb{D})$ can be written as a product $f = \theta u$ where $u$ is an outer function and $\theta$ is an inner function.

**Inner and Outer functions in $H^2(\mathbb{D}^n)$** When $n > 1$, the definitions of both inner function and outer function are naturally extended. $\theta$ is an inner function if $\theta \in H^2(\mathbb{D}^n)$ and $|\theta(z)| = 1$ a.e. on $\mathbb{T}^n$. $u$ is an outer function if $u \in H^2(\mathbb{T}^n)$ such that

$$\log|u(0)| = \int_{\mathbb{T}^n} \log|u| dm.$$ 

Although Hardy space on polydisc $H^2(\mathbb{D}^n)$ and its inner and outer functions inherit their definitions from $H^2(\mathbb{D})$, they have much more complicated structure. There are more differences than analogues, and many classical theorems are no longer true. For example,

1. In $H^2(\mathbb{D})$, a function $u$ is outer if and only if $H^2(\mathbb{D}) = [u]$, where $[u] = \text{clos}\{uf : f \in A(\mathbb{D})\}$. However, for $H^2(\mathbb{D}^2)$, it is no longer true. W. Rudin showed that there exists outer function $h \in H^2(\mathbb{D}^n)$ such that $[u] \nsubseteq H^2(\mathbb{D}^n)$ [32].

2. The zero set of $H^2(\mathbb{D})$ function satisfies the Blaschke condition. It has been proved that there is no simple mass measure characterization of zero set of $H^2(\mathbb{D}^n)$ functions in several variables [62].

### 2.2. Submodules of Hardy Space over Bidisk

For a Hilbert space $\mathcal{H}$, denote $B(\mathcal{H})$ be all the bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, a closed subspace $M$ of $\mathcal{H}$ is called invariant subspace of $T$ if $TM \subseteq M$. The following is probably the most famous unsolved problem in Functional Analysis.
The Invariant Subspace Problem (as it stands today): If $T$ is a bounded linear operator on an infinite-dimensional separable Hilbert space $\mathcal{H}$, does it follow that $T$ has a non-trivial invariant subspace?

The study of invariant subspace problem has lasted for more than 70 years. On Hilbert space of finite dimension, the problem is trivial. For infinite dimensional Hilbert space, this problem is so complicated that people turned to operators with special conditions, such as compact operators and normal operators.[2]

In 1949, the classic a paper of A. Beurling[3] initiated an interesting line of research. In the language of Hilbert module[6], the Hardy space $H^2(D)$ is a module over the disk algebra $A(D)$ with multiplication defined pointwise on $T$. A invariant subspace of $T_z$ on $H^2(D)$, which is closed under multiplication by polynomials, is a submodule in $H^2(D)$.

**Beurling Theorem**: A closed subspace $M$ of $H^2(D)$ is a submodule if and only if $M = \theta H^2(D)$ for some inner function $\theta$.

**Beurling-Lax-Halmos Theorem** Later, Nagy-Foias built the general structure of operator model theory. Sz.-Nagy's dilation theorem, proved in 1953, states that for any contraction $T$ on a Hilbert space $\mathcal{H}$, there exists an isometric dilation to a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. Generalizations of Sz.-Nagy-Foias model theory are available to treat contractions by employing the machinery of vector-valued Hardy space $H^2(E)$[8].

**Definition 2.2.1.** Let $E$ be a separable Hilbert space of infinite dimension, then the $E$-valued Hardy space $H^2(E)$ is

$$H^2(E) := \{ u(z) = \sum_{j=0}^{\infty} z^j x_j : \sum_{j=0}^{\infty} \| x_j \|^2_E < \infty, z \in D \}.$$  

One remarkable result on $H^2(E)$ is Beurling-Lax-Halmos theorem, an extension of Beurling’s theorem made by P. Lax[9] for finite dimensional $E$. 

and by P. Halmos\cite{10} for infinite dimensional $E$. The $B(E)$-valued analytic function $\Theta(z)$ on $\mathbb{D}$ is said to be inner if $\|\Theta(z)\| \leq 1$ for each $z \in \mathbb{D}$ and $\Theta(z)$ is an isometry almost everywhere on $\mathbb{T}$ (with respect to the Lebesgue measure on $\mathbb{T}$).

**Theorem 2.2.2. (Beurling-Lax-Halmos Theorem)** $M \subset H^2(E)$ is invariant if and only if there exists a $B(E', E)$-valued inner function $\Theta$ such that $M = \Theta(z)H^2(E')$.

This theorem gives an abstract representation of shift invariant subspaces of general Hardy spaces, including Hardy space on polydiscs. However, in $H^2(\mathbb{D}^n)$, subspaces like $\Theta(z)H^2(E)$ is only invariant for $T_z$ and is not necessarily a submodule.

**Submodule over Bidisk** On $H^2(\mathbb{D}^2)$, a closed subspace $M$ is said to be a submodule if $zM \subset M$ and $wM \subset M$. Although the existence of inner functions in this context is obvious, one quickly sees that a Beurling like characterization is not possible. Let $[X] := \text{span}\{A(\mathbb{D}^2)X\}$ denote the submodule generated by $X \subset H^2(\mathbb{D}^2)$, it is shown that $[z - w]$ can’t be written as $\varphi(z, w)H^2(\mathbb{D}^2)$ for an inner function $\varphi$. The earliest result concerning submodules of $H^2(\mathbb{D}^n)$ is about submodules of finite codimension, due to Ahern and Clark \cite{11}. Let $\mathcal{R}$ denote the ring of all polynomials on $\mathbb{C}^n$.

**Theorem 2.2.3.** Suppose $M$ is a submodule of $H^2(\mathbb{D}^n)$ of codimension $k < \infty$. Then $\mathcal{R} \cap M$ is an ideal in $\mathcal{R}$ such that

(a) $\mathcal{R} \cap M$ is dense in $M$;
(b) $\dim \mathcal{R}/\mathcal{R} \cap M = k$;
(c) the set of common zeros (in $\mathbb{C}^n$) of the members of $\mathcal{R} \cap M$ is finite and lies in $\mathbb{D}^n$.

Conversely, if $I$ is an ideal in $\mathcal{R}$ whose common zeros form a finite subset $V_I$ of $\mathbb{D}^n$, and if $M$ is the closure of $I$ in $H^2(\mathbb{D}^n)$, the $M$ is a submodule of finite codimension, and $I = \mathcal{R} \cap M$. 

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In general, it seems unlikely to obtain a fully characterization of $M$.

**Unitary equivalence and rigidity phenomena** Unitary equivalence is a well-known equivalence relation in set of submodules. Two submodules $M_1$ and $M_2$ are said to be unitary equivalent if there is a unitary module map between them. By Beurling’s theorem submodules of $H^2(\mathbb{D})$ are all unitary equivalent. But it is not true in $H^2(\mathbb{D}^2)$. In [12], Agrawal, Clark and Douglas obtained that two submodules of finite codimension are unitary equivalent only if they are equal. In [13], K. Guo used results of characteristic spaces to completely classified the submodule generated by polynomials.

The ACD result shows that unitary equivalence has the so-called rigidity phenomenon. One typical example of their result is the following

**Example 2.2.4.** For $\lambda=(\lambda_1, \lambda_2) \in \mathbb{D}^2$, let submodule $M_\lambda = \{ f \in H^2 : f(\lambda) = 0 \}$. Then $M_\alpha$ is unitary equivalent to $M_\beta$ if and only if $\alpha = \beta$.

Even though $M_\alpha$ and $M_\beta$ are intuitively the “same type” of submodules, they are not unitary equivalent. Further, Douglas and Yan[16] proved that under some technical hypotheses on the zero varieties, two submodules that are quasi-similar must be equal. By using localization techniques, Douglas, Paulsen, Sah and Yan[21] gave a generalization of the rigidity theorem.

This phenomena indicates that the unitary equivalence, which is very sensitive to perturbations of zero sets, lack the flexibility one might need for a classification of submodules. Other equivalence relations are developed. For instance, R. Yang[23] defined congruence relation, through the core operator. We’ll discuss this in next section.

Another approach to the submodules of $H^2(\mathbb{D}^n)$ is through study of particular submodules. Some examples, which showed how pathological submodule of $H^2(\mathbb{D}^n)$ can be, were constructed in Rudin’s textbook[32]. One of which is a submodule that is infinitely generated. In 2007, M. Seto[33] showed
that such a submodule is generated by a sequence of inner functions. This showed that it in fact has a simple structure. I shall say more about it later in the thesis.

**Other applications** In addition, techniques and viewpoint from the multivariate study have enriched the one variable theory. For example, Bergman space $L^2_a(D)$ can be imbedded in to $H^2(D^2)$\[53\]. J. Hu, S. Sun, X. Xu and D. Yu\[52\] studied the structure of reducing subspace of analytic Toeplitz operator with finite Blaschke product symbol on the Bergman space via techniques from Hardy space over the bidisk. More have been done by S. Sun, D. Zheng and C. Zhong\[54\]. Later S. Sun and D. Zheng\[55\] gave a new proof of the Beurling type theorem for $L^2_a(D)$ by expressing the Bergman shift up as the compression of $(T_z, T_w)$ on the orthogonal complement of a submodule in $H^2(D^2)$. This idea makes $H^2(D^2)$ relevant to invariant subspace problem.

## 2.3. Operators on the Hardy Space over Bidisk

Operator theory on Hilbert spaces has two important components: normal operator theory and non-normal operator theory. A milestone of normal operator theory is the spectral decomposition, which completely characterizes the structure of normal operators. However, the study of non-normal operators is far from complete. One important model for non-normal operators is the isometry operator.

**Wold-von Neumann decomposition theorem** An operator $S \in B(\mathcal{H})$ such that $\|Sx\| = \|x\|$ for any $x \in \mathcal{H}$. Wold-von Neumann decomposition theorem states that every isometry on a Hilbert space is either a shift, or a unitary, or a direct sum of shift and unitary.

**Theorem 2.3.1.** (Wold-von Neumann Decomposition Theorem) Let $S$ be an isometry on Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ has a unique decomposition $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_N$.
\( \mathcal{H}_U \), where \( \mathcal{H}_S \) and \( \mathcal{H}_U \) are \( S \)-reducing subspaces of \( \mathcal{H} \) and \( S|_{\mathcal{H}_S} \) is a shift and \( S|_{\mathcal{H}_U} \) is unitary. Moreover,

\[
\mathcal{H}_S = \bigoplus_{k=0}^{\infty} S^k \mathcal{W}, \quad \mathcal{H}_U = \bigcap_{k=0}^{\infty} S^k \mathcal{H}.
\]

where \( \mathcal{W} = \text{ran}(I - SS^*) \) is the wandering subspace for \( S \).

Let \( M \) be a submodule in \( H^2(\mathbb{D}) \), \( N = H^2(\mathbb{D}) \ominus M \) be the quotient module of \( M \). Then for \((z, w) \in \mathbb{D}^2\), \( T_z^* N \subset N \) and \( T_w^* N \subset N \). \( N \) is called a backward shift invariant subspace of \( H^2 \). By Wold-von Neumann decomposition,

**Theorem 2.3.2.** Let \( M \) be a submodule of \( H^2(\mathbb{D}^n) \), denote \( M_i = M \ominus z_i M \), \( M_i^\infty = \bigcap_{k=1}^{\infty} f \in M : \bar{z}_i^k f \in M, i = 1, 2 \), then

\[
M_i^\infty = 0, \quad M = \sum_{k=0}^{\infty} \bigoplus T_{z_i}^k M_i, i = 1, 2
\]

**Compressions on submodule and quotient submodule**

**Definition 2.3.3.** Let \( R_z \) and \( R_w \) be the compression of \( T_z \) and \( T_w \) to \( M \), \( S_z \) and \( S_w \) be the compression of \( T_z \) and \( T_w \) to \( N \), e.g.

\[
R_z = P_M T_z |_M; \quad R_w = P_M T_w |_M;
\]

\[
S_z = P_N T_z |_N; \quad S_w = P_N T_w |_N.
\]

It is easy to see that \((R_z, R_w)\) is a commuting isometric pair, \((S_z, S_w)\) is a commuting contraction tuple. The cross commutator \([R_z^*, R_w]\) and the self commutators \([R_z^*, R_z]\), \([R_w^*, R_w]\) capture key information about the submodule \( M \). Here for two operators \( S \) and \( T \), \([S, T] = ST - TS\).

For the study of cross commutator, Mandrekar\[37\] proved that \([R_z^*, R_w]\) = 0 if and only if \( M \) is of Beurling type, that is, \( M = \theta H^2 \) for some two variable inner function \( \theta \). In \[30\], Yang showed that \([R_z, R_w^*]\) is Hilbert-Schmidt.
under a mild condition. In [41, 42], K. Izuchi studied submodules \( M \) satisfying \( \text{rank}[R_z, R_w^*] = 1 \). And it was shown that \([S_z^*, S_w] = 0\) if and only if \( M = \theta_1 H^2 + \theta_2 H^2(\text{cf [31]}), \) for one-variable inner functions \( \theta_1(z) \) and \( \theta_2(w) \).

**Core operator** However, as manifested by Rudin, Beurling type submodules are few and very special. Hence attention is directed to find intrinsic notions appropriate of characterizing general submodules.

Denote the reproducing kernel of \( H^2(D) \) as \( K(\lambda, \eta; z, w) = \frac{1}{(1-\lambda z)(1-\eta w)} \) and the reproducing kernel of \( M \) as \( K^M(\lambda, \eta; z, w) \), then the core function is \( G^M(\lambda, \eta; z, w) \) for \( M \) is defined [23] by

\[
G^M(\lambda, \eta; z, w) := \frac{K^M(\lambda, \eta; z, w)}{K(\lambda, \eta; z, w)}
\]

and the core operator on \( H^2(D) \) is given by

\[
C^M(f)(z, w) := \int_{\mathbb{T}^2} G^M(\lambda, \eta; z, w)f(\lambda, \eta)d\lambda d\eta, \quad z \in \mathbb{T}^2,
\]

Two submodules \( M_1 \) and \( M_2 \) are congruent if there is a bounded invertible operator \( J \) from \( M_2 \) to \( M_1 \) such that \( C^M_1 = JC^M_2J^* \). It was shown that this equivalence relation effectively classifies submodules with a finite rank core operator [25, 30]. In [50], one relation between core operator \( C^M \) and the pair \((R_z, R_w)\) is displayed by the formula:

\[
C = 1 - R_z R_z^* - R_w R_w^* + R_z R_w R_z^* R_w^*
\]

A submodule \( M \) is said to be Hilbert–Schmidt if its core operator \( C \) is Hilbert–Schmidt. It is worth pointing out that it was shown in [24] that almost all known examples of submodules in \( H^2(D) \) is Hilbert-Schmidt.

**Spectrum of \( S_z \) and \( S_w \)** The joint spectrum of \((S_z, S_w)\) is another important subject. In \( H^2(D) \), the compression \( S(\theta) \) of the unilateral shift to the quotient space \( N = H^2(D) \ominus \theta H^2(D) \) is called a Jordan block. It is known that the spectrum and essential spectrum of \( S(\theta) \) is determined by zeros and singular
measure of inner function $\theta$, and the Fredholm index is always 0. Two variable analogue of the Jordan block in the Hardy space over the bidisk is given by the pair $(S_z, S_w)$. It is then natural to ask if the similar property holds.

The classical spectra of $S_z, S_w$ relates to other problems. The Berger-Shaw theorem states that an operator $T$ which is hyponormal and possesses a finite rational cyclic set has a trace-class self commutator $[T^*, T]$, and the trace of $[T^*, T]$ is controlled by the spectrum of $T$. In multi-variable operator theory, efforts have been made to generalize of Berger-Shaw theorem. For $(R_z, R_w)$, properties of its commutators were used to define two numerical invariants for submodules, namely

$$\Sigma_0 = \| [R_z^*, R_z] [R_w^*, R_w] \|_{H.S.}^2,$$
$$\Sigma_1 = \| [R_z^*, R_w] \|_{H.S.}^2.$$

where $\| \cdot \|_{H.S.}$ means the Hilbert-Schmidt norm. It is easy to see that $\Sigma_0$ and $\Sigma_1$ are invariants under unitary equivalence of submodules. Let $\sigma_c(A)$ be the continuous spectrum of operator $A$, e.g., $\lambda \notin \sigma_c(A)$ if and only if $A - \lambda I$ has closed range and finite dimensional kernel. R.Yang proved that if $\sigma_c(S_z) \cap \sigma_c(S_w) \neq \emptyset$, both $[R_z^*, R_z] [R_w^*, R_w]$ and $[R_z^*, R_w]$ are Hilbert-Schmidt. Further, since the spectrum(or joint spectrum) of $S_z$ and $S_w$ are known to be connected with the zero set of $M$, it would be interesting to study $\sigma(S_z, S_w)$ and $\sigma_c(S_z, S_w)$ for submodules generated by zero sets.

**Evaluation operator** The evaluation operator is very important in the study. Let $H_z^2$ and $H_w^2$ be the Hardy spaces with variable $z$ and $w$ respectively. Then for $\lambda \in \mathbb{D}$, we define the left evaluation $L(\lambda)$ from $H^2(\mathbb{D}^2)$ to $H_w^2$ and the right evaluation $R(\lambda)$ from $H^2(\mathbb{D}^2)$ to $H_z^2$

$$L(\lambda)f(z, w) = f(\lambda, w), \quad R(\lambda)f(z, w) = f(z, \lambda), \quad f \in H^2(\mathbb{D}^2)$$

It was shown in [23, 25] that evaluation operators play important role in the study of compression operators, which we will state in chapter 4.
There is an observation we need to make now.

For submodule $M$ in $H^2(D^2)$,

$$M = \sum_{n=0}^\infty \oplus z^n(M \ominus zM)$$

For a fixed $\lambda \in D$ and $f \in M$, we write $f = \sum_{n=0}^\infty z^n f_n$ for some $\{f_n\}$ in $M \ominus zM$, then

$$f = \sum_{n=0}^\infty \oplus \lambda^n f_n + \sum_{n=0}^\infty \oplus (z^n - \lambda^n) f_n$$

means $f = g_1 + (z - \lambda)g_2$ for some $g_1 \in M \ominus zM$ and $g_2 \in M$. Therefore,

$$M = (M \ominus zM) + (z - \lambda)M.$$ 

So we have

(3.3.1) \quad \text{ran}(L(\lambda)|_{M \ominus zM}) = \text{ran}(L(\lambda)|_M).
CHAPTER 3

Submodule over Bidisk via BLH Theorem

In this chapter, we make a study of submodules of $H^2(D^2)$ by Beurling-Lax-Halmos theorem. First, we recall Beurling-Lax-Halmos theorem and its spectral implication. Then we fully characterize two types of submodules. Lastly, we use those characterization to obtain the spectral of $S_z$ and $S_w$.

3.1. Submodules over Bidisk and some examples

Recall BLH theorem states that

Compression of $T_z$ to the quotient space $H^2(E) \ominus M$ shall be denoted by $S_z$. It is a classical theorem that every bounded linear operator on a separable Hilbert space is of the form $S_z$ up to a scalar multiple and unitary equivalence. There exists a spectral connection.

Let $G_1 = \{ \lambda \in \mathbb{D} : \Theta(\lambda) \text{ is not invertible} \}$, and $G_2$ be the collection of $\lambda \in \partial \mathbb{D}$ such that $\Theta$ has analytic no extension to a neighborhood $U$ of $\lambda$ such that $\Theta$ is unitary-valued on $U \cap \partial \mathbb{D}$. Then the spectrum of $\Theta$ is defined as $\sigma(\Theta) = G_1 \cup G_2$. By the model theory of Nagy and Foias, it follows

**Lemma 3.1.1.** $\sigma(S_z) = \sigma(\Theta)$.

When $E = H^2(D)$, $H^2(E)$ is the Hardy space over the bidisk $H^2(D^2)$. Let $H^2_z$ (or $H^2_w$) be the Hardy space with variable $z$ (or $w$). Recall that a submodule over bidisk is a closed subspace $M \subset H^2(D^2)$ that is invariant for both $T_z$ and $T_w$. Let $\mathbb{C}[z, w]$ be the polynomial ring.

**Example 3.1.2.** *If $J$ is an ideal in $\mathbb{C}[z, w]$, then $\overline{J}$ is a submodule.*
The second example is the so-called inner-sequence based submodules, originally considered by Rudin in late 1950s.

**Example 3.1.3.** Let $M$ the submodule consisting of all functions in $H^2$ which have a zero of order greater than or equal to $n$ at $(\alpha_n, 0) = (1 - n^{-3}, 0)$ for every positive integer $n$. M. Seto and R. Yang\[35] showed that $M$ is generated by a sequence of inner functions:

$$M = \bigoplus_{j=0}^{\infty} q_j(z) H^2_z w$$

where we set $b_n(z) = (\alpha_n - z)/(1 - \alpha_n z)$, $q_0(z) = \prod_{n=1}^{\infty} b^n_n(z)$ and $q_j(z) = q_{j-1}(z)/\prod_{n=j}^{\infty} b_n(z)$ for any positive integer $j$.

**Definition 3.1.4.** An infinite sequence of inner functions \{\phi_j(z)\}_{j \geq 0} in $H^2_z$ is called an nontrivial inner sequence if it satisfies one of the following conditions:

1. \{\phi_j(z)\}_{j \geq 0} is decreasing, that is, every $\phi_j/\phi_{j+1}$ is a nonconstant inner function;
2. \{\phi_j(z)\}_{j \geq 0} is increasing, that is, every $\phi_{j+1}/\phi_j$ is a nonconstant inner function;

**Lemma 3.1.5.** Given a decreasing inner sequence \{\phi_j(w)\}_{j \geq 0}, define

(3.1.1) $$M = \bigoplus_{l=0}^{\infty} \phi_l(w) H^2_w z^l.$$ 

Then $M$ is a submodule.

This form of definition and a general study were made in [35]. They showed that this type of invariant subspace has direct connections with the so-called $C_0$ class contractions.

Moreover, one can define two-inner-sequence-based submodules.
Lemma 3.1.6. Let \( \{\varphi_j(z)\}_{j \geq 0} \) be a decreasing inner sequence and \( \{\phi_j(w)\}_{j \geq 0} \) be an increasing inner sequence. Then

\[
M = \sum_{j=1}^{\infty} \varphi_j(z) H^2_z \otimes (\phi_j(w) H^2_w \ominus \phi_{j+1}(w) H^2_w)
\]

and

\[
M = \sum_{j=1}^{\infty} (\varphi_j(z) H^2_z \ominus \varphi_{j-1}(z) H^2_z) \otimes \phi_j(w) H^2_w
\]

are submodules.

Another type of submodule we’ll focus on is generated by two inner functions.

**Lemma 3.1.7.** Suppose \( \phi_1(z) \) and \( \phi_2(w) \) are two inner functions (or 0) with variables \( z \) and \( w \) respectively. Define

\[
M = \phi_1(z) H^2(D^2) + \phi_2(w) H^2(D^2)
\]

Then \( M \) is a submodule.

This submodule emerges in the work of Izuchi, Nakazi and Seto in [43] as they study the commutator \( [S_z^*, S_w] \). They showed that a submodule is of this form if and only if \( [S_z^*, S_w] = 0 \). R. Yang used this type of submodule to address several issues regarding the unitary equivalence of submodules.

### 3.2. Inner-sequence-based submodules

Now we focus on inner-sequence-based submodules. Recall that by BLH theorem every submodule can be written as \( \Theta(z) H^2(E) \), where \( E = H^2_w \) in the setting of \( H^2(D^2) \). Notice that \( \Theta(z) \) is analytic on \( \mathbb{D} \). So it has power series representation as \( \Theta(z) = \sum_{l=0}^{\infty} z^l P_l \), where \( z \in \mathbb{D} \) and \( P_l \in B(H^2_w) \). For an inner sequence \( \{\phi_l(w) : l \geq 0\} \), we let \( M = \bigoplus_{l=0}^{\infty} \phi_l(w) H^2_w z^l \).
One note needs to be made here. First, if we let $\phi_\infty$ be the greatest common divisor of \{\phi_l(w) : l \geq 0\}, then up to some scalar normalization, the sequence \{\phi_l(w) : l \geq 0\} converges to $\phi_\infty$ in $H^2_w$. If $\phi_\infty$ is nontrivial then we can write $M = \phi_\infty(w)M'$, where $M'$ is an inner-sequence-based submodule with the inner sequence having greatest common divisor equal to 1. Since $M$ and $M'$ are unitary equivalent, without loss of generality we assume that $\phi_\infty = 1$. The following theorem completely characterizes inner-sequence-based submodules in terms of $\Theta(z)$.

**Theorem 3.2.1.** Let $\Theta(z) = \sum_{l=0}^\infty z^l P_l$ be an operator valued analytic function and $M = \Theta(z)H^2(T^2)$ is a submodule. Then $M$ is inner-sequence-based if and only if $P_l$, $l \geq 0$, are orthogonal projections on $H^2_w$ with perpendicular ranges.

**Proof.** $\Rightarrow$: Since $M = \bigoplus_{l=1}^\infty \phi_l(w)H^2_w z^l$, it’s easy to compute that

$$M \ominus zM = \phi_0 H^2_w \bigoplus \bigoplus_{l=1}^\infty z^l (\phi_l H^2_w \ominus \phi_{l-1} H^2_w).$$

For simplicity, we let $N_l = \phi_l H^2_w \ominus \phi_{l-1} H^2_w$, $l \geq 1$. Let $P_l$ be the orthogonal projection from $H^2_w$ onto $N_l$, $l \geq 1$, $P_0$ be the orthogonal projection from $H^2_w$ onto $\phi_0 H^2_w$, and set $\Theta(z) = \sum_{l=0}^\infty z^l P_l$. Two observations need to be made here. First, if there is a non-negative integer $k$ such that $\phi_j(w)H^2_w = \phi_k(w)H^2_w$ for any $j \geq k$, then $P_j = 0$ for $j \geq k$, and in this case $\Theta(z)$ is a polynomial. Second, by the assumption that $\phi_\infty = 1$,

$$H^2_w = \phi_0 H^2_w \bigoplus \bigoplus_{l=1}^\infty N_l,$$

hence $\sum_{l=0}^\infty P_l$ converges strongly to the identity operator $I$. We now show that $\Theta(z)$ is an isometry a.e. on $T$. Check that for any $g(w) \in H^2_w$, $\Theta(z)g =$
\[ \sum_{l=0}^{\infty} z^l P_l g, \text{ and} \]

\[
\| \Theta(z) g(w) \|_2^2 = \sum_{l=0}^{\infty} \| P_l g(w) \|_2^2 \\
= \| \sum_{l=0}^{\infty} P_l g(w) \|_2^2 \\
= \| g(w) \|_2^2.
\]

This concludes that \( \Theta(z) \) is an operator inner function. Further, since \( zM = \Theta(z) H^2_w, M = \Theta(z) H^2(\mathbb{T}^2) \).

\( \Leftarrow: \) Suppose \( P_l, l \geq 0, \) are orthogonal projections on \( H^2_w \) with perpendicular ranges, and \( \Theta(z) = \sum_{l=0}^{\infty} z^l P_l \). Then

\[
M = \Theta(z) H^2(\mathbb{D}^2) \\
= \sum_{l=0}^{\infty} z^l P_l (\bigoplus_{k=0}^{\infty} z^k H^2_w) \\
= \sum_{n=0}^{\infty} z^n \bigoplus_{l=0}^{n} P_l H^2_w). \tag{3.2.1}
\]

Since \( T_w, M \subset M \), each closed subspace \( \bigoplus_{l=0}^{n} P_l H^2_w, \ n \geq 0 \) is an invariant subspace for \( T_w \). By Beurling’s Theorem, there exists an inner function \( \phi_n(w) \) such that

\[
\bigoplus_{l=0}^{n} P_l H^2_w = \phi_n(w) H^2_w.
\]

Clearly, \( \phi_n|\phi_{n-1}, \ n \geq 1, \) hence \( \{ \phi_n \} \) is an inner sequence, and by (3.2.1),

\[
M = \bigoplus_{n=0}^{\infty} z^n \phi_n H^2_w.
\]

\( \square \)

Since \( M \) is also invariant under \( T_w \), there is an operator inner function \( \Gamma(w) \) such that \( M = \Gamma H^2(\mathbb{T}^2), \) where \( \Gamma(\lambda) \) is an operator from \( H^2_z \) to \( M \ominus wM \) for each \( \lambda \in \mathbb{D} \). This \( \Gamma \) can also be determined.
Theorem 3.2.2. Let $M = \bigoplus_{l=0}^{\infty} \phi_l(w)H^2_wz^l$. If $\Gamma(w)$ is the operator inner function such that $M = \Gamma(w)H^2(\mathbb{D}^2)$, then $\Gamma(w) = \sum_{l=0}^{\infty} \phi_l(w)P_l$, where $P_l$ is the projection from $H^2_z$ to $\mathbb{C}z^l$.

Proof. It’s not hard to verify that $M \ominus wM = \bigoplus_{l=0}^{\infty} \phi_l(w)\mathbb{C}z^l$. So if one defines

$$\Gamma(w) = \sum_{l=0}^{\infty} \phi_l(w)P_l,$$

then $\Gamma(w)H^2_z = M \ominus wM$, and hence $M = \Gamma(w)H^2(\mathbb{D}^2)$. Further, for every $g \in H^2_z$,

$$\|\Gamma(w)g\|^2 = \sum_{l=0}^{\infty} |\phi_l(w)|^2 \|P_l g\|^2$$

$$= \sum_{l=0}^{\infty} \|P_l g\|^2$$

$$= \|g\|^2$$

for almost every $w \in \partial \mathbb{D}$. This verifies that $\Gamma$ is inner. $\Box$

Y. Yang made a study of two inner-sequence-based submodule [36]. It can be checked by using analogous technique that it has similar structure.

Let $M = \sum_{j=1}^{\infty} \varphi_j(z)H^2_z \otimes (\phi_j(w)H^2_w \ominus \phi_{j+1}(w)H^2_w)$, where $\{\varphi_j(z)\}_{j \geq 0}$ is decreasing inner sequence and $\{\phi_j(w)\}_{j \geq 0}$ is increasing inner sequence. Here we assume $\varphi_\infty(z) = 1$ and $\phi_0(w) = 1$. Then

$$M \ominus zM = \sum_{j=0}^{\infty} \mathbb{C} \cdot \varphi(z) \otimes (\phi_j(w)H^2_w \ominus \phi_{j+1}(w)H^2_w)$$

$$M \ominus wM = \sum_{j=0}^{\infty} (\varphi_j(z)H^2_z \ominus \varphi_{j-1}(z)H^2_z) \otimes \mathbb{C} \cdot \phi_j(w)$$

Using similar strategy as in Theorem 3.2.1 and 3.2.2, Y. Yang obtain the explicit formula of $\Theta(z)$ and $\Phi(w)$.

Theorem 3.2.3.
(1) Let $\Theta(z) = \sum_{j=0}^{\infty} \varphi_j(z)P_j$ and $M = \Theta(z)H^2(D^2)$. $M$ is a submodule of form (3.1.2) for an increasing inner sequence $\{\phi_j\}_{j=0}^{\infty}$ if and only if $\{P_j\}$ are orthogonal projections on $H_w^2$ such that

$$P_j : H_w^2 \rightarrow \phi_j(w)H_w^2 \ominus \phi_{j+1}H_w^2;$$

(2) $M$ is a submodule of form (3.1.2), then $M = \Gamma(w)H^2(D^2)$ for $\Gamma(w) = \sum_{j=0}^{\infty} \phi_j(w)Q_j$ where $\{Q_j\}$ are orthogonal projections on $H_z^2$ such that

$$Q_j : H_z^2 \rightarrow \varphi_j(z)H_z^2 \ominus \varphi_{j-1}(z)H_z^2.$$

3.3. Submodules generated by two inner functions

Suppose $\phi_1(z)$ and $\phi_2(w)$ are two inner functions with variables $z$ and $w$ respectively. In this section we consider the submodule of the form (3.1.3).

**Theorem 3.3.1.** Let $\Theta(z)$ be the operator inner function for a submodule $M$. Then $M$ is of the form (3.1.3) if and only if $\Theta(z) = \phi_1(z)P_0 + P_1$, where $P_0$ and $P_1$ are complementary projections on $H_w^2$, e.g., $P_0P_1 = 0$ and $P_0 + P_1 = I$.

**Proof.** Suppose $M$ is of the form (3.1.3). In [31], it was shown that

$$M \ominus zM = \phi_1(z)(H_w^2 \ominus \phi_2(w)H_w^2) \oplus \phi_2(w)H_w^2.$$

Set $\Theta(z) = \phi_1(z)P_0 + P_1$, where $P_0 : H_w^2 \rightarrow H_w^2 \ominus \phi_2(w)H_w^2$ is the orthogonal projection and $P_1 = I - P_0$. Then for every $g \in H_w^2$ by Pythagorean theorem

$$\|\Theta(z)g\|^2 = |\phi_1(z)|^2\|P_0g\|^2 + \|P_1g\|^2$$

$$= \|P_0g\|^2 + \|P_1g\|^2$$

$$= \|g\|^2$$
a.e. on $\mathbb{T}$. This shows that $\Theta$ is an operator inner function. Further, it is not hard to check $\Theta(z)H^2_w = M \ominus zM$, and hence

$$M = \bigoplus_{n=0}^{\infty} z^n(M \ominus zM) = \Theta(z)H^2(\mathbb{D}^2).$$

On the other hand, suppose $\Theta(z) = \phi_1(z)P_0 + P_1$, where $\phi_1$ is inner, and $P_0$ and $P_1$ are complementary projections on $H^2_w$. Then

$$M = \Theta(z)H^2(\mathbb{D}^2) = \phi_1(H^2_z \otimes P_0H^2_w) \oplus (H^2_z \otimes P_1H^2_w).$$

(3.3.1)

First, we will show that $T_w$ and $P_1$ commute on $M$. Denote $M_0 = \phi_1H^2_z \otimes P_0H^2_w$ and $M_1 = H^2_z \otimes P_1H^2_w$, and let $P_{M_0}$ and $P_{M_1}$ stand for the projections from $H^2(\mathbb{D}^2)$ to $M_0$ and $M_1$ respectively. Then, with respect to the decomposition (2.3.1), we rewrite $T_w$ on $M$ as

$$T_w = \begin{pmatrix} P_{M_0}T_wP_{M_0} & P_{M_0}T_wP_{M_1} \\ P_{M_1}T_wP_{M_0} & P_{M_1}T_wP_{M_1} \end{pmatrix}.$$ 

Since $M$ is invariant under $T_w$, we have

$$\begin{pmatrix} P_{M_0}T_wP_{M_0} & P_{M_0}T_wP_{M_1} \\ P_{M_1}T_wP_{M_0} & P_{M_1}T_wP_{M_1} \end{pmatrix} \begin{pmatrix} \phi_1M_0 \\ M_1 \end{pmatrix} \subset \begin{pmatrix} \phi_1M_0 \\ M_1 \end{pmatrix}$$

i.e.

(3.3.2)

$$\begin{pmatrix} \phi_1P_{M_0}T_wM_0 + P_{M_0}T_wM_1 \\ \phi_1P_{M_1}T_wM_0 + P_{M_1}T_wM_1 \end{pmatrix} \subset \begin{pmatrix} \phi_1M_0 \\ M_1 \end{pmatrix}.$$ 

Consider the first line in (3.3.2). It is clear that $\phi_1P_{M_0}T_wM_0 \subset \phi_1M_0$, and hence $P_{M_0}T_wM_1 \subset \phi_1M_0$. Check that $P_{M_0}T_wM_1 = H^2_z \otimes P_0wP_1H^2_w$, and $\phi_1M_0 = \phi_1H^2_z \otimes P_0H^2_w$. Therefore, since $\phi_1$ is nontrivial, the first inclusion in (3.3.2) holds only if $P_0wP_1H^2_w = \{0\}$. This implies $T_wP_1H^2_w \subset P_1H^2_w$. By Beurling Theorem, there exists an inner function $\phi_2(w)$ such that $P_1H^2_w = \phi_2(w)H^2_w$. 

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Thus

\[ M = \phi_1(H_z^2 \otimes P_0H_w^2) \oplus (H_z^2 \otimes \phi_2H_w^2) = \phi_1H^2(\mathbb{D}^2) + \phi_2H^2(\mathbb{D}^2). \]

Because of symmetry, the operator inner function \( \Gamma(w) \) for \( M \) in the variable \( w \) is \( P_0 + \phi_2(w)P_1 \).

\[ \square \]

### 3.4. Some applications of BLH characterization

For a submodule \( M \), the quotient module \( N = H^2 \ominus M \) is defined as the orthogonal complement of \( M \) in \( H^2(\mathbb{D}^2) \). Recall that \( S_z \) (respectively \( S_w \)) denote the compression of \( T_z \) (respectively \( T_w \)) to \( N \), i.e. \( S_z = P_NT_z|_N \) (respectively \( S_w = P_NT_w|_N \)), where \( P_N \) denotes the orthogonal projection from \( H^2(\mathbb{D}^2) \) onto \( N \). Lemma 3.1.2 indicates that the spectra of \( S_z \) and \( S_w \) can be calculated through the corresponding operator inner functions \( \Theta(z) \) and \( \Gamma(w) \). This section will make a use of the results in Sections 2 and 3 to this end. All the results in this section are known\(^{34} \). But the work here will show how elegantly these results can be obtained through operator inner functions.

**Proposition 3.4.1.** Let \( M \) be an inner-sequence-based invariant subspace defined in (3.1.1), then \( \sigma(S_z) \) is either \( \{0\} \) or \( \mathbb{D} \), and \( \sigma(S_w) = \sigma(\phi_0) \).

**Proof.** For \( \lambda \in \mathbb{D} \), \( \lambda \) is in \( \sigma(S_z) \) if and only if \( \Theta \) is not invertible at \( \lambda \). We know by Theorem 3.2.1 that \( \Theta(z) = \sum_{l=0}^{\infty} z^lP_l \) with \( P_l \) orthogonal. There are two cases. First, if there is a non-negative integer \( k \) such that \( P_l = 0 \) for all \( l \geq k \), then \( \Theta(z) \) is a polynomial, and by (3.2.1)

\[ M = \sum_{n=0}^{k-1} z^n \left( \bigoplus_{l=0}^{n} P_lH_w^2 \right) \oplus z^kH^2(\mathbb{D}^2). \]

This implies that \( (S_z)^k = 0 \), hence \( \sigma(S_z) = \{0\} \). Second, if \( \Theta(z) \) is not a polynomial, then \( \Theta(\lambda) \) is not invertible for all \( \lambda \in \mathbb{D} \), because we can select \( \{e_n\}_n \) with \( e_n \in \text{Range}(P_n) \) and \( \|e_n\| = 1 \) (or 0 if \( P_n = 0 \)), then

\[ \|\Theta(\lambda)e_n\| = |\lambda^n| \to 0. \]
Now since $D$ is dense in $\overline{D}$ and $\sigma(S_z)$ is closed, we have $\overline{D} \subseteq \sigma(S_z)$. Also since $S_z$ is a contraction, $\sigma(S_z) \subseteq \overline{D}$. Therefore $\sigma(S_z) = \overline{D}$.

For $\sigma(S_w)$, we recall that $\Gamma(w) = \sum_{m=0}^{\infty} \phi_m(w)P_m$, where $P_m$ is the orthogonal projection from $H^2_z$ to $\mathbb{C}z^m$. So for a point $w$ in $D$, $\Gamma(w)$ is invertible if and only if $|\phi_m(w)|$ is bounded below by some positive constant for all $m \geq 0$. Since $|\phi_m(w)| \geq |\phi_0(w)|$ for each $m$, this means $\phi_0(w) \neq 0$. For $w \in \partial D$, $w \notin G_2$ (defined above Lemma 3.1.1) if and only if $\phi_m$ has an analytic continuation to a neighborhood of $w$ for each $m$. Clearly, this is so if and only if $\phi_0$ has an analytic continuation to a neighborhood of $w$. In conclusion, $\sigma(S_w) = \sigma(\Gamma) = \sigma(\phi_0)$. □

For the two-inner-sequence-based submodule $M$ of form (3.1.2), if $\{ \phi_j(w) \}$ is an infinite inner sequence satisfying $\phi_{j+1}/\phi_j$ is an nontrivial inner function for any $j \geq 0$, then $\lim_{j \to \infty} \phi_j(\lambda) = 0$ for any $\lambda \in D$.

Since $\Theta(z) = \sum_{j=0}^{\infty} \varphi_j(z)P_j$ where $P_j$ are orthogonal projections on $H^2_w$ with $\sum_{j=0}^{\infty} P_j = 1$, $\Gamma(z) = \sum_{j=0}^{\infty} \phi_j(w)Q_j$ where $Q_j$ are orthogonal projections on $H^2_z$ with $\sum_{j=0}^{\infty} Q_j = 1$, by the proof in Proposition 3.4.1 we have

**Proposition 3.4.2.** Let $M$ be the two-inner-sequences-based submodule of form (3.1.2), then $\sigma(S_z) = \sigma(\varphi_0)$ and $\sigma(S_w) = \overline{D}$ if $\phi_{j+1}/\phi_j \neq 1$ for any $j \geq 0$, $\sigma(S_w) = \sigma(\phi_N)$ if $\phi_j = \phi_N$ for $j \geq N$.

Next, we look at the submodule generated by two inner functions. In this case since $\Theta(z) = \phi_1P_0 + P_1$ and $\Gamma(w) = P_0 + \phi_2(w)P_1$ for some complementary projections $P_0$ and $P_1$, the the next proposition is immediate.

**Proposition 3.4.3.** Let $M$ be a submodule of the form (3.1.2). Then $\sigma(S_z) = \sigma(\phi_1)$ and $\sigma(S_w) = \sigma(\phi_2)$.

**Remark 3.4.4.** In this chapter, one sees that if the structure of $M \ominus zM$ is clear, it is possible to write $\Theta(z)$ explicitly. However, this is hard to tackle in general. By Beurling-Lax-Halmos, $M$ is an invariant subspace of $H^2_{H^2(D)}(\mathbb{D})$.
if and only if $M = \Theta H^2_E(\mathbb{D})$ for some closed subspace $E \subseteq H^2(\mathbb{D})$ and inner function $\Theta \in H^\infty_{L(E,H^2(\mathbb{D}))}$. A question arises naturally, given the closed subspace $E \subseteq H^2_w$ for what inner function $\Theta$ the invariant subspace $M = \Theta H^2(\mathbb{D}^2)$ is a submodule, i.e. invariant of $T_w$ as well? This question was partially answered in , where the author made a classification for the class of co-doubly commuting submodules of $H^2(\mathbb{D}^2)$.
CHAPTER 4

Zero-Based Submodule and Rudin’s Second Example

We begin this chapter by introducing basic concepts of zero-based submodule and the joint spectrum of operator tuple \((S_z, S_w)\). Then we mainly focus on Rudin’s example, a submodule in \(H^2(\mathbb{D}^2)\) contains no bounded functions. We show that under a mild condition such a pathological submodule is also Hilbert-Schmidt.

4.1. Zero-based Submodule

For holomorphic function \(f\), let \(Z(f)\) be the set of zeros of \(f\). A subset \(Z \subseteq \mathbb{D}^2\) is called a zero set of \(H^2(\mathbb{D}^2)\) if there is a nontrivial function \(f \in H^2(\mathbb{D}^2)\) such that \(Z = Z(f) \cap \mathbb{D}^2\). For a zero set \(Z\), let \(H_Z = \{h \in H^2(\mathbb{D}^2) | h(z, w) = 0 \text{ for } (z, w) \in Z\}\). Clearly, \(H_Z\), called zero-based submodule, is a nontrivial submodule in \(H^2(\mathbb{D}^2)\).

**Example 4.1.1.** Let \(M = [z - w]\). Then it is a zero based submodule with zero set \(Z = \{(\lambda, \lambda) : \lambda \in \mathbb{D}^2\}\).

It is well known that the zero sets \(Z(f)\) of \(H^p(\mathbb{D})\) function \(f\) are the same for all the \(p \geq 1\): \(Z \subseteq \mathbb{D}\) is a zero set for some \(f \in H^p(\mathbb{D})\) if and only if \(Z\) satisfies the Blaschke condition. The analogous question of two variable case were naturally considered and \([62]\) indicates that the holomorphic \(H^p(\mathbb{D}^2)\) spaces have different zero set for different values of \(p\). Many attempts has been made to characterize the zero sets by different points of views, such as geometric conditions of a zero set.
Example 4.1.2. Rudin [32] showed a sufficient condition for zero set of \( H^\infty(\mathbb{D}^2) \). Assume \( Z = Z(f) \) for some holomorphic function \( f \) in \( \mathbb{D}^2 \), and assume that no point of \( \mathbb{T}^2 \) is a limit point of \( Z \). Then there exists \( F \in H^\infty(\mathbb{D}^2) \) such that

1. \( Z = Z(F) \)
2. \( \frac{1}{F} \) is bounded near \( \mathbb{T}^2 \). The requirement of \( Z \) being away from distinguished boundary \( \mathbb{T}^2 \) is equivalent to: \( \exists r < 1 \) so that \( \text{dist}(Z, Q^2) > 0 \), where \( Q = \{ \lambda : r < |\lambda| < 1 \} \).

Other than above example, very little results were obtained for \( H^p(\mathbb{D}^2) \). In single variable case, for an inner function \( \theta \) and a submodule \( \theta H^2(\mathbb{D}) \) the Jordan block \( S(\theta) \) is defined on the quotient module \( N_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D}) \) by

\[
S_\theta f = P_\theta z f, \quad f \in N_\theta
\]

where \( P_\theta \) is the orthogonal projection from \( H^2(\mathbb{D}) \) onto \( N_\theta \). It is known that spectrum of \( S_\theta \) is determined by zeros of \( \theta \) and singular measure of \( \theta \) (if any).

For submodules on bidisk, we call \((S_z, S_w)\) the two variable Jordan block. One purpose of study of joint spectrum of \((S_z, S_w)\) on zero based submodule is to understand the zero set of \( H^2 \) function.

Let’s first give the definition of the joint spectrum.

Recall that \( S_z, S_w \) are commuting on the quotient module \( N = H^2(\mathbb{D}^2) \ominus M \). Let \( d_1 \) be the map from \( N \) to \( N \oplus N \) such that

\[
d_1 f = (-S_w f, S_z f), \quad f \in N
\]

and \( d_2 \) be the map from \( N \oplus N \) to \( N \) such that

\[
d_2 (f, g) = S_z f + S_w g, \quad f, g \in N
\]
We consider the short sequence $K(S_z, S_w)$:

$$0 \longrightarrow N \xrightarrow{d_1} N \oplus N \xrightarrow{d_2} N \longrightarrow 0$$

We say that $(\lambda, \eta)$ is in the joint resolvent set $\rho(S_z, S_w)$ if the short sequence $K(S_z - \lambda, S_w - \eta)$ is exact. The Taylor joint spectrum $\sigma(S_z, S_w)$ is $\mathbb{C} \setminus \rho(S_z, S_w)$. Cowen and Rubel [67] made a study of the spectral properties of the operators $S_1, S_2, \ldots, S_n$ on $H^2(\mathbb{D}^n)$ and showed that for $M = [f]$ the joint right spectrum in $\mathbb{D}^n$ equals the zeros of generator $f$. In [20], R. Douglas and R. Yang proved for submodule $[h]$, if $h$ is holomorphic in a neighborhood of $\mathbb{D}^2$ and satisfies certain condition, then $\sigma(S_z, S_w) = Z(h) \cap \mathbb{D}^2$.

Here for zero-based submodule $H_z$, the inclusion $Z \subset \sigma(S_z, S_w) \cap \mathbb{D}^2$ is easy to check.

**Proposition 4.1.3.** If $(\lambda, \eta) \in \mathbb{Z}$, then $(\lambda, \eta) \in \sigma(S_z, S_w)$.

**Proof.** Since $f(\lambda, \eta) =< f, k(\lambda, \eta) >= 0$ for $(\lambda, \eta) \in \mathbb{Z}$ and $f \in H_Z$, $k(\lambda, \eta) \in N$. Clearly $< P_N[(z - \lambda)f_1 + (w - \eta)f_2], k(\lambda, \eta) >= 0$, for $\forall f_1, f_2 \in H^2(\mathbb{D}^2)$. So we have $S_z - \lambda N + S_w - \eta N \neq N$, i.e. $(\lambda, \eta) \in \sigma(S_z, S_w)$. $\square$

However, the other direction is hard. The followings are some properties of $S_z$ and $S_w$.

**Proposition 4.1.4.** For submodule $H_Z$, $\ker(S_z - \alpha) \cap \ker(S_w - \beta) = \{0\}$, $\forall (\alpha, \beta) \in \mathbb{D}^2$.

**Proof.** Suppose $h \in H^2(\mathbb{D}^2) \ominus H_Z$, $h \neq 0$ and $h \in \ker(S_z - \alpha) \cap \ker(S_w - \beta)$, then $\exists (\lambda, \eta) \in \mathbb{Z}$ such that $h(\lambda, \eta) \neq 0$. That is, there exists a neighborhood $U$ of $(\lambda, \eta)$ such that $h(z, w) \neq 0$ for $\forall (z, w) \in U$.

Since $U \cap \mathbb{Z}$ is not a single point, we can pick a point $(\lambda', \eta') \in U \cap \mathbb{Z}$ such that $\lambda' \neq \alpha$ or $\eta' \neq \beta$. Since $h \in \ker(S_z - \alpha) \cap \ker(S_w - \beta)$, we have $(z - \alpha)h(z, w) \in H_Z$ and $(w - \beta)h(z, w) \in H_Z$. It implies $(\lambda' - \alpha)h(\lambda', \eta') = 0$, $(\eta' - \beta)h(\lambda', \eta') = 0$, i.e. $h(\lambda', \eta') = 0$ for $(\lambda', \eta') \in U$. Contradiction. $\square$
For $X \subset \mathbb{D}^2$, let $\pi_1(X) = \{ z \in \mathbb{D} : (z, w) \in X \}$.

**Proposition 4.1.5.** If $Z$ is a zero set such that there is a point $(\alpha, \beta) \in Z$ with a neighborhood $U$ such that $\pi_1(U \cap Z) = \{ \alpha \}$, then $S_z - \alpha$ is injective.

**Proof.** Assume $h \in \ker(S_z - \alpha)$ and $h \neq 0$, then $(z - \alpha)h \in H_Z$. Since $h \in H^2(\mathbb{D}^2) \ominus H_Z$, there exists $(\lambda', \eta') \in Z$ and a neighborhood $U$ of $(\lambda', \eta')$ such that $h(z, w) \neq 0$ for $\forall (z, w) \in U$. On the other hand, $(z - \alpha)h(z, w) = 0$ for $(z, w) \in U \cap Z$ and by the assumption $h(z, w) = 0$. \[ \square \]

**Proposition 4.1.6.** $S_z N + S_w N$ is closed.

**Proof.** By the definition of $S_z$ and $S_w$, we have $S_z N + S_w N = P_N (z H^2(\mathbb{D}^2) + w H^2(\mathbb{D}^2))$. Since $I - S_z S^*_z - S_w S^*_w + S_z S_w S^*_z S^*_w = P_N 1 \otimes P_N 1$ (cf. [23]), we have $S_z S^*_z + S_w S^*_w - S_z S_w S^*_z S^*_w$ is Fredholm with cokernel of dimension $\leq 1$. Moreover, $\text{ran}(S_z S^*_z + S_w S^*_w - S_z S_w S^*_z S^*_w) \subset S_z N + S_w N$, i.e. $(S_z N + S_w N)^\perp \subset \text{ran}(S_z S^*_z + S_w S^*_w - S_z S_w S^*_z S^*_w)^\perp$. So $S_z N + S_w N$ is closed with $\text{codim} \leq 1$. \[ \square \]

### 4.2. Hilbert-Schmidt submodule

For a submodule $M \subset H^2(\mathbb{D}^2), \lambda \in D$. The Core operator $C$ on $M$ is defined as [50]

$$C = 1 - R_z R^*_z - R_w R^*_w + R_z R_w R^*_z R^*_w$$

A submodule $M$ is said to be Hilbert-Schmidt if its core operator $C$ is Hilbert-Schmidt, i.e. $\| C \|^2_{HS} - Tr(C^* C) := \sum_{n=1}^{\infty} \| C e_n \|^2$ is finite for orthonormal basis $\{ e_n \}_{n=1}^{\infty}$ of $M$.

Classification is an essential problem in every branch of mathematics. In chapter 2, we have discussed the complicated yet intriguing structure of submodules of $H^2(\mathbb{D}^2)$. The core operator is useful tool in the study of submodule.
Recall

\[ \Sigma_0 = \| [R^*_z, R_z][R^*_w, R_w] \|^2_{H.S}; \quad \Sigma_1 = \| [R^*_z, R_z][R^*_w, R_w] \|^2_{H.S}. \]

It is well known that \([R^*_z, R_z]\) and \([R^*_w, R_w]\) are orthogonal projections with range \(M \ominus zM\) and \(M \ominus wM\), respectively. It is easy to compute that \(\text{tr}C^2 = \Sigma_0 + \Sigma_1\).

For a bounded operator \(T\), we define \(\rho_c(T)\) to be the collection of those complex numbers \(\lambda\) for which \(T - \lambda I\) has closed range and finite dimensional kernel, and we let \(\sigma_c(T) = \mathbb{C} \setminus \rho_c\). It is clear that \(\sigma_c(T)\) is a subset of the essential spectrum \(\sigma_e(T)\). Define \(\sigma_c(M) = \sigma_c(S_z) \cap \sigma_c(S_w)\). The following two theorem were given \([30]\).

**Theorem 4.2.1.** If \(M\) is a submodule and \(D\) is not a subset of \(\sigma_c(M)\), then (a) \([R^*_z, R_z]\) is Hilbert-Schmidt; (b) \([R^*_w, R_w][R^*_z, R_z]\) is Hilbert-Schmidt.

**Theorem 4.2.2.** If \(M\) is \(z\)-invariant and \(\lambda \in D\), then \(S_z - \lambda\) is Fredholm on \(H^2(\mathbb{D}^2) \ominus M\) if and only if \(L(\lambda)|_{M \ominus zM}\) is Fredholm. Moreover, \(\dim(\ker(S_z - \lambda)) = \dim(\ker(L_\lambda))\) and \(\dim(\text{coker}(S_z - \lambda)) = \dim(\text{coker}(L_\lambda))\).

The first theorem says if we’re able to find a \(\lambda \in D\) such that \(\lambda \not\in \sigma_c(S_z)\), then \(M\) is Hilbert-Schmidt. Then the next theorem provides a tool. So now the question of Hilbert-Schmidtness is reduced to: is there a \(\lambda\) such that \(L(\lambda)|_{M \ominus zM}\) has closed range and finite dimensional kernel?

Here, we’ll use this effective tool to prove that the submodule generated by the zero set \(Z\) in Example 4.1.2 is Hilbert-Schmidt. We first prove the following lemma.

**Lemma 4.2.3.** Let \(F\) be the \(H^\infty\) function in Example 4.1.2, then \(L(\lambda)|_{[F] \ominus z[F]}\) has closed range and \(\ker(L(\lambda)) = \{0\}\) some \(\lambda \in \mathbb{D}\).

**Proof.** By Suppose \(\lambda \in Q\), first we prove \(\ker L(\lambda) = 0\).
Since $F \in H^\infty(\mathbb{D}^2)$, for $f \in [F]$ we have $f = F\phi(z, w)$ for some $\phi$ holomorphic in $\mathbb{D}^2$. Consider sequence of polynomials $P_n(z, w)$ such that $\lim_{n \to \infty} FP_n = F\phi$, then $L(\lambda)(F\phi)(z, w) = \lim_{n \to \infty} F(\lambda, w)P_n(\lambda, w)$.

Notice $F(\lambda, w)$ is bounded below on $w \in Q$, we have $\lim_{n \to \infty} P_n(\lambda, w) = \phi(\lambda, w)$, then $\lim_{n \to \infty} F(\lambda, w)P_n(\lambda, w) = F(\lambda, w)\phi(\lambda, w)$. If $L(\lambda)f = 0$, then $\phi(\lambda, w) = 0$.

Therefore

$$\lim_{n \to \infty} F\frac{P_n - P_n(\lambda, w)}{z - \lambda} = F\frac{\phi(z, w)}{z - \lambda} \in H^2(\mathbb{D}^2)$$

i.e. $\frac{F}{z - \lambda} \in [F]$. Moreover, for any polynomial $Q_n$, since $f \in [F] \oplus z[F]$,

$$< \frac{f}{z - \lambda}, FQ_n > = < f, \frac{1}{z - \lambda} FQ_n > = < f, z - \frac{1}{1 - \lambda z} FQ_n > = 0$$

hence $f = 0$.

Next, we’ll show $L(\lambda)$ has closed range.

By(3.4.1), $L(\lambda)([F] \oplus z[F]) = L(\lambda)[F]$ and $|F(\lambda, w)| \geq \delta > 0$ on $\mathbb{T}$ for some $\delta$, we have $\text{ran}L(\lambda) = F(\lambda, w)H^2_w$, which is closed in $H^2_w$. Notice $\lambda \in Q$, $F(\lambda, w)$ has only finitely many zeros, then $F(\lambda, w)H^2_w$ has finite codimension, i.e. $L(\lambda)$ has closed range. \(\square\)

Notice that in Example 4.1.2, there exists $r < 1$ so that $\text{dist}(Z, Q^2) > 0$ where $Q = \{\lambda : r < |\lambda| < 1\}$. Fix a $(\lambda, \eta) \in Q^2$, there exists a neighborhood $U$ of $(\lambda, \eta)$ such that either $\pi_1(U \cap Z) = \{\lambda\}$ or $\pi_2(U \cap Z) = \{\eta\}$.

**Theorem 4.2.4.** The zero base submodule $H_Z$ is Hilbert-Schmidt.

**Proof.** For $\lambda \in Q$, since $[F] \subset H_Z$, $L(\lambda)[F] \subset L(\lambda)H_Z \subset H^2_w$. By the proof in Lemma 3.2.3, $\text{ran}L(\lambda)$ is closed and of finite codimension, We have $\text{ran}L(\lambda)H_Z \subset H^2_w$ is a closed subspace of finite codimension. Therefore, $H_Z$ is Hilbert-Schmidt. \(\square\)
Remark 4.2.5. Since $H_Z$ is Hilbert-Schmidt, we have $\text{tr} C^2$ is finite. It is thus natural to ask if there exists an estimation of $\text{tr} C^2$ by the $\text{dist}(Z, Q^2)$.

4.3. Rudin’s Pathological Example

By Beurling’s Theorem, we know that the every submodule in $H^2(\mathbb{D})$ contains some bounded functions. The follow example shown by Rudin \[32\] is a zero based submodule in $H^2(\mathbb{D}^2)$ which contains no bounded function.

Example 4.3.1. Fix a number $R > 1$, let $f(z, w) = \prod_{k=1}^{\infty} (1 - R(\frac{z + \alpha_k w}{2})^{n_k})$, where $|\alpha_k| = 1$ such that the value of each $\alpha_k$ repeats infinite many times in the sequence $(a_k)$, and $n_k$ are chosen such that $f \in H^2(\mathbb{D}^2)$. Define $M = \{f\}$

Rudin showed that the common zeros of $M$ force out non-zero bounded function in $M$, and the power $n_k$ can be chosen such that $f$ is in $L^2(\mathbb{T}^2)$.

We’ll show that the above pathological submodule is Hilbert-Schmidt. We start the study with an estimation of the generator $f$.

Theorem 4.3.2. For a suitable selection of $\{n_l\}_{l=1}^{\infty}$, there exist $0 < k \leq K < \infty$ such that

$$k \leq \int_{\mathbb{D}} |\prod_{l=1}^{\infty} (1 - R(\frac{z + \alpha_l w}{2})^{n_l})|^2 |dz| \leq K$$

a.e. $w \in \mathbb{T}$, where $|dz|$ is normalized Lebesgue measure.

Proof. Denote $J_N(w) = \int_{\mathbb{T}} |\prod_{l=1}^{N} (1 - R(\frac{z + \alpha_l w}{2})^{n_l})|^2 |dz|$. For $0 < \delta < 1$, let $\Gamma_{\pi w}(\delta)$ be an arc of $\mathbb{T}$ such that

$$|\frac{z + \alpha_l w}{2}| = \begin{cases} < \delta & z \in \mathbb{T} \setminus \Gamma_{\pi w}(\delta) \\ \geq \delta & z \in \Gamma_{\pi w}(\delta) \end{cases}$$

Since $|\frac{z + \alpha_l w}{2}| \leq 1$ and equality holds only when $z = \alpha_l w$ i.e. a set of measure 0. So it’s easy to show, for any $\alpha_l w \in \mathbb{T}$ and $\delta \in \left[\frac{0}{10}, 1\right)$, such a $\Gamma_{\pi w}(\delta)$ exists and its length $|\Gamma_{\pi w}(\delta)| \to 0$ as $\delta \to 1$. For simplicity, we write as $\Gamma_l(\delta)$. 

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For $N = 1$, $J_1(w) = \int_\Gamma |1 - R \frac{z + \alpha_I w}{2}|^2|dz|$. Fix a $\delta \in \left( \frac{\delta}{10}, 1 \right)$, pick $n_1$ large enough such that

$$
J_1(w) = \int_{\Gamma_i(\delta)} |1 - R \frac{z + \alpha_I w}{2}|^{n_1}|^2|dz| + \int_{\Gamma \setminus \Gamma_i(\delta)} |1 - R \frac{z + \alpha_I w}{2}|^{n_1}|^2|dz| \\
\geq \int_{\Gamma \setminus \Gamma_i(\delta)} |1 - R \frac{z + \alpha_I w}{2}|^{n_1}|^2|dz| \\
\geq (1 - |\Gamma_i(\delta)|)(1 - R^{n_1}) = k_1
$$

On the other hand $J_1(w) = \int_\Gamma |1 - R \frac{z + \alpha_I w}{2}|^{n_1}|^2|dz| \leq (1 + R)^2 := K_1$. For any $N$ and $\delta < 1$, by hypothesis $|\frac{z + \alpha_I w}{2}| < \delta$ on $\Gamma \setminus \Gamma_N(\delta)$, then

$$
|J_N(w) - J_{N-1}(w)| \leq \int_\Gamma \prod_{i=1}^{N-1} |1 - R \frac{z + \alpha_I w}{2}|^{n_i}|^2||1 - R \frac{z + \alpha_I w}{2}|^{n_1}|^2 - 1||dz| \\
= \int_{\Gamma \setminus \Gamma_N(\delta)} \prod_{i=1}^{N-1} |1 - R \frac{z + \alpha_I w}{2}|^{n_i}|^2||1 - R \frac{z + \alpha_I w}{2}|^{n_1}|^2 - 1||dz| + \int_{\Gamma_N(\delta)} \prod_{i=1}^{N-1} |1 - R \frac{z + \alpha_I w}{2}|^{n_i}|^2||1 - R \frac{z + \alpha_I w}{2}|^{n_1}|^2 - 1||dz| \\
\leq \int_{\Gamma \setminus \Gamma_N(\delta)} \prod_{i=1}^{N-1} |1 - R \frac{z + \alpha_I w}{2}|^{n_i}|^2||(1 + R^{n_N})^2 - 1||dz| + \int_{\Gamma_N(\delta)} \prod_{i=1}^{N-1} |1 - R \frac{z + \alpha_I w}{2}|^{n_i}|^2||(1 + R)^2 - 1||dz|
$$

Denote

$$I_1 = \int_\Gamma \prod_{i=1}^{N-1} |1 - R \frac{z + \alpha_I w}{2}|^{n_i}|^2||(1 + R^{n_N})^2 - 1||dz|$$

$$I_2 = \int_{\Gamma_N(\delta)} \prod_{i=1}^{N-1} |1 - R \frac{z + \alpha_I w}{2}|^{n_i}|^2||(1 + R)^2 - 1||dz|$$

First pick $\delta$ such that $I_2 \leq \frac{k_1}{2^{N+1}}$, then choose $n_N$ such that $I_1 \leq \frac{k_1}{2^{N+1}}$. It follows

$$- \frac{k_1}{2^{N+1}} \leq J_N(w) - J_{N-1}(w) \leq \frac{k_1}{2^{N+1}}$$
Therefore, for any \( w \in \mathbb{T} \)

\[
\frac{k_1}{2} \leq \lim_{N \to \infty} J_N(w) = \sum_{i=1}^{\infty} (J_{i+1}(w) - J_i(w)) + J_1(w) \leq K_1 + \frac{k_1}{2}
\]

So there exists \( 0 < k \leq K < \infty \) such that

\[
k \leq \int_{\mathbb{T}} \left| \prod_{l=1}^{\infty} (1 - R(\frac{\alpha_l w}{2})^{n_k}) \right|^2 |dz| \leq K
\]

We obtain the following corollary immediately.

**Corollary 4.3.3.** For the \( f \) in Example 4.3.1, there exist \( k \) and \( K \) such that

\[
0 < k \leq \inf_{w \in \mathbb{T}} \int_{\mathbb{T}} |f(z, w)|^2 |dz| < \sup_{w \in \mathbb{T}} \int_{\mathbb{T}} |f(z, w)|^2 |dz| \leq K < \infty,
\]

\[
0 < k \leq \inf_{z \in \mathbb{T}} \int_{\mathbb{T}} |f(z, w)|^2 |dw| < \sup_{z \in \mathbb{T}} \int_{\mathbb{T}} |f(z, w)|^2 |dw| \leq K < \infty.
\]

Then we obtain \([f]\) is Hilbert-Schmidt by showing \( L(0)|_{M \oplus \bar{z}M} \) is bounded below.

**Theorem 4.3.4.** If \( M = [f], f = \prod_{k=1}^{\infty} \left(1 - R(\frac{\alpha_k w}{2})^{n_k}\right) \) where \( 1 < R < 2 \), then \( L(0)|_{M \oplus \bar{z}M} \) is bounded below.

**Proof.** For \( g \in M \ominus zM \), we have \( g = fh \) for some \( h \) holomorphic on \( \mathbb{D}^2 \). Rewrite as \( fh = f(h - h(0, w)) + fh(0, w) \), we have \( \|f(h - h(0, w))\|^2 + \|fh\|^2 = \|fh(0, w)\|^2 \) since \( fh \in M \ominus zM \) and \( f(h - h(0, w)) \in zM \), i.e. \( \|fh\|^2 \leq \|fh(0, w)\|^2 \). Notice that

\[
f(0, w) = \prod_{k=1}^{\infty} \left(1 - R(\frac{\alpha_k w}{2})^{n_k}\right)
\]
If $R < 2$, $\frac{\alpha_k w^2}{2} \leq 1$ so $f(0, w)$ has no zero on $\overline{D}$. This implies $|f(0, w)|$ is bounded below on $\overline{D}$. If $R \geq 2$, we may pick $n_1$ such that $\frac{R}{2n_1} < 1$. Without loss of generality, we assume $1 < R < 2$.

$$\|L(0)f h\|^2 = \|f(0, w)h(0, w)\|^2 = \int_{\mathcal{T}} |f(0, w)h(0, w)|^2|dw|$$

$$\geq \frac{c^2}{K} \int_{\mathcal{T}^2} |f(z, w)h(0, w)|^2|dz||dw|$$

$$\geq \frac{c^2}{K} \|fh\|^2$$

Here $c = \inf_{w \in D} |f(0, w)|^2$. \hfill \Box

**Remark 4.3.5.** In this chapter, one sees that mostly we are dealing with singly generated submodule, although the motivation is zero-based submodule. In general, $H_Z \neq [F]$ with $Z(F) = Z$. For example, $[z - w] \neq [(z - w)^2]$ but they have the same zero set. Many obstacles lie ahead on the road. One of the major problem is that the structures of $H_Z$ and $H^2 \ominus H_Z$ are not clear. The former has no universal presentation, and it is hard to find a Riesz basis of the latter in order to do any computation.
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