Projective spectrum and cyclic cohomology

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PROJECTIVE SPECTRUM
AND
CYCLIC COHOMOLOGY

by

Patrick Gene Cade

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ABSTRACT

For a tuple $A = (A_1, A_2, \ldots, A_n)$ of elements in a unital topological algebra $\mathcal{B}$, the projective spectrum, $P(A)$ is the set of $z \in \mathbb{C}^n$ such that the linear pencil $A(z) = z_1 A_1 + z_2 A_2 + \cdots + z_n A_n$ is not invertible in $\mathcal{B}$. The Maurer-Cartan type $\mathcal{B}$-valued one-form $\omega_A := (A(z))^{-1} dA(z)$ appears to contain much information about the tuple $A$. Here, $\omega_A$ will establish a Jacobi type formula in the finite dimensional case. Furthermore, $\omega_A$ gives rise to a map between the cyclic cohomology, $HC^*(\mathcal{B}_A)$ and the de Rham cohomology of the projective resolvent $H^*_d(P^c(A))$. 
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I dedicate this dissertation to my loving family; Gene, Judith, Anne and Lindsey.
Chapter 0

Introduction

We call a topological algebra $B$ over a field $K$ a Banach algebra if $B$ is complete with respect to the metric induced by the topology. Generally and for the rest of this dissertation $K = \mathbb{C}$. If $B$ has a unit then we call it a unital Banach algebra. A standard approach to study the properties of $B$ is to investigate the set of units, or invertible elements of $B$. More generally, for an element $A$ of $B$ one can define a set in the coefficient field $\mathbb{C}$ called the spectrum, denoted $\sigma(A)$ that in some sense measures how the element $A$ relates to the unit $I$. The spectrum is defined to be $\sigma A := \{ \lambda \in \mathbb{C} | \lambda I - A \text{ is not invertible} \}$, the compliment of this set is referred to as the resolvent set. The spectrum has been used to great success in exploring the properties of $B$, many of these are outlined below.

Little work done on studying a single operator can be generalized to tuples of operators. The results in this dissertation are work towards a linking of the topological world the spectrum lives in and the algebraic world the elements of Banach algebras live in. One classic example of this type of connection is Jacobi’s formula [7], if $A$ is a differential $n \times n$ matrix function then $tr(A^{-1}dA) = d\log det(A)$. Here, Jacobi was able to use the trace and Maurer-Cartan type object $A^{-1}dA$ to connect the matrix $A$ sitting inside the algebra of matrices with a differential form. We will use this idea to make a similar connection for arbitrary Banach algebras. For a $n$-tuple, $A$, of elements from a unital Banach algebra we define the linear pencil $A(z) = z_1 A_1 + z_2 A_2 + \cdots + z_n A_n$. In turn, we are able to define the projective spectrum, $P(A) = \{ z \in \mathbb{C}^n | A(z) \text{ is not}$
invertible}, and the projective resolvent $P^c(A) = \mathbb{C} \setminus P(A)$. This sub-set of $\mathbb{C}^n$ can be thought of as a measure of the relationship between the elements $\{A_i\}$.

On the projective resolvent, $P^c(A)$, the $\mathcal{B}$-valued one form $\omega_A = A(z)^{-1}dA(z)$, where $d = \partial + \overline{\partial}$ is the exterior derivative, is well defined and appears to contain much information about the projective spectrum. We can use this one form, as Jacobi did, to establish properties of the projective resolvent set. It should be noted that for a tuple of matrices, $A$, Jacobi’s formula holds and furthermore that $P(A) = \{z \in \mathbb{C}^n | det A(z) = 0\}$.

It was shown by R. Yang that the one-form $\omega_A$ establish’s a homomorphism between the algebra of invariant linear functionals and the de Rham cohomology generated by the compliment of the projective spectrum, $H^*_d(P^c(A), \mathbb{C})$.

But this is not the whole story, since $\mathcal{B}$ is a topological algebra we can use methods from noncommutative geometry to attack and help us unravel the properties of $\mathcal{B}$. This dissertation will establish a homomorphism between the cyclic cohomology defined on the smallest inversion closed sub-algebra that contains the $n$-tuple $A$ and $H^*_d(P^c(A), \mathbb{C})$. A basic outline of the dissertation follows.

In Chapter 1, we start with a discussion of the basic tools and procedures used in standard functional analysis. Also included are some examples that illustrate the ideas and properties defined. For a more thorough treatment of the material in this chapter see [?], [?].

In Chapter 2, we include a basic discussion of exterior algebras so that we can define the de Rham cohomology of a complex manifold, see [?], [?]. Then we define, $HC^n(A)$, Connes’ cyclic cohomology of an algebra $A$ [?].

In Chapter 3, we discuss the work that has been done classically by Taylor and others, [?], [?], [?] and [?], to lift results for one element in a Banach algebra to an $n$-tuple of elements in a Banach algebra. More recently Dr. Yang, [?], has defined a projective spectrum that attempts to work around the obstructions that lie in the Taylor spectrum.

In Chapter 4, we will explore the properties of the homomorphism $\tau : \mathcal{F}^*(\mathcal{B})$ into $H^*_d(P^c(A), \mathbb{C})$ defined by evaluation at the one-form $\omega_A$, [?]. Where $\mathcal{F}^*(\mathcal{B})$ is the
algebra of invariant multilinear functionals on $\mathcal{B}$. Here, we will establish a higher order Jacobi type formula for elements in a unital Banach algebra.

In Chapter 5, we define a map $\kappa : \text{HC}^n(\mathcal{B}_A) \xrightarrow{\kappa} H^{n+1}_d(P^c(A))$, from the cyclic cohomology of $\mathcal{B}_A$ to the de Rham cohomology of $P^c(A)$. We will then explore this map and discuss some future considerations.
Chapter 1

Basic Operator Theory

We start with a basic reminder of the relevant properties for one operator. What is included here is standard, for a thorough treatment see [?] and [?].

1.1 Operators

Definition 1.1.1. A unital Banach algebra is a complex linear algebra $\mathcal{B}$ with a norm $\| \cdot \| : \mathcal{B} \to \mathbb{C}$ and identity $I$ satisfying

- $\|I\| = 1$
- $\|f\| = 0$ if and only if $f = 0$
- $\|\lambda f\| = |\lambda|\|f\|$ for $\lambda \in \mathbb{C}$ and $f$ in $\mathcal{B}$
- $\|f + g\| \leq \|f\| + \|g\|$ for $f, g$ in $\mathcal{B}$
- $\|fg\| \leq \|f\|\|g\|

such that $\mathcal{B}$ is complete with respect to this norm.

There are many useful examples of Banach algebras.

Example 1.1.2. Let $X$ be a Banach space. The set $\mathcal{L}(X, X)$, of all bounded linear maps from $X$ to itself forms a Banach algebra under composition of maps. The
elements of \( \mathcal{L}(X, X) \) are often referred to as bounded linear operators and \( \mathcal{L}(X, X) \) is often denoted \( \mathcal{L}(X) \). The norm on \( \mathcal{L}(X) \) is called the operator norm and is given by the following:

\[
\|T\| = \sup_{\|x\|=1} \|Tx\|, \text{ for } T \in \mathcal{L}(X), x \in X
\]

**Example 1.1.3.** Let \( \mathcal{B} = M_n(\mathbb{C}) \) be the algebra of \( n \times n \)-matrices over the complex numbers.

**Example 1.1.4.** The disk algebra \( A(\mathbb{D}) \), which consists of all analytic functions on the open unit disk, with the supremum norm:

\[
\|f\| = \sup_{\lambda \in \mathbb{D}} |f(\lambda)|.
\]

**Example 1.1.5.** Let \( X \) be a compact Hausdorff space. Then the space of all continuous \( \mathbb{C} \)-valued functions on \( X \), denoted \( C(X) \), is a Banach algebra with respect to pointwise multiplication and the supremum norm.

**Definition 1.1.6.** An element \( f \) of a unital Banach algebra \( \mathcal{B} \) is called invertible if it has an inverse \( g = f^{-1} \) in \( \mathcal{B} \):

\[
gf = I = fg.
\]

\( f \) is said to have a left inverse \( g \), respectively a right inverse \( h \) if;

\[
gf = I \quad I = fh.
\]

Note that if \( f \) has a left inverse \( g \) and a right inverse \( h \) then \( g = h \), since

\[
g = g(fh) = (gf)h = h.
\]

### 1.2 Spectrum

To more closely study the properties of the element \( f \) in a Banach algebra we look at how \( f \) relates to the identity \( I \).
Definition 1.2.1. The spectrum of \( f \) in \( \mathcal{B} \) is the set of complex numbers \( \lambda \) for which

\[
\lambda I - f
\]

is not invertible. This set is denoted by \( \sigma(f) \). We call the complement of the spectrum, \( \mathbb{C} \setminus \sigma(f) \) the resolvent set and denote it by \( \rho(f) \).

Theorem 1.2.2. The spectrum, \( \sigma(f) \), is a closed, bounded, nonempty set in \( \mathbb{C} \).

The proof of this theorem is non trivial [?].

Definition 1.2.3. The spectral radius of \( f \), denoted as \( |\sigma(f)| \), is defined as

\[
|\sigma(f)| = \max_{\lambda \in \sigma(f)} |\lambda|.
\]

Example 1.2.4. If we let \( M \in M_n(\mathbb{C}) \) then the spectrum of \( M \) is the set of all eigenvalues of \( M \). Moreover, the spectral radius is the eigenvalue with the largest norm.

Bounded linear maps with one dimensional range, the so called bounded linear functionals, will prove to be extremely useful.

Definition 1.2.5. Let \( X \) be a complex Banach space. Then we say a mapping \( \phi \) from \( X \) to \( \mathbb{C} \) is a bounded linear functional if the following hold

1) \( \phi(ax + y) = a\phi(x) + \phi(y) \) for \( a \in \mathbb{C} \) and \( x, y \in X \);

2) \( |\phi(x)| \leq c|x| \) for all \( x \in X \) and some \( c \in \mathbb{C} \).

Definition 1.2.6. The collection of all bounded linear functionals on a Banach space \( X \) is denoted \( X^* \).

Theorem 1.2.7. Let \( X \) be a Banach space. Then \( X^* \) is a Banach space with respect to the operator norm

\[
\|l\| := \sup_{|x|=1} |l(x)|.
\]

One important class of Banach algebras are those arising from the set of bounded linear maps from a Hilbert space \( \mathcal{H} \) to itself. Before we can define what a Hilbert space is we need to define an inner product.
Definition 1.2.8. An inner product on a complex linear space $V$ is a function $\theta$ from $V \times V$ to $\mathbb{C}$ such that:

1) $\theta(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \theta(f_1, g) + \alpha_2 \theta(f_2, g)$ for $\alpha_1, \alpha_2$ in $\mathbb{C}$ and $f_1, f_2, g$ in $V$;

2) $\theta(f, \beta_1 g_1 + \beta_2 g_2) = \overline{\beta}_1 \theta(f, g_1) + \overline{\beta}_2 \theta(f, g_2)$ for $\beta_1, \beta_2$ in $\mathbb{C}$ and $f, g_1, g_2$ in $V$;

3) $\theta(f, g) = \overline{\theta(g, f)}$ for $f, g$ in $V$; and

4) $\theta(f, f) \geq 0$ for $f$ in $V$ and $\theta(f, f) = 0$ if and only if $f = 0$.

An inner product is usually denoted $(\cdot, \cdot)$.

Definition 1.2.9. A complex linear space, $L$, equipped with a inner product, $(\cdot, \cdot)$ is called an inner product space, we may define a norm on $L$ by $\|f\|^2 = (f, f)$. If we complete $L$ with respect to this norm we obtain the Banach space $H$ called a Hilbert space.

It should be noted here that Hilbert spaces are tied to their inner products. In fact, one can recover the inner product from the norm in a Hilbert space via the Parallelogram law, [?].

Definition 1.2.10. A subset, $\{e_i\}_{i \in I}$ of a Hilbert space $H$ is said to be an orthonormal basis if the following hold:

i) The smallest closed subspace containing $\{e_i\}_{i \in I}$ is $H$;

ii) $(e_i, e_i) = 1$ for all $i \in I$;

i) $(e_i, e_j) = 0$ for all $i \neq j$.

Theorem 1.2.11. Every non trivial Hilbert space $H, (\neq \{0\})$, has an orthonormal basis.

Theorem 1.2.12. (Riesz Representation Theorem) If $\phi$ is a bounded linear functional on a Hilbert space $H$, then there exists a unique $g$ in $H$ such that $\phi(f) = (f, g)$ for $f$ in $H$. 

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This allows us to identify the space, \( \mathcal{H}^* \), with \( \mathcal{H} \). Furthermore, this allows us to define the so called adjoint operator.

**Proposition 1.2.13.** If \( T \) is a bounded operator on the Hilbert space \( \mathcal{H} \) then there exists a unique bounded operator \( S \) on \( \mathcal{H} \) such that:

\[
(Tf, g) = (f, Sg) \text{ for all } f \text{ and } g \in \mathcal{H}.
\]

This unique bounded operator is denoted \( T^* \).

**Theorem 1.2.14.** Let \( T \) be a bounded linear operator on \( \mathcal{H} \). Then we have the following:

\[
\sigma(T^*) = \overline{\sigma(T)}.
\]

**Example 1.2.15.** Let \( \mathcal{B} \) be the Banach algebra \( l^2 \), consisting of vectors \( x \) with complex components such that

\[
x = (a_0, a_1, \ldots), \quad \|x\| := \sum |a_i|^2 < \infty.
\]

The right shift \( R \) and the left shift \( L \) are defined by

\[
Rx = (0, a_0, a_1, \ldots), \quad Lx = (a_1, a_2, \ldots).
\]

We have that \( LR = I \), but \( RL \neq I \), thus neither \( L \) or \( R \) are invertible. Further calculation leads to the following: the spectrum of \( R \) and \( L \) consists of the unit disk, \( |\lambda| \leq 1 \).

The spectrum is an important tool in decoding information about an operator and its connection to the rest of the algebra.

One important property we are interested in is the the existence of a trace. For Hilbert spaces there is not a well defined trace for all operators on the Hilbert space. But for specific operators we can define a trace.

**Definition 1.2.16.** If \( T \) is an operator on the Hilbert space \( \mathcal{H} \), then \( T \) is positive if \( (Tf, f) \geq 0 \) for all \( f \) in \( \mathcal{H} \).
Proposition 1.2.17. If \( P \) is a positive operator on the Hilbert space \( \mathcal{H} \), then there exists a unique operator \( Q \) such that \( Q^2 = P \).

Proposition 1.2.18. If \( T \) is an operator on the Hilbert space \( \mathcal{H} \), then \( T^*T \) is a positive operator.

With the previous two results we can define a trace in Hilbert spaces

Definition 1.2.19. Let \( \mathcal{H} \) be a Hilbert space and let \( \{e_j\} \) be an orthonormal basis. Then we say a bounded operator \( T \) on \( \mathcal{H} \) is trace class if:

\[
\text{tr}(|T|) := \sum_j \langle (T^*T)^{1/2} e_j, e_j \rangle < \infty.
\]

Trace class operators have many important properties. For example; for \( A \in M_n(\mathbb{C}) \) we have that \( \text{det}(A) = \prod_{\lambda \in \sigma(A)} (\lambda) \). This motivates the following.

Definition 1.2.20. If \( T \) is trace class then we can define the determinant of \( I + T \) follows:

\[
\text{det}(I + T) := \prod_{\lambda \in \sigma(T)} (1 + \lambda).
\]

The condition for an operator to be trace class is precisely the condition required for the product \( \prod_{\lambda \in \sigma(T)} (1 + \lambda) \) to converge.

1.3 Gelfand-Naimark Theorem

Here we recall an important structure theorem for Banach algebras.

Definition 1.3.1. Let \( \mathcal{B} \) be a Banach algebra. A complex linear functional \( \phi \) on \( \mathcal{B} \) is said to be multiplicative if:

1) \( \phi(fg) = \phi(f)\phi(g) \) for \( f, g \) in \( \mathcal{B} \); and

2) \( \phi(I) = 1 \).

The set of all multiplicative linear functionals on \( \mathcal{B} \) is denoted by \( M = M_\mathcal{B} \).

We can define a topology on \( M \).
Definition 1.3.2. Let $\mathcal{B}$ be a Banach algebra and $M$ the set of all multiplicative linear functionals on $\mathcal{B}$. The weak star topology on $M$ is the weakest topology such that all maps on $M$ of the form, $\hat{f}(\phi) := \phi(f)$ for $f \in \mathcal{B}$, are continuous.

Now we can make the following definition.

Definition 1.3.3. For the Banach algebra $\mathcal{B}$, if $M \neq \emptyset$, the Gelfand transform is the function $\Gamma : \mathcal{B} \rightarrow C(M)$ given by $\Gamma(f)(\phi) = \phi(f)$ for $\phi$ in $M$.

When $\mathcal{B}$ is a commutative Banach algebra then we have the following proposition.

Proposition 1.3.4. Let $\mathcal{B}$ be a commutative Banach algebra. The set of all multiplicative linear functionals on $\mathcal{B}$ is in a one-to-one correspondence with the set of all maximal ideals in $\mathcal{B}$.

With the previous proposition we have the following far reaching theorem.

Theorem 1.3.5 (Gelfand). If $\mathcal{B}$ is a commutative Banach algebra, $M$ is its maximal ideal space, and $\Gamma : \mathcal{B} \rightarrow C(M)$ is the Gelfand transform, then:

1) $M$ is not empty;

2) $\Gamma$ is an algebra homomorphism;

3) $\|\Gamma f\|_\infty \leq \|f\|$ for $f$ in $\mathcal{B}$; and,

4) $f$ is invertible in $\mathcal{B}$ if and only if $\Gamma(f)$ is invertible in $C(M)$.

The existence of the adjoint to an operator in Hilbert spaces is one of the properties that makes Hilbert spaces so useful. But the existence of the adjoint is not unique to Hilbert spaces.

Definition 1.3.6. If $\mathcal{B}$ is a Banach algebra, then an involution on $\mathcal{B}$ is a mapping $T \rightarrow T^*$ which satisfies:

i) $T^{**} = T$ for $T$ in $\mathcal{B}$;

ii) $(\alpha S + \beta T)^* = \bar{\alpha}S^* + \bar{\beta}T^*$ for $S, T$ in $\mathcal{B}$ and $\alpha, \beta$ in $\mathbb{C}$;
ii) $(ST)^* = T^*S^*$ for $S, T$ in $B$.

If, in addition, $\|T^*T\| = \|T\|^2$ for $T$ in $B$, then we say $B$ is a $C^*$-algebra.

We call an operator $T$ on a Hilbert space $\mathcal{H}$ normal if $TT^* = T^*T$. In this case the Gelfand transform gives us a connection with the spectrum of the operator.

**Theorem 1.3.7.** If $\mathcal{H}$ is a Hilbert space and $T$ is a normal operator on $\mathcal{H}$, then the $C^*$-algebra $\mathcal{C}_T$ generated by $T$ and $T^*$ is commutative. Moreover, the maximal ideal space of $\mathcal{C}_T$ is homeomorphic to $\sigma(T)$, and hence the Gelfand transform is an isometric isomorphism of $\mathcal{C}_T$ onto $C(\sigma(T))$ that also respects adjoints.
Chapter 2

Cohomology Theories

One useful tool in investigating the structure of Banach algebras is cohomology. To define cohomology we must first start with exterior algebras. Then we will define the de Rham cohomology of a smooth manifold and look at some examples. Then we will look at Connes’s cyclic cohomology, a tool to study the structure of noncommutative algebras.

2.1 Exterior Algebra

We include a straightforward discussion of this object for review, there is a good discussion in [?].

Definition 2.1.1. We say that \( \Lambda \) is the exterior algebra on \( n \) generators \( e_1, e_2, \ldots, e_n \) with identity \( e_0 = 1 \) in the following case. Let \( \Lambda \) be the algebra of forms in \( e_1, e_2, \ldots, e_n \) with complex coefficients, subject to the collapsing property,

\[
e_i e_j + e_j e_i = 0 \quad \text{for} \quad 0 \leq i, j \leq n.
\]

Note that this collapsing property implies that \( e_i e_i = 0 \) for all \( i \).

The term, form, in the above definition is used to denote the objects:

\[
\{e_{i_1} e_{i_2} \cdots e_{i_k} | 1 \leq i_1 < i_2 < \cdots i_k \leq n\}
\]

for \( 1 \leq k \leq n \).
If one declares \( \{ e_{i_1} e_{i_2} \ldots e_{i_k} | 1 \leq i_1 < i_2 < \cdots i_k \leq n \} \) for \( 1 \leq k \leq n \) to be
an orthonormal basis, the exterior algebra \( \Lambda \) becomes a Hilbert space, admitting an
orthogonal decomposition \( \Lambda = \bigoplus_{k=1}^{n} \Lambda^k \) where \( \dim \Lambda^k = \binom{n}{k} \). Here \( \Lambda^k \) is the set of all
\( k \)-forms, \( \Lambda^k := \{ e_{i_1} e_{i_2} \ldots e_{i_k} | 1 \leq i_1 < i_2 < \cdots i_k \leq n \} \).

2.2 de Rham Cohomology

A thorough treatment in the real case is done in [?], [?] is a good resource in the
complex case. Let \( z_1, z_2, \ldots, z_n \) be the linear coordinates of \( \mathbb{C}^n \). We define \( \Omega^* \) to be
the exterior algebra generated by \( dz_1, dz_2, \ldots, dz_n \) and \( d\bar{z}_1, d\bar{z}_2, \ldots, d\bar{z}_n \).

**Definition 2.2.1.** The product in \( \Omega^* \) is called the wedge product and is denoted \( \wedge \). Thus, when there no the potential for confusion we will use \( dz_i dz_j = dz_i \wedge dz_j \).

Let \( M \) be a smooth complex manifold. Then we define \( \Omega^*(M) := C^\infty(M) \otimes \mathbb{C} \Omega^* \), where \( C^\infty(M) \) is the space of infinitely differentiable \( \mathbb{C} \)-valued functions on \( M \). We expand the use of the term, form, to denote any object in \( \Omega^*(M) \). Thus a \( p \)-form is
an element of \( \Omega^p(M) \).

Because of the natural grading on \( \Omega^*(M) \) we define the following map.

**Definition 2.2.2.** The operator \( \partial \) is the \( \mathbb{C} \)-linear map from \( \Omega^p(M) \rightarrow \Omega^{p+1}(M) \) that satisfies the following properties,

- \( \partial f = \sum \frac{\partial f}{\partial z_i} dz_i \) for smooth functions \( f \) on \( M \),
- \( \partial(\partial f) = 0 \) for any smooth function \( f \) on \( M \),
- \( \partial(\alpha \wedge \beta) = \partial \alpha \wedge \beta + (-1)^p \alpha \wedge \partial \beta \) where \( \alpha \) is a \( p \)-form.

**Definition 2.2.3.** The operator \( \bar{\partial} \) is the conjugate linear map from \( \Omega^p(M) \rightarrow \Omega^{p+1}(M) \) that satisfies the following properties,

- \( \bar{\partial} f = \sum \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i \) for smooth functions \( f \) on \( M \),
- \( \bar{\partial}(\bar{\partial} f) = 0 \) for any smooth function \( f \) on \( M \),
\[ \overline{\partial}(\alpha \wedge \beta) = \partial \alpha \wedge \beta + (-1)^p \alpha \wedge \overline{\partial} \beta \] where \( \alpha \) is a \( p \)-form.

**Definition 2.2.4.** The exterior derivative \( d \) is the map from \( \Omega^p(M) \to \Omega^{p+1}(M) \) given by:

\[ d = \partial + \overline{\partial}. \]

The situation is pictured as the following sequence:

\[ 0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots. \]

**Definition 2.2.5.** We say a \( p \)-form \( f \) is closed if \( df = 0 \), and it is exact if \( f = dg \) for some smooth \( (p - 1) \)-form \( g \).

The following lemma is a straightforward calculation.

**Lemma 2.2.6.** \( d^2 = 0 \).

Many questions about the manifold \( M \) can be answered by those forms which are in some sense nontrivial with respect to the exterior derivative. One way to view forms is as a differential equation in the variables of the manifold. Then looking for those forms which are closed is the same as looking for solutions to a differential equation. Since exact forms are the trivial or non-interesting solutions we look to find those solutions that are interesting, closed but not exact. To help solve this question we further reduce the picture above.

**Definition 2.2.7.** The \( p \)-th de Rham cohomology is the vector space,

\[ H_d^p(M, \mathbb{C}) := \{ \text{closed } p \text{-forms} \}/\{ \text{exact } p \text{-forms} \}, \quad p \geq 0. \]

The wedge product allows the de Rham cohomology to admit the same algebra structure as the algebra of general forms.

**Definition 2.2.8.** Let \( H_d^*(M, \mathbb{C}) := \bigoplus H_d^p(M, \mathbb{C}) \) denote the algebra of the de Rham cohomology under the wedge product.

It should be noted that \( f \) is a holomorphic function if and only if \( \overline{\partial}f = 0 \), hence \( df = \partial f \). Furthermore, if \( M \) is a Stein domain, i.e. a domain of holomorphy, then \( H_d^p(M, \mathbb{C}) \) is generated by holomorphic forms [?].
Example 2.2.9. Let $D = \{ z \in \mathbb{C}^n | 0 < |z_j| < 2, 1 \leq j \leq n \}$. For $1 \leq r \leq n$ define the $r$-cycle $\gamma_r$ in $D$ by $\gamma_r(t_1, \ldots, t_r) = (e^{it_1}, \ldots, e^{it_r}, 1, \ldots, 1), 0 \leq t_j \leq 2\pi$; the holomorphic $r$-form

$$f_r = \frac{dz_1 \wedge \cdots \wedge dz_r}{z_1 \cdots z_r}$$

on $D$ is $d$-closed. Since

$$\int_{\gamma_r} f_r = (2\pi i)^r \neq 0,$$

$f_r$ is not exact. Thus $H^r_d(D, \mathbb{C}) \neq 0$ for $1 \leq r \leq n$.

2.3 Cyclic Cohomology

A good resource for the following material can be found in [?]. Let $\mathcal{A}$ be a topological algebra over $\mathbb{C}$ that is associative but not necessarily commutative nor unital.

Definition 2.3.1. A continuous $n$-dim Hochschild cochain, $\phi$, on $\mathcal{A}$ is a continuous $(n + 1)$-multi-linear functional $\phi : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathbb{C}$.

This leads to the following definition.

Definition 2.3.2. Let $C^n(\mathcal{A})$ be the $\mathbb{C}$-vector space of all continuous $n$-dim cochains.

Example 2.3.3. $C^0(\mathcal{A}) = \mathcal{A}^*$

Definition 2.3.4. The Hochschild coboundary map is the linear map $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ given by,

$$(b\phi)(a_0, a_1, \ldots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1} \phi(a_{n+1} a_0, \ldots, a_n)$$

for every $\phi \in C^n(\mathcal{A}), a_0, a_1, \ldots, a_{n+1} \in \mathcal{A}$.

Example 2.3.5. If $\phi \in C^0(\mathcal{A})$ then $(b\phi)(a_0, a_1) = \phi(a_0 a_1) - \phi(a_1 a_0) = \phi([a_0, a_1])$.

Lemma 2.3.6. $b^2 = 0$. 
The proof is a routine exercise. This allows us to define the cohomology of the complex of these spaces.

**Definition 2.3.7.** The Hochschild complex of \( \mathcal{A} \) is the cochain complex \( C^n(\mathcal{A}) \) with respect to the coboundary map \( b \). The \( p \)-th Hochschild cohomology space is,

\[
HH^p(\mathcal{A}) := \{ \ker b \}/\{ \im b \}, \quad p \geq 0.
\]

**Example 2.3.8.** \( HH^0(\mathcal{A}) = \ker(b : C^0(\mathcal{A}) \to C^1(\mathcal{A})) = \{ \text{traces on } \mathcal{A} \} \).

**Definition 2.3.9.** A continuous \( n \)-dim Hochschild cochain \( \phi \in C^n(\mathcal{A}) \) is cyclic if

\[
\phi(a_0, a_1, \ldots, a_n) = (-1)^n \phi(a_n, a_0, a_1, \ldots, a_{n-2}).
\]

We will denote the space of all continuous \( n \)-dim cyclic cochains on \( \mathcal{A} \) as \( C^n_\lambda(\mathcal{A}) \).

A key observation made by [?] is the following Lemma.

**Lemma 2.3.10.** If \( \phi \) is a cyclic cochain then \( b\phi \) is a cyclic cochain.

Hence we can define a complex \( C_\lambda^* \) with \( b \) as the coboundary map.

**Definition 2.3.11.** The cyclic complex of \( \mathcal{A} \) is the cochain complex of \( C^n_\lambda(\mathcal{A}) \) with respect to the coboundary map \( b \). The \( p \)-th cohomology space is denoted \( HC^p(\mathcal{A}) \) and called the cyclic cohomology of \( \mathcal{A} \).

**Example 2.3.12.** \( HC^0(\mathcal{A}) = HH^0(\mathcal{A}) \)

We can connect these two cohomologies with the following construction.

**Definition 2.3.13.** Let \( M \) be a smooth manifold. Then a \( k \)-current \( \phi \) is a continuous linear functional on \( \Omega_c^k(M) \), the space of compact de Rham \( k \)-forms on \( M \). We denote the space of all \( k \)-currents by \( D_k(M) \).

Here we mean continuity in the sense that, \( T \) is continuous if \( a_n \) is a sequence of smooth forms all defined on the same compact set, such that the derivatives of the coefficients tend uniformly to zero when \( n \) tends to \( \infty \), then \( T(a_n) \) tends to zero.

Hence, we have the following.
Theorem 2.3.14. If $M$ is a compact manifold $M$, there is a canonical isomorphism between the continuous Hochschild cohomology group $HH^k(C^\infty(M))$ and the space $D_k(M)$ of de Rham $k$-currents on $M$.

Furthermore, if we look at the $C^*$ algebra $C(M)$. Then we have that

\[ HC^{2n}(C(M)) = HC^0(C(M)) = \{\text{traces on } C(M)\}, \]

\[ HC^{2n+1}(C(M)) = 0. \]
Chapter 3

Multivariate Operator Theory

3.1 Introduction

The study of a single operator $M$ in a unital Banach algebra has been very fruitful. What is a new and exciting area of study is the study of an $n$-tuple of operators. Now we are no longer dealing with one variable techniques. We need to use, often more complicated, multivariable techniques.

3.2 Taylor’s Spectrum

There have been many attempts to extend the notion of the spectrum of an operator to an $n$-tuple of operators. If we have an $n$-tuple, $A$, whose elements pairwise commute then there is an established joint spectrum for $A$ called the Taylor joint spectrum or simply the Taylor spectrum, see; [?], [?], [?], and [?]. A tuple $A$ is said to be commuting if $A_i A_j = A_j A_i$, $\forall \ 1 \leq i, j \leq n$.

Definition 3.2.1. Let $X$ be a normed space and let $A$ be a commuting $n$-tuple of bounded operators on $X$. We set $\Lambda(X) := X \otimes \Lambda$ where $\Lambda$ is the exterior algebra defined on $n$ generators, $e_1, e_2, \ldots, e_n$.

Definition 3.2.2. For an $n$-tuple $A = (A_1, A_2, \ldots, A_n)$ one defines the coboundary map $D_A : \Lambda(X) \to \Lambda(X)$ by $D_A := \sum_{i=1}^{n} A_i \otimes E_i$. Where $E_i : \Lambda \to \Lambda$ is the creation
operator, given by $E_i \beta = e_i \beta$.

The following lemma follows directly from the collapsing property.

**Lemma 3.2.3.** We have that $D_A^2 = 0$.

Hence we have that the $\text{range} D_A \subset \text{kernel} D_A$. This fact leads to the following definition.

**Definition 3.2.4.** The commuting tuple $A$ is said to be non-singular on $X$ if $\text{range} D_A = \text{kernel} D_A$. The Taylor joint spectrum, or simply the Taylor spectrum of $A$ on $X$ is the set,

$$\sigma_T(A, X) := \{ \lambda \in \mathbb{C}^n | A - \lambda I \text{ is singular} \}.$$

There is an alternative definition for the Taylor spectrum. The decomposition $\Lambda = \bigoplus_{k=1}^n \Lambda^k$ gives rise to a cochain complex $K(A, X)$, the so-called Koszul complex associated to $A$ on $X$, which is defined as follows.

**Definition 3.2.5.** The Koszul complex associated to a commuting $n$-tuple $A$ on a normed space $X$ is:

$$K(A, X) : 0 \rightarrow \Lambda^0(x) \xrightarrow{D_A^0} \Lambda^1(x) \xrightarrow{D_A^1} \cdots \xrightarrow{D_A^{n-1}} \Lambda^n(x) \rightarrow 0$$

where $D_A^k$ denotes the restriction of $D_A$ to the subspace $\Lambda^k(X)$.

Thus we have the following:

**Theorem 3.2.6.**

$$\sigma_T(A, X) = \{ \lambda \in \mathbb{C}^n | K(A - \lambda I, X) \text{is not exact} \}.$$

One fact one should always check is nontrivially.

**Theorem 3.2.7.** (Taylor)? If $X$ is a Banach space, then $\sigma_T(A, X)$ is compact and non-empty.
Example 3.2.8. If $A$ is a commuting $n$-tuple of compact operators acting on a Banach space $X$, then $\sigma_T(A,X)$ is countable, with $(0,0,\ldots,0)$ as the only accumulation point.

Any candidate for a spectrum of a tuple of operators should in some way contain many of the same properties that made the spectrum an object worth studying. Taylor’s spectrum has many similar properties to those listed for the spectrum of a single operator, see [?] for a thorough list. The main restriction on Taylor’s spectrum is the requirement that the tuple commute. What follows is a discussion of a new spectrum defined for any tuple of elements of a unital Banach algebra.

### 3.3 Projective Spectrum

Much of what follows is established in [?]. We let $z = (z_1, z_2, \ldots, z_n)$ denote a general point in $\mathbb{C}^n$. The group $\mathbb{C}^\times$ of nonzero complex numbers acts on $\mathbb{C}^n$ by scalar multiplication. The $(n - 1)$-dimensional projective space $\mathbb{P}^{n-1}$ is the quotient $(\mathbb{C}^n \setminus \{0\})/\mathbb{C}^\times$. We will denote this quotient map by $\rho$. With topology induced from $\rho$, $\mathbb{P}^{n-1}$ is a compact complex manifold. $[z_1, z_2, \ldots, z_n]$ denotes the homogeneous coordinate of a general point in $\mathbb{P}^{n-1}$.

Unless stated otherwise, $A = (A_1, A_2, \ldots, A_n)$ always stands for an $n$-tuple of general elements in an unital Banach algebra $\mathcal{B}$.

In order to study the tuple $A$ we will form in the multilinear pencil associated to the tuple $A$.

**Definition 3.3.1.** The $\mathcal{B}$-valued linear function $A(z) = z_1A_1 + z_2A_2 + \cdots + z_nA_n$ is the multivariate pencil of $A$.

Without loss of generality, we assume the elements $A_1, A_2, \ldots, A_n$ are linearly independent, hence the range of $A(z)$ is an $n$ dimensional subspace of $\mathcal{B}$.

**Definition 3.3.2.** A sub-algebra of $\mathcal{B}$ is said to be inversion-closed if for an element $a$ in the sub-algebra, if $a$ is invertible in $\mathcal{B}$ then $a^{-1}$ is also in the sub-algebra.
Thus we associate an algebra to our tuple \(A\).

**Definition 3.3.3.** For a tuple \(A = (A_1, A_2, \ldots, A_n)\), we let \(B_A\) denote the smallest inversion-closed sub-algebra of \(B\) that contains \(A_1, A_2, \ldots, A_n\). Clearly, \(A(z)\) is invertible in \(B\) if and only if it is invertible in \(B_A\).

With this construction we make the following definition.

**Definition 3.3.4.** For a tuple \(A\), we let

\[P(A) = \{z \in \mathbb{C}^n : A(z) \text{ is not invertible in } B\}\]

The projective spectrum \(p(A)\) of \(A\) is \(\rho(P(A))\), i.e.,

\[p(A) = \{z = [z_1, z_2, \ldots, z_n] \in \mathbb{P}^{n-1} : A(z) \text{ is not invertible in } B\}\]

For simplicity, we also refer to \(P(A)\) as projective spectrum. The projective resolvent sets refer to their complements \(P^c(A) = \mathbb{P}^{n-1} \setminus p(A)\) and \(P^c(A) = \mathbb{C}^n \setminus P(A)\).

**Proposition 3.3.5.** Let \(B\) be a Banach algebra. Then for any tuple \(A\), \(p(A)\) is a nonempty compact subset of \(\mathbb{P}^{n-1}\).

The above proposition is an interesting fact, since the elements \(A_1, A_2, \ldots, A_n\) may have nothing to do with each other. Note that this result is trivial if one of the \(A_i\)’s is not invertible, but the projective spectrum is non-empty even if all of the elements are invertible. Let us look at a few examples.

**Example 3.3.6.** When \(B\) is the matrix algebra \(M_k(\mathbb{C})\), \(A = (A_1, A_2, \ldots, A_n)\) is a tuple of \(k \times k\) matrices. Then \(A(z)\) is invertible if and only if \(\det A(z) \neq 0\). Since \(\det A(z)\) is homogenous of degree \(k\),

\[p(A) = \{z = [z_1, z_2, \ldots, z_n] \in \mathbb{P}^{n-1} : \det A(z) = 0\}\]

is a projective hypersurface of degree \(k\). \(P^c(A)\) in this case is a hypersurface complement.
Example 3.3.7. Let $A_1$ be any element in $B$, and $A_2 = -I$. Then for the tuple $A = (A_1, A_2)$, $A(z) = z_1 A_1 - z_2 I$. Clearly, if $[z_1, z_2]$ is in $p(A)$ then $z_1 \neq 0$, and $p(A)$, under the affine coordinate $z_2/z_1$, is the classical spectrum $\sigma(A_1)$. So Proposition 3.3.2 in fact implies the nontriviality of the classical spectrum.

Example 3.3.8. Now consider $L^2(\mathbb{T}, m)$, where $m$ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. $\{w^n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T}, m)$. Let $\theta$ be an irrational number and set $\lambda = \exp(2\pi \sqrt{-1} \theta)$. Consider the two unitaries defined by

$$A_1 f(w) = wf(w), \quad A_2 f(w) = f(\lambda w), \quad f \in L^2(\mathbb{T}, m).$$

Then

$$A_2 A_1 = \lambda A_1 A_2,$$

and we let $B$ be the $C^*$-algebra generated by $A_1$ and $A_2$. Clearly, $A(z)$ is invertible if and only if $z_1 A_1 A_2^* + z_2 I$ is invertible. So by Example 2, $p(A) = -\sigma(A_1 A_2^*)$. One checks that $A_2^* f(w) = f(\bar{\lambda} w)$, hence

$$A_1 A_2^* w^n = w(\bar{\lambda} w)^n = \bar{\lambda}^n w^{n+1}.$$

So $A_1 A_2^*$ is a unitary bilateral weighted shift. Since $A_2 A_1 = \lambda A_1 A_2$,

$$A_2^* A_1 A_2 = A_2^* A_1 = \bar{\lambda} A_1 A_2^*,$$

hence $\sigma(A_1 A_2^*)$ is invariant under multiplication by $\bar{\lambda}$, which implies $\sigma(A_1 A_2^*) = \mathbb{T}$. Therefore

$$p(A) = \mathbb{T}, \quad \text{and} \quad P(A) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2|\}.$$

In this case, $P^c(A)$ consists of two connected components:

$$\Omega_1 = \{|z_1| > |z_2|\}, \quad \text{and} \quad \Omega_2 = \{|z_1| < |z_2|\}.$$

The $C^*$ algebra generated by $A_1$ and $A_2$ is the irrational rotation algebra often denoted by $A_\theta$, [?].

It is of interest to compare the projective spectrum to the Taylor spectrum in the commutative case. Moreover, when $A$ is a commuting tuple, $\mathcal{B}_A$ is a commutative algebra. In this case, the maximal ideal space $\mathcal{B}_A$ shall be denoted by $M_A$. 

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Theorem 3.3.9 (Yang). [?] If $A$ is a commuting tuple in a Banach algebra $\mathcal{B}$, then $P(A)$ is a union of hyperplanes. For simplicity, we let $H_\phi = \{z \in \mathbb{C}^n : \sum_{j=1}^n z_j \phi(A_j) = 0\}$, and one sees that

$$P(A) = \bigcup_{\phi \in M_A} H_\phi.$$  

In fact, as recently observed by M. Putinar, a finer representation of $P(A)$ can be given through the Taylor spectrum $\sigma_T(A)$. Let $f_z(w) = z_1 w_1 + z_2 w_2 + \cdots + z_n w_n$. Then $A(z) = f_z(A)$, and hence by spectral mapping theorem [?]

$$\sigma(A(z)) = f_z(\sigma_T(A)),$$

which implies that $A(z)$ is not invertible if and only if there exists a $w \in \sigma_T(A)$ such that $z_1 w_1 + z_2 w_2 + \cdots + z_n w_n = 0$. Therefore, the above can be rewritten as

$$P(A) = \bigcup_{w \in \sigma_T(A)} H_w,$$

where $H_w = \{z \in \mathbb{C}^n : z_1 w_1 + z_2 w_2 + \cdots + z_n w_n = 0\}$.

In the case when $P(A)$ is a union of a finite number of hyperplanes, $P(A)$ is also called a central hyperplane arrangement.

Example 3.3.10. Let $\mathcal{B} = A(\mathbb{D})$, the disk algebra. Thus we have $f(w) \in A(\mathbb{D})$ if $f(w)$ is holomorphic on the disk and continuous on the closed disk. The Gelfand transform tells us that an element, $g(w)$ of $A(\mathbb{D})$, is invertible if and only if $g(w) \neq 0$ for all $w \in \overline{\mathbb{D}}$. If we let $A = (1, w^1, w^2, \ldots, w^n)$ then we have that $A(z)$ has a zero if $z_1 = 0$. Thus we know that $P^c(A) \subset \{z | z_1 \neq 0\}$. Hence if we let $\xi_j = \frac{z_{j+1}}{z_1}$ then $A(z) = z_1 (1 + \xi_1 w + \xi_2 w^2 + \cdots + \xi_n w^n)$. Hence we have that $P^c(A) = \mathbb{C}^x \times p^c(A)$ where $p^c(A)$ is contractable.

If we change the Banach algebra to a similar space we can see how this changes the projective spectum

Example 3.3.11. Let $\mathcal{B} = C(\mathcal{T})$ be the algebra of all continuous functions on the circle. Then we know $C(\mathcal{T})$ is generated by $\{1, w, \bar{w}\}$. Let the tuple $A = (1, w, \bar{w})$.  

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Then by Gelfand we know that \( p(A) = \{ z | z_1 + z_2w + z_3\bar{w} \text{ has a zero in } \mathcal{T} \} \). In this case the projective resolvent set \( p^c(A) = \{(a,b,c) \in \mathbb{P}^3 | aw^2 + bw + c \text{ has no zero in } \mathcal{T} \} \) has three connected components given by

\[
\begin{align*}
\Omega_1 &:= \{(a,b,c) \in \mathbb{P}^3 | aw^2 + bw + c \text{ has no zero in } \mathbb{D} \} \\
\Omega_2 &:= \{(a,b,c) \in \mathbb{P}^3 | aw^2 + bw + c \text{ has one zero in } \mathbb{D} \} \\
\Omega_3 &:= \{(a,b,c) \in \mathbb{P}^3 | aw^2 + bw + c \text{ has two zeros in } \mathbb{D} \}.
\end{align*}
\]
Chapter 4

An Operator Valued 1-form

Let $A = (A_1, A_2, \ldots, A_n)$ be an $n$-tuple of elements of a unital Banach algebra $\mathcal{B}$. To make a study of $P^c(A)$ we resort to the Maurer-Cartan type $\mathcal{B}$-valued 1-form $\omega_A(z) := A(z)^{-1}dA(z)$ and multilinear functionals on $\mathcal{B}$.

The $\mathcal{B}$-valued 1-form $\omega_A(z)$ appears to contain much topological information about $P^c(A)$. Bounded linear functionals on $\mathcal{B}$ are good tools to decode it. First, one observes that for a $\phi \in \mathcal{B}^*$, $\phi(\omega_A(z)) := \sum_{j=1}^{n} \phi(A^{-1}(z)A_j)dz_j$ is a holomorphic 1-form on $P^c(A)$. Likewise, for a $k$-linear functional $F$, $F(\omega_A(z), \omega_A(z), \ldots, \omega_A(z))$ is a holomorphic $k$-form on $P^c(A)$. Here, since everything is holomorphic $d = \partial$. One interesting class of linear functionals are the following.

4.1 Introduction

Definition 4.1.1. A $k$-linear functional $F$ on $\mathcal{B}$ is said to be invariant if

$$F(a_1, a_2, \ldots, a_k) = F(ga_1g^{-1}, ga_2g^{-1}, \ldots, ga_kg^{-1})$$

for all $a_1, a_2, \ldots, a_k$ in $\mathcal{B}$ and every invertible $g$ in $\mathcal{B}$. An invariant 1-linear functional is usually called a trace.

Definition 4.1.2. If $F_1$ and $F_2$ are invariant $k$-linear, and respectively, $m$-linear
functionals, then an associative product $F_1 \times F_2$ can be defined by

$$(F_1 \times F_2)(a_1, a_2, ..., a_{k+m}) = F_1(a_1, a_2, ..., a_k)F_2(a_{k+1}, a_{k+2}, ..., a_{k+m}).$$

Clearly, $F_1 \times F_2$ is an invariant $(k + m)$-linear functional. We let $F^0 = \mathbb{C}$, and $F^k$ be the space of invariant $k$-linear functionals on $\mathcal{B}$, $k \geq 1$, and set

$$F^*(\mathcal{B}) = \bigoplus_{k=0}^{\infty} F^k(\mathcal{B}).$$

Then $F^*$ is a graded algebra over $\mathbb{C}$ with respect to above-defined product. Now consider the map $\tau$ from $F^*$ to holomorphic forms on $P^c(\mathcal{A})$ defined by

$$\tau(1) = 1, \text{ and } \tau(F) = F(\omega_\mathcal{A}(z), \omega_\mathcal{A}(z), ..., \omega_\mathcal{A}(z)).$$

**Theorem 4.1.3 (Yang).** [?] $\tau$ is a homomorphism from $F^*(\mathcal{B})$ into $H^*_d(P^c(\mathcal{A}), \mathbb{C})$.

This result brings to mind the well known theorem by C.G.J. Jacobi, that for $\mathcal{B} = M_n(\mathbb{C})$ we have

$$tr(\omega_\mathcal{A}) = d \log \det(\mathcal{A}(z)).$$

Seen in this light, the study of this map is the study of an extension of Jacobi’s formula.

One of the important facts used in the proof of the previous proposition is the following.

**Lemma 4.1.4.** $d(\omega_\mathcal{A}) = -\omega_\mathcal{A} \wedge \omega_\mathcal{A}$

**Proof.** First, we have that for each $j$,

$$\frac{\partial}{\partial z_j} A^{-1}(z) = -A^{-1}(z) A_j A^{-1}(z),$$
hence

\[ d\omega_A(z) = \sum_{j=1}^{n} dA^{-1}(z)A_j dz_j \]

\[ = \sum_{i,j=1}^{n} \frac{\partial A^{-1}(z)}{\partial z_i} A_j dz_i \wedge dz_j \]

\[ = \sum_{i,j=1}^{n} -A^{-1}(z)A_i A^{-1}(z)A_j dz_i \wedge dz_j \]

\[ = \sum_{i<j} (-A^{-1}(z)A_i A^{-1}(z)A_j - A^{-1}(z)A_j A^{-1}(z)A_i) dz_i \wedge dz_j \]

\[ = -\omega_A(z) \wedge \omega_A(z). \]

\[ \square \]

If \( \mathcal{B} \) is a Banach algebra with a trace \( \phi \), then

\[ F(a_1, a_2, \ldots, a_s) := tr(a_1a_2 \cdots a_s) \]

is in \( \mathcal{F}^s \), and

\[ F(\omega_A(z), \ldots, \omega_A(z)) = \phi(\omega^s_A(z)) \in H^s_d(P^c(A), \mathbb{C}). \]

If \( s \) is even, say \( s = 2k \) where \( k \geq 1 \), then because of the equality \( d\omega_A(z) = -\omega_A(z) \wedge \omega_A(z) \),

\[ \phi(\omega^s_A(z)) = (-1)^k tr((d\omega_A(z))^k) \]

\[ = (-1)^{k+1} d\phi(\omega_A(z)) (d\omega_A(z))^{k-1}), \]

which shows that \( \phi(\omega^s_A(z)) \) is trivial in \( H^s_d(P^c(A), \mathbb{C}) \). However, \( \phi(\omega^s_A(z)) \) can be nontrivial when \( s \) is odd.

The result is summarized in the following theorem.

**Theorem 4.1.5.** [??] If \( A \) is an \( n \)-tuple of elements from a Banach algebra with trace \( \phi \) and \( m \) a positive integer, \( \phi(\omega^{2m}_A) = 0. \)
The previous discussion makes it clear that there is much to be learned by studying larger powers of $\omega_A$. First we want to look at an arbitrary size power. Here we have that $\phi(\omega_A^p) \in H^p_d(P^c(A), \mathbb{C})$. A simple calculation using properties of a trace yields the following lemma.

For simplicity we will reserve

$$W_i := (A(z))^{-1}A_i.$$ 

**Lemma 4.1.6.** If $A$ is an $n$-tuple of elements in a Banach algebra $B$ with trace $\phi$. Let $G_J$ be the permutation group of $\{i_2, i_3, \ldots, i_m\}$, for $m \leq n$ and if $\pi \in G_J$ then $W_\pi = W_{\pi(i_2)} \cdot W_{\pi(i_3)} \cdots W_{\pi(i_m)}$. Then

$$\phi(\omega_A^m) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} I_J d z_{i_1} \wedge \cdots \wedge d z_{i_m}$$

where $J$ is the ordered list $J = \{i_1, i_2, \ldots, i_m\}$ and $I_J = m \cdot \phi(\sum_{\pi \in G_J} (-1)^{sgn \pi} W_{i_1} W_{\pi})$.

**Proof.** When we multiply the forms we get only those terms who do not have a repeated term. Thus they are all of the terms in the permutation group. Once the forms are put in the canonical basis the sign of the term is the same as the sign of the permutation. Then, since the trace cycles, we can move it so that the same term is the leading term. This leads to many similar terms, $m$, since after collecting we have all the permutations of $m - 1$ objects. Then we are done.

$$\phi(\omega_A^m) = \phi(\omega_A \wedge \omega_A \wedge \cdots \wedge \omega_A)$$

$$= \phi(\sum_k W_k d z_k) \wedge (\sum_k W_k d z_k) \wedge \cdots \wedge (\sum_k W_k d z_k))$$

$$= \sum_R \phi(W_{k_1} W_{k_2} \cdots W_{k_m}) d z_{k_1} \wedge \cdots \wedge d z_{k_m}$$

Here $R$ denotes the set of $m$ tuples of $\{W_k\}$ where each element is distinct. Also note that the $m$-form is not ordered. Now we want to order our differentials and use the cyclicity of the trace to order our $W_k$’s. Because we may cycle elements in a trace we may classify this collection by it’s smallest element. But after reordering we have $m$ copies of the same tuple, for every tuple. Because by fixing one element we have
gone from $m!$ permutations of a given collection of $m$ elements of $\{W_i\}$ to $(m - 1)!$ permutations. Hence we have:

$$\phi(\omega_m^A) = \sum_R \phi(W_{i_1}W_{i_2}\cdots W_{i_m})dz_{i_1}\wedge\cdots\wedge dz_{i_m}$$

$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} m \cdot \phi(\sum_{\pi \in G_j} (-1)^{sgn\pi}W_{i_1}W_{i_2}\cdots W_{i_m})dz_{i_1}\wedge\cdots\wedge dz_{i_m}$$

$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} I_j dz_{i_1}\wedge\cdots\wedge dz_{i_m}$$

The lemma 4.1.6 has some interesting consequences. Recall that when the tuple $A$ is commuting we can recover the Taylor spectrum.

**Lemma 4.1.7.** Let $A$ be a commuting tuple of operators from $\mathcal{B}$ with trace $\phi$. Then $\phi(\omega_A^n) = 0$ for all $m \geq 2$.

**Proof.** It will suffice to show that $\omega_A \wedge \omega_A = 0$ for $A$ a commuting tuple. Let $A = (A_1, A_2, \ldots, A_n)$ be a commuting $m$-tuple of operators in a unital Banach algebra $\mathcal{B}$. Then

$$\omega_A \wedge \omega_A = \left(\sum_{i=1}^n W_idz_i\right) \wedge \left(\sum_{j=1}^n W_jdz_j\right)$$

$$= \sum_{1 \leq i,j \leq n} (W_iW_j)dz_i \wedge dz_j = \sum_{1 \leq i,j \leq n} (W_iW_j - W_jW_i)dz_i \wedge dz_j = 0$$

The previous result is not true for $m = 1$.

**Example 4.1.8.** Let $\mathcal{B} = C(T)$ the space of continuous functions on the circle. Since $C(T)$ is commutative all bounded linear functionals on $C(T)$ are traces. Moreover, $C(T)^* = \{\mu | \mu \text{ is a regular Borel measure on } T\} (= HC^0(C(T)))$. Let $A = (1, w, w)$, we have seen that $p^c(A) = \cup_{i=1}^3 \Omega_i$. Thus, on $\Omega_1$ then

$$\int_T \omega_A d\mu = \int_T \frac{dz_1 + wdz_2 + \bar{w}dz_3}{z_1 + w \cdot z_2 + \bar{w} \cdot z_3} d\mu.$$
Now the integrand is continuous for \( w \in \mathbb{D} \), so the integral is just the value at zero. For the standard borel measure we get

\[
\int_{T} \omega_A dw = \frac{dz_1}{z_1} \neq 0.
\]

The lemma 4.1.6 allows us to work with the elements and calculate the following Theorem.

**Theorem 4.1.9.** If \( A \) is an \( n \)-tuple of elements from a Banach algebra with trace \( \phi \) we have that \( \phi(\omega_A^{n-1}) = f(z)s(z) \) for some holomorphic function on \( P^c(A) \) and \( s(z) = \sum_{j=1}^{n} (-1)^j z_j \cdot d\overline{z}_j \).

**Proof.** The idea of the proof is to add and subtract the same value so that we can cancel one of the \((A(z))^{-1}\), i.e. that \( \sum_{k=1}^{n} z_k W_k = I \). In this case the subsets \( j \) defined in the previous lemma have all but one term. Hence we will use \( I_j \) to denote the \( I_k \) that does not include \( j \). Thus the lemma gives us;

\[
\phi(\omega_A^{n-1}) = \sum_{j=1}^{n} I_j d\overline{z}_j.
\]

We have already seen that \( \phi(\omega_A^{2k}) = 0 \). Thus we may assume \( n \) is even. Let \( G_j \) be the permutations of the \( n - 2 \) elements, \( \{2, 3, \ldots, j - 1, j + 1, \ldots, n\} \). We have the following for \( 1 < i < n \);

\[
\phi\left( \sum_{\pi \in G_{\mathbb{R}}} (-1)^{sgn\pi} W_i W_{\pi} \right) = 0
\]

This follows because the sign of the permutation changes when we switch two elements. But the trace allows us to cycle the elements without changing the sign. Let \( \pi_0 \in G_{\mathbb{R}} \) then there exits a \( \pi_1 \in G_{\mathbb{R}} \) such that \( \pi_0(i) = \pi_1(i + 1) \) then \( sgn\pi_0 = -sgn\pi_1 \).

But if \( 1 < i < n \), \( W_i \) is an element that \( G_{\mathbb{R}} \) is permuting, thus

\[
\phi(W_i W_{\pi_0}) = \phi(W_i W_{\pi_1}).
\]

Thus we have the statement above.
Using that fact we have the following:

\[
 z_1 \cdot I_\bar{n} = \phi(z_1 W_1 \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) \\
= \phi(z_1 W_1 \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) + \phi(z_2 W_2 \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) \\
+ \phi(z_3 W_3 \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) + \cdots + \phi(z_{n-1} W_{n-1} \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) \\
+ \phi(z_n W_n \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) - \phi(z_n W_n \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) \\
= \phi((\sum_{k=1}^{n} z_k W_k) \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) - \phi(z_n W_n \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) \\
= \phi(\sum_{\pi \in G_\bar{n}} W_\pi) - (-1)^{n-1} z_n \cdot I_\bar{n} \\
= (-1)^{n} z_n \cdot I_\bar{n}
\]

Since \( n \) is even \( \phi(\omega_n^{n-2}) = 0 \). We know \( \phi(\sum_{\pi \in G_\bar{n}} W_\pi) \) is an element of \( \phi(\omega_n^{n-2}) \) by the lemma 4.1.6. Thus, \( \phi(\sum_{\pi \in G_\bar{n}} W_\pi) = 0 \). Hence \( z_1 \cdot I_\bar{n} = z_n \cdot I_\bar{n} \).

The other terms are similar, the only change is that \( \phi(z_2 W_2 \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) = z_2 I_\bar{1} \) and \( \phi(z_2 W_2 \cdot (\sum_{\pi \in G_\bar{n}} W_\pi)) = -z_2 I_\bar{1} \)

Thus the complete formula is

\[
z_i \cdot I_\bar{n} = (-1)^i z_n \cdot I_\bar{i}
\]

for all \( 1 \leq i < n \). Thus we have that

\[
\phi(\omega^{n-1}) = \sum_{j=1}^{n} I_j dz_j = \sum_{j=1}^{n} I_\bar{n} \cdot \frac{z_i}{z_n} dz_j
\]

If we let \( s(z) = \sum_{j=1}^{n} (-1)^j z_j \cdot dz_j \) and let \( f(z) = \frac{1}{z_n} \cdot I_\bar{n} \). Then \( f(z) \) is a holomorphic function on \( P^c(A) \) because if \( z_n = 0 \) then \( I_\bar{n} = 0 \). Hence we have the desired formula

\[
\phi(\omega^{n-1}) = f(z) \cdot s(z).
\]

\[\Box\]

Note that \( d(s(z)) = n \cdot dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \).

The proof of the previous Theorem leads to the following Corollary.
Corollary 4.1.10. Let \( A \) be an \( n \)-tuple of elements from a Banach algebra with trace \( \phi \). Then \( \phi(\omega_A^n) = 0 \) for \( m \geq n > 1 \).

Proof. It can be seen from the proof the previous theorem that if \( A \) was in fact a \((n - 1)\)-tuple then there would be no non-zero extra term. Thus adding zero will get you zero. Hence the term you started with must be zero. \( \square \)

Moreover we have the following condition for \( \phi(\omega_A^n) \) to be zero.

Corollary 4.1.11. If \( A \) is a linearly dependent \( n \)-tuple with \( m \) or less independent terms then \( \phi(\omega_A^n) = 0 \).

Proof. Suppose \( X_i = A_i \) for \( 1 \leq i \leq s \) are the linearly independent terms of the \( n \)-tuple \( A = (A_1, A_2, \ldots, A_n) \) with \( s \leq m \). Suppose \( A_{s+l} = \sum_{j=1}^{s} c_j A_j \) for \( 1 \leq l \leq n - s \). Let \( \xi_j = z_j + \sum_{l=1}^{n-s} c_j z_{s+l} \). Then we have that \( A(z) = X(\xi) \) and \( dA(z) = dX(\xi) \). Hence \( \omega_A^n(z) = \omega_X^n(\xi) \) for all \( m \). But if \( m \geq s \) then we have that \( \phi(\omega_X(\xi)) = 0 \), thus \( \phi(\omega_A(z)) = 0 \). \( \square \)

There is also a relation when we are dealing with self-adjoint operators.

Corollary 4.1.12. Suppose \( \mathcal{B} \) is a \( C^* \)-algebra. If \( A \) is an \( n \)-tuple of self adjoint operators and \( \phi \) is a trace on \( \mathcal{B} \), then \( \phi(\omega_A^{n-1}) = f(z)s(z) \) where \( f(z) = -\overline{f(z)} \).

Proof. Note that we assume \( n \) is even, if not then the function \( f(z) = 0 \) and the statement is vacuously true. Since \( A \) is self adjoint we have that \( A(z)^* = A(z) \) and we have that \( W_i(z)^* = W_i(\bar{z}) \). In a \( C^* \)-algebra we have that \( \phi(A^*) = \overline{\phi(A)} \). This follows from the positivity of \( \phi \). We also know that \( f(z) = (n - 1) \cdot I_n \) from 4.1.6. If

\[ f(z) = -\overline{f(z)} \]
we let $G_n$ denote the permutation group of $\{2, \ldots, n-1\}$ then;

$$\frac{f(\bar{z})}{n-1} = \phi\left( \sum_{\pi \in G_n} (-1)^{sgn\pi} W_1(\bar{z}) W_\pi(\bar{z}) \right)$$

$$= \phi\left( \sum_{\pi \in G_n} (-1)^{sgn\pi} W_1^*(z) W_\pi^*(z) \right)$$

$$= \phi\left( \sum_{\pi \in G_n} (-1)^{sgn\pi} (W_\pi(z) W_1(z))^* \right)$$

$$= \phi\left( \sum_{\pi \in G_n} (-1)^{sgn\pi} (W_\pi(z) W_1(z)) \right)$$

$$= \phi\left( \sum_{\pi \in G_n} (-1)^{sgn\pi+1} (W_1(z) W_\pi(z)) \right)$$

$$= - \frac{f(z)}{4}$$

\[\square\]

### 4.2 $\phi(\omega^3_A)$

It is enlightening to look at the first non trivial case of the previous theorems. In that case we have the following. The proofs have been included for clarity.

**Lemma 4.2.1.** If $A = (A_1, A_2, A_3, A_4)$ is a 4-tuple of elements in a Banach algebra $B$ with trace, $\phi$. Let $W_k = (A(z))^{-1} A_k$ Then $\phi(\omega^3_A) = \sum_{1 \leq i < j < k \leq 4} I_{ijk} dz_i \wedge dz_j \wedge dz_k$. Where $I_{ijk} = 3 \cdot \phi(W_i W_j W_k - W_i W_k W_j)$

**Proof.** Recall that for $A, C \in B$ and $x \in \mathbb{C}$ we have $\phi(AC) = \phi(CA)$ and $\phi(xA) = x\phi(A)$. Thus we have the following;

$$\phi(\omega^3_A) = \phi(\omega_A \wedge \omega_A \wedge \omega_A)$$

$$= \phi\left( \sum_{i=1}^{4} (W_idz_i) \wedge \sum_{i=1}^{4} (W_idz_i) \wedge \sum_{i=1}^{4} (W_idz_i) \right)$$
\[ \phi((W_1 W_2 W_3 - W_1 W_3 W_2 - W_2 W_1 W_3 + W_2 W_3 W_1 + W_3 W_1 W_2 - W_3 W_2 W_1)dz_1 \wedge dz_2 \wedge dz_3 \\
+ (W_1 W_2 W_4 - W_1 W_4 W_2 - W_2 W_1 W_4 + W_2 W_4 W_1 + W_4 W_1 W_3 - W_4 W_3 W_1)dz_1 \wedge dz_2 \wedge dz_4 \\
+ (W_1 W_3 W_4 - W_1 W_4 W_3 - W_3 W_1 W_4 + W_3 W_4 W_1 + W_4 W_1 W_3 - W_4 W_3 W_1)dz_1 \wedge dz_3 \wedge dz_4 \\
+ (W_2 W_3 W_4 - W_2 W_4 W_3 - W_3 W_2 W_4 + W_3 W_4 W_2 + W_4 W_2 W_3 - W_4 W_3 W_2)dz_2 \wedge dz_3 \wedge dz_4) \\
= 3 \cdot \phi((W_1 W_2 W_3 - W_1 W_3 W_2)dz_1 \wedge dz_2 \wedge dz_3 + (W_1 W_2 W_4 - W_1 W_4 W_2)dz_1 \wedge dz_2 \wedge dz_4 \\
+ (W_1 W_3 W_4 - W_1 W_4 W_3)dz_1 \wedge dz_3 \wedge dz_4 + (W_2 W_3 W_4 - W_2 W_4 W_3)dz_2 \wedge dz_3 \wedge dz_4) \\
\]

With this intermediate step accomplished we have the following corollary

**Corollary 4.2.2.** If \( A = (A_1, A_2, A_3, A_4) \) is a 4-tuple of elements in a Banach algebra \( B \) with trace, \( \phi \). Then

\[ \phi(\omega^3_A) = f(z)s(z) \]

for the three form \( s(z) = z_1 dz_2 \wedge dz_3 \wedge dz_4 - z_2 dz_1 \wedge dz_3 \wedge dz_4 + z_3 dz_1 \wedge dz_2 \wedge dz_4 - z_4 dz_1 \wedge dz_2 \wedge dz_3 \). Moreover, \( f(z) \) is holomorphic on \( P^e(A) \).

**Proof.** As per the lemma 4.1.6 we are able to write

\[ \phi(\omega^3_A) = \sum_{1 \leq i < j < k \leq 4} I_{ijk} dz_i \wedge dz_j \wedge dz_k. \]

Furthermore we have the following identity, \( \frac{I_{123}}{z_4} = -\frac{I_{143}}{z_4} \), this is seen by the fol-
following calculation;

\[
\frac{z_4}{3} I_{123} = \phi(W_1 W_2 W_3 - W_1 W_3 W_2)
= \phi(W_1 W_2 z_3 W_3 - W_1 z_3 W_3 W_2)
+ W_1 W_2 z_1 W_1 - W_1 z_1 W_1 W_2
+ W_1 W_2 z_2 W_2 - W_1 z_2 W_2 W_2
+ W_1 W_2 z_4 W_4 - W_1 z_4 W_4 W_2
- W_1 W_2 z_4 W_4 + W_1 z_4 W_4 W_2)
= \phi(W_1 W_2 A(z)^{-1} A(z) - W_1 A(z) A(z)^{-1} W_2)
- \phi(W_1 W_2 z_4 W_4 - W_1 z_4 W_4 W_2)
= \phi(W_1 W_2 - W_1 W_2)
- z^4 \phi(W_1 W_2 z_4 W_4 - W_1 z_4 W_4 W_2)
= -\frac{z_4}{3} I_{124}.
\]

A similar calculation shows that \( \frac{I_{123}}{z_4} = -\frac{I_{124}}{z_3} = \frac{I_{134}}{z_2} = -\frac{I_{234}}{z_1} \). Since \( \phi(\omega^3_A) = I_{123} dz_1 \wedge dz_2 \wedge dz_3 + I_{124} dz_1 \wedge dz_2 \wedge dz_4 + I_{134} dz_1 \wedge dz_3 \wedge dz_4 + I_{234} dz_2 \wedge dz_3 \wedge dz_4 \). It follows that \( \phi(\omega^3_A) = -\frac{I_{123}}{z_4} s \). Note, if \( z_4 = 0 \) then the above calculation shows \( I_{123} = 0 \). Hence \( -\frac{I_{123}}{z_4} \) is holomorphic on \( P^c(A) \).

\[\square\]

### 4.3 \( M_n(\mathbb{C}) \)

As we will see below, in the matrix algebra case we will make use of the determinant of a matrix. In certain circumstances we may also define a determinant of an operator.

This might lead to further reductions of the characterization of higher order trace class formulae. We want to first look at what can be done in the matrix case. The hope for future research is that insights into the matrix case will give us formulae in the trace class case.

Recall the well known theorem by C.G.J. Jacobi [?], that for \( B = M_n(\mathbb{C}) \) we have \( tr(\omega_A) = d \log det(A(z)) \). We have seen what happens in the case of general
Banach algebras, it is interesting to discover what we can say about the case Jacobi was interested in, the matrix algebra, \( M_n(\mathbb{C}) \). There are many possible approaches to matrices, but the theorem most relevant to us is one of Jacobi’s theorems on determinants of square matrices.

What follows is a statement of a formula of Jacobi as seen in Charles Dodgson’s *Elementary Treatise on Determinants*.

**Theorem 4.3.1** (Jacobi). If there be a square block of the \( n \)th degree, and if in it any minor of the \( m \)th degree be selected: the determinant is the corresponding minor in the adjugate block is equal \( \cdots \) to the product of the \((m-1)\)th power of the determinate of the first block, multiplied by the determinant of the minor complimentary to the one selected.

Recall that if a matrix \( B \) is invertible then \( B^{-1} = \frac{1}{\det(B)} \cdot B^\# \). We call the matrix \( B^\# \) the adjugate matrix. The entries of the adjugate matrix are given by \( B_{i,j}^\# = (-1)^{i+j} \det(C_{i,j}) \) where \( C_{i,j} \) is the minor matrix obtained by deleting the \( i \)th row and the \( j \)th column of \( B \). How this is used is summarized in the following Lemma.

**Lemma 4.3.2** (Jacobi). Let \( B \) be an \( n \times n \) matrix. Then

\[
B_{i,j}^\# B_{p,q}^\# - B_{i,q}^\# B_{p,j}^\# = \det(B) \cdot \det(C_{(i,j),(p,q)})
\]

where \( C_{(i,j),(p,q)} \) is the matrix obtained from \( B \) by removing rows \( i, p \) and columns \( j, q \).

Using this lemma we are able to further characterize \( \text{tr} (\omega_A^3) \).

**Theorem 4.3.3.** If \( A = (A_1, A_2, A_3, A_4) \) is a tuple of elements of the Banach algebra \( M_n(\mathbb{C}) \), then

\[
\text{tr} (\omega_A^3) = \frac{p(z)}{\det^2(A(z))} s(z)
\]

where \( p(z) \) is a homogeneous polynomial of degree \( 2n - 4 \).

**Proof.** First we make the observation that \( \omega_A \) is homogeneous of degree 0. Thus \( \text{tr} (\omega_A^3) \) is homogeneous of degree 0. Further \( \det(A(z)) \) is homogeneous of degree \( 2n \)
and \(s\) is homogeneous of degree 4. Thus \(p(z)\) must be a homogeneous polynomial of degree \(2n - 4\). What is left to prove is that the holomorphic function \(f(z)\) factors as

\[
f(z) = \frac{p(z)}{(\det(A(z)))^2}.
\]

Following the notation from above we can say \(f(z) = -I_{123}z^4\). Hence we strive to simplify the term \(I_{123}\). First we make the following decomposition \(A(z)^{-1} = \frac{1}{\det(A(z))} \cdot B\) where \(B = (A(z))^\#\). Thus we have the following result;

\[
I_{123} = \frac{1}{(\det(A(z)))^3} \cdot \text{tr}(BA_1BA_2BA_3 - BA_1BA_3BA_2).
\]

Hence what remains to be proven is that \(\text{tr}(BA_1BA_2BA_3 - BA_1BA_3BA_2) = p(z) \cdot \det(A(z))\) First we note that since the trace is linear we can work on elementary matrices. In particular if we denote \(A_r = [a_{i,j}^r]\) and let \(E_{i,j}\) to be the matrix with 1 in the \(i\)th row \(j\)th column and zero otherwise. Then we have;

\[
\text{tr}(BA_1BA_2BA_3 - BA_1BA_3BA_2) = \sum_{1 \leq p,q,k,l,r,s \leq n} a_{p,q}^1 a_{k,l}^2 a_{r,s}^3 \cdot \text{tr}(BE_{p,q}BE_{k,l}BE_{r,s} - BE_{p,q}BE_{r,s}BE_{k,l})
\]

Basic matrix multiplication leads to the fact that;

\[
\text{tr}(BE_{p,q}BE_{k,l}BE_{r,s} - BE_{p,q}BE_{r,s}BE_{k,l}) = B_{s,p}B_{q,k}B_{l,r} - B_{t,p}B_{q,r}B_{s,k}
\]

Now we use Jacobi’s formula repeatedly;

\[
B_{s,p}B_{q,k}B_{l,r} - B_{t,p}B_{q,r}B_{s,k}
\]

\[
= B_{s,p}B_{q,k}B_{l,r} - B_{q,r}(B_{l,k}B_{s,p} + \det(A(z)) \cdot \det(C_{(l,p),(s,k)}))
\]

\[
= B_{s,p}B_{q,k}B_{l,r} - B_{q,r}B_{l,k}B_{s,p} - B_{q,r}\det(A(z)) \cdot \det(C_{(l,p),(s,k)}))
\]

\[
= B_{s,p}(B_{q,k}B_{l,r} - B_{q,r}B_{l,k}) - B_{q,r}\det(A(z)) \cdot \det(C_{(l,p),(s,k)}))
\]

\[
= B_{s,p}(\det(A(z)) \cdot \det(C_{(q,k),(l,r)})) - B_{q,r}\det(A(z)) \cdot \det(C_{(l,p),(s,k)}))
\]

\[
= \det(A(z))(B_{s,p} \cdot \det(C_{(q,k),(l,r)})) - B_{q,r} \cdot \det(C_{(l,p),(s,k)}))
\]

Hence we have that for any values of \(p, q, k, l, r, s\) we can factor out \(\det(A(z))\). Thus we have:

\[
I_{123} = \frac{1}{(\det(A(z)))^2} \cdot \sum_{1 \leq p,q,k,l,r,s \leq n} a_{p,q}^1 a_{k,l}^2 a_{r,s}^3 \cdot (B_{s,p} \cdot \det(C_{(q,k),(l,r)})) - B_{q,r} \cdot \det(C_{(l,p),(s,k)}))
\]

\(\square\)
We have the following interesting corollary to the previous theorem.

**Corollary 4.3.4.** If $A$ is a tuple of upper triangular matrices then $\text{tr}(\omega_A^3) = 0$

**Proof.** We can see from the above formula that

\[
I_{123} = \frac{1}{(\text{det}(A(z)))^2} \cdot \sum_{1 \leq p,q,k,l,r,s \leq n} a^1_{p,q} a^2_{k,l} a^3_{r,s} \cdot (B_{s,p} \cdot \text{det}(C_{(q,k),(l,r)}) - B_{q,r} \cdot \text{det}(C_{(l,p),(s,k)})).
\]

In the case of upper triangular matrices the matrices $A_i$ all have the property that $a^i_{p,q} = 0$ if $p > q$. The matrices $C_{(i,p),(s,k)}$ are also upper triangular. Furthermore, the matrix $B$ is upper triangular.

For an upper triangular matrix the determinant is the product of the diagonal terms, thus the only way for $\text{det}(C_{(i,p),(s,k)}) \neq 0$ is for $p = l, s = k$. But we have that $p \leq q, k \leq l, r \leq s$ from the original matrices and $q \leq r$ from the adjugate matrix. Hence we have that $q \leq r \leq s = k \leq l \leq q$. Thus the only non-zero terms are those who have the same indices. But if $B_{q,r} \cdot \text{det}(C_{(l,p),(s,k)}) \neq 0$ then so is $B_{s,p} \cdot \text{det}(C_{(q,k),(l,r)})$. But we are subtracting them, thus we get that each term is zero and so $I_{123} = 0$. 

**4.3.1 $M_2(\mathbb{C})$**

There is a further reduction possible in the case $\mathcal{B} = M_2(\mathbb{C})$.

**Example 4.3.5.** When $\mathcal{B} = M_2(\mathbb{C})$ we have that the polynomial $p(z)$ is a constant.

Specifically, let $A_j = \begin{bmatrix} a^j_1 & a^j_2 \\ a^j_3 & a^j_4 \end{bmatrix}$ for $j \leq n$. Let $\sigma$ be a one-to-one map from $\{1, 2, 3, 4\}$ to $\{1, 2, 3, \ldots, n\}$ and $S$ the collection of all such $\sigma$. Then we can define matrices

\[
c_\sigma = \begin{pmatrix} a^{(1)}_1 & a^{(2)}_1 & a^{(3)}_1 & a^{(4)}_1 \\ a^{(1)}_2 & a^{(2)}_2 & a^{(3)}_2 & a^{(4)}_2 \\ a^{(1)}_3 & a^{(2)}_3 & a^{(3)}_3 & a^{(4)}_3 \\ a^{(1)}_4 & a^{(2)}_4 & a^{(3)}_4 & a^{(4)}_4 \end{pmatrix}
\]

such that $p(z) = \sum_{\sigma \in S} \text{det}(C_\sigma)$.

To illustrate the previous theorems and show the difficulty of calculation by hand we include the entire result.
If $W = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$ then, $dW = \begin{bmatrix} dw_1 & dw_2 \\ dw_3 & dw_4 \end{bmatrix}$ and

$W^{-1} = \frac{1}{w_1w_4 - w_2w_3} \begin{bmatrix} w_4 & -w_2 \\ -w_3 & w_1 \end{bmatrix}$.

Thus;

$\omega_W = W^{-1}dW = \frac{1}{w_1w_4 - w_2w_3} \begin{bmatrix} w_4dw_1 - w_2dw_3 & w_4dw_2 - w_2dw_4 \\ w_1dw_3 - w_3dw_1 & w_1dw_4 - w_3dw_2 \end{bmatrix}$.

Now we have that;

$(\omega^2_W)_{1,1} = \frac{1}{(w_1w_4 - w_2w_3)^2} (w_4dw_1 - w_2dw_3 \wedge (w_4dw_1 - w_2dw_3)
+ (w_4dw_2 - w_2dw_4) \wedge (w_1dw_3 - w_3dw_1))$

$= \frac{1}{(w_1w_4 - w_2w_3)^2} (w_3w_4dw_1 \wedge dw_2 + w_1w_4dw_2 \wedge dw_3
- w_2w_3dw_1 \wedge dw_4 + w_1w_2dw_3 \wedge dw_4)$.

$(\omega^2_W)_{1,2} = \frac{1}{(w_1w_4 - w_2w_3)^2} (w_4dw_1 - w_2dw_3 \wedge (w_4dw_2 - w_2dw_4)
+ (w_4dw_2 - w_2dw_4) \wedge (w_1dw_4 - w_3dw_2))$

$= \frac{1}{(w_1w_4 - w_2w_3)^2} (w_4w_4dw_1 \wedge dw_2 - w_2w_4dw_1 \wedge dw_4 + w_2w_4dw_2 \wedge dw_3
+ w_2w_2dw_3 \wedge dw_4 + (w_1w_4 - w_2w_3)dw_2 \wedge dw_4)$.

$(\omega^2_W)_{2,1} = \frac{1}{(w_1w_4 - w_2w_3)^2} ((w_1dw_3 - w_3dw_1) \wedge (w_4dw_1 - w_2dw_3)
+ (w_1dw_4 - w_3dw_2) \wedge (w_1dw_3 - w_3dw_1))$

$= \frac{1}{(w_1w_4 - w_2w_3)^2} (-w_3w_3dw_1 \wedge dw_2 - w_1w_3dw_2 \wedge dw_3
+ w_1w_3dw_1 \wedge dw_4 - w_1w_1dw_3 \wedge dw_4 + (w_2w_3 - w_1w_4)dw_1 \wedge dw_3)$.
\[ (\omega_{W}^{2})_{2,2} = \frac{1}{(w_{1}w_{4} - w_{2}w_{3})^{2}}((w_{1}dw_{3} - w_{3}dw_{1}) \wedge (w_{4}dw_{1} - w_{2}dw_{3}) + (w_{1}dw_{4} - w_{3}dw_{2}) \wedge (w_{4}dw_{1} - w_{2}dw_{3})) \]

Therefore;

\[ (\omega_{W}^{3})_{1,1} = \frac{1}{(w_{1}w_{4} - w_{2}w_{3})^{3}}(((\omega_{W}^{2})_{1,1}) \wedge (w_{4}dw_{1} - w_{2}dw_{3}) + ((\omega_{W}^{2})_{1,2}) \wedge (w_{1}dw_{3} - w_{3}dw_{1})) \]
\[ = \frac{1}{(w_{1}w_{4} - w_{2}w_{3})^{3}}(w_{1}w_{4}w_{1}dw_{w_{4}} \wedge dw_{2} \wedge dw_{3} + w_{1}w_{2}w_{4}dw_{1} \wedge dw_{3} \wedge dw_{4} - w_{2}w_{3}w_{4}dw_{1} \wedge dw_{2} \wedge dw_{3} - w_{2}w_{3}w_{4}dw_{1} \wedge dw_{3} \wedge dw_{4} - w_{2}w_{3}w_{4}dw_{1} \wedge dw_{2} \wedge dw_{4} - w_{1}w_{2}w_{3}w_{4}dw_{1} \wedge dw_{3} \wedge dw_{4} + w_{1}w_{2}w_{3}w_{4}dw_{1} \wedge dw_{3} \wedge dw_{4} + w_{1}w_{2}w_{3}w_{4}dw_{1} \wedge dw_{2} \wedge dw_{4}) \]

\[ (\omega_{W}^{3})_{2,2} = \frac{1}{(w_{1}w_{4} - w_{2}w_{3})^{3}}(((\omega_{W}^{2})_{2,1}) \wedge (w_{4}dw_{2} - w_{2}dw_{4}) + ((\omega_{W}^{2})_{2,2}) \wedge (w_{1}dw_{4} - w_{3}dw_{2})) \]
\[ = \frac{1}{(w_{1}w_{4} - w_{2}w_{3})^{3}}(w_{1}w_{4}w_{1}dw_{w_{4}} \wedge dw_{2} \wedge dw_{3} - w_{2}w_{3}w_{4}dw_{1} \wedge dw_{2} \wedge dw_{4} - w_{2}w_{3}w_{4}dw_{1} \wedge dw_{3} \wedge dw_{4} + w_{1}w_{2}w_{3}w_{4}dw_{1} \wedge dw_{3} \wedge dw_{4} + w_{1}w_{2}w_{3}w_{4}dw_{1} \wedge dw_{2} \wedge dw_{4} + w_{1}w_{2}w_{3}w_{4}dw_{1} \wedge dw_{3} \wedge dw_{4} - w_{1}w_{2}w_{3}w_{4}dw_{1} \wedge dw_{2} \wedge dw_{4} - w_{1}w_{2}w_{3}w_{4}dw_{1} \wedge dw_{3} \wedge dw_{4}) \]
Hence;

\[
tr \omega^3_W = \frac{3}{(w_1 w_4 - w_2 w_3)^3} (w_1 w_4^2 - w_2 w_3 w_4) dw_1 \wedge dw_2 \wedge dw_3 \\
+ (w_2 w_3^2 - w_1 w_4 w_3) dw_1 \wedge dw_2 \wedge dw_4 + (w_1 w_2 w_4 - w_2 w_3^2) dw_1 \wedge dw_3 \wedge dw_4 \\
+ (w_1 w_2 w_3 - w_1^2 w_4) dw_2 \wedge dw_3 \wedge dw_4)
\]

= \frac{3}{(w_1 w_4 - w_2 w_3)^3} (w_4 dw_1 \wedge dw_2 \wedge dw_3 - w_3 dw_1 \wedge dw_2 \wedge dw_4 \\
+ w_2 dw_1 \wedge dw_3 \wedge dw_4 - w_1 dw_2 \wedge dw_3 \wedge dw_4).

If we now substitute \( w_i = \sum_{j=1}^n a_i^j z_j \) we have \( W = A(z) \) where \( A_j = \begin{bmatrix} a_1^j & a_2^j \\ a_3^j & a_4^j \end{bmatrix} \)

For simplification let \( \sigma \) be a one-to-one map from \( \{1, 2, 3, 4\} \) to \( \{1, 2, \ldots, n\} \) such that \( \sigma(2) < \sigma(3) < \sigma(4) \) and let \( S \) be the collection of all such \( \sigma \). Moreover let

\[
c_\sigma = det \left( \begin{array}{cccc}
a_1^{(1)} & a_1^{(2)} & a_1^{(3)} & a_1^{(4)} \\
a_2^{(1)} & a_2^{(2)} & a_2^{(3)} & a_2^{(4)} \\
a_3^{(1)} & a_3^{(2)} & a_3^{(3)} & a_3^{(4)} \\
a_4^{(1)} & a_4^{(2)} & a_4^{(3)} & a_4^{(4)} \end{array} \right).
\]

Another straightforward, if lengthy, calculation leads to the following formula:

\[
tr(\omega^3_A(z)) = \frac{3}{(detA(z))^2} \left( \sum_{\sigma \in S} c_\sigma z_\sigma(1) dz_\sigma(2) \wedge dz_\sigma(3) \wedge dz_\sigma(4) \right).
\]

Given the formula in \( w \) above substitute the following, \( w_i = \sum_{j=1}^n a_i^j z_j \). This substitution gives \( W = A(z) \). Thus:

\[
w_4 dw_1 \wedge dw_2 \wedge dw_3 = \left( \sum_{j=1}^n a_1^j z_j \right) \left( \sum_{j=1}^n a_2^j dz_j \right) \wedge \left( \sum_{j=1}^n a_3^j dz_j \right) \wedge \left( \sum_{j=1}^n a_4^j dz_j \right)
\]

= \left( \sum_{j=1}^n a_1^j \right) \left( \sum_{r<s<t} ((a_1^r a_2^s a_3^t - a_2^r a_1^s a_3^t + a_3^r a_2^s a_1^t - a_1^r a_3^s a_2^t + a_2^r a_3^s a_1^t - a_3^r a_1^s a_2^t) dz_r \wedge dz_s \wedge dz_t)
\]

= \sum_{j=1}^n \sum_{r<s<t} ((a_1^r a_2^s a_3^t - a_2^r a_1^s a_3^t + a_3^r a_2^s a_1^t - a_1^r a_3^s a_2^t + a_2^r a_3^s a_1^t - a_3^r a_1^s a_2^t) z_j dz_r \wedge dz_s \wedge dz_t).
\[ w_3 dw_1 \wedge dw_2 \wedge dw_4 = \left( \sum_{j=1}^{n} a_j^3 z_j \right) \left( \sum_{j=1}^{n} a_j^4 dz_j \right) \wedge \left( \sum_{j=1}^{n} a_j^4 dz_j \right) \wedge \left( \sum_{j=1}^{n} a_j^4 dz_j \right) \]

\[ = \left( \sum_{j=1}^{n} a_j^3 z_j \right) \left( \sum_{r<s<t} (a_1^r a_2^s a_3^j - a_2^r a_1^s a_3^j + a_3^r a_1^s a_2^j - a_1^r a_3^s a_2^j + a_2^r a_3^s a_1^j - a_3^r a_2^s a_1^j) dz_r \wedge dz_s \wedge dz_t \right) \]

\[ w_2 dw_1 \wedge dw_3 \wedge dw_4 = \left( \sum_{j=1}^{n} a_j^2 z_j \right) \left( \sum_{j=1}^{n} a_j^2 dz_j \right) \wedge \left( \sum_{j=1}^{n} a_j^2 dz_j \right) \wedge \left( \sum_{j=1}^{n} a_j^2 dz_j \right) \]

\[ = \left( \sum_{j=1}^{n} a_j^2 z_j \right) \left( \sum_{r<s<t} (a_1^r a_3^s a_4^j - a_3^r a_1^s a_4^j + a_4^r a_1^s a_3^j - a_1^r a_4^s a_3^j + a_4^r a_3^s a_1^j - a_3^r a_4^s a_1^j) dz_r \wedge dz_s \wedge dz_t \right) \]

\[ w_1 dw_2 \wedge dw_3 \wedge dw_4 = \left( \sum_{j=1}^{n} a_j^1 z_j \right) \left( \sum_{j=1}^{n} a_j^2 dz_j \right) \wedge \left( \sum_{j=1}^{n} a_j^2 dz_j \right) \wedge \left( \sum_{j=1}^{n} a_j^2 dz_j \right) \]

\[ = \left( \sum_{j=1}^{n} a_j^1 z_j \right) \left( \sum_{r<s<t} (a_1^r a_2^s a_3^j - a_2^r a_1^s a_3^j + a_3^r a_1^s a_2^j - a_1^r a_3^s a_2^j + a_3^r a_2^s a_1^j - a_2^r a_3^s a_1^j) dz_r \wedge dz_s \wedge dz_t \right) \]

Thus we see:

\[ w_4 dw_1 \wedge dw_2 \wedge dw_3 - w_3 dw_1 \wedge dw_2 \wedge dw_4 + w_2 dw_1 \wedge dw_3 \wedge dw_4 - w_1 dw_2 \wedge dw_3 \wedge dw_4 \]

\[ = \sum_{j=1}^{n} \sum_{r<s<t} \left( (a_1^r a_2^s a_3^j - a_2^r a_1^s a_3^j + a_3^r a_1^s a_2^j - a_1^r a_3^s a_2^j + a_3^r a_2^s a_1^j - a_2^r a_3^s a_1^j) \right) \]

\[ - (a_1^r a_2^s a_3^j - a_2^r a_1^s a_3^j + a_3^r a_1^s a_2^j - a_1^r a_3^s a_2^j + a_3^r a_2^s a_1^j - a_2^r a_3^s a_1^j) \]

\[ + (a_1^r a_2^s a_3^j - a_2^r a_1^s a_3^j + a_3^r a_1^s a_2^j - a_1^r a_3^s a_2^j + a_3^r a_2^s a_1^j - a_2^r a_3^s a_1^j) \]

\[ - (a_2^r a_3^s a_4^j - a_3^r a_2^s a_4^j + a_4^r a_2^s a_3^j - a_2^r a_4^s a_3^j + a_4^r a_3^s a_2^j - a_3^r a_4^s a_2^j) dz_r \wedge dz_s \wedge dz_t \]

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Note, if \( j \) equals any of the \( r, s, t \) then our sum is zero. WLOG assume \( j = r \) then:

\[
((a_1^r a_2^s a_3^t a_4^r - a_2^s a_1^r a_3^t a_4^r + a_3^t a_1^r a_2^s a_4^r - a_1^r a_3^t a_2^s a_4^r + a_2^s a_3^t a_1^r a_4^r) - (a_1^r a_2^s a_3^t a_4^r - a_2^s a_1^r a_3^t a_4^r + a_3^t a_1^r a_2^s a_4^r - a_1^r a_3^t a_2^s a_4^r + a_2^s a_3^t a_1^r a_4^r) + (a_1^r a_2^s a_3^t a_4^r - a_2^s a_1^r a_3^t a_4^r + a_3^t a_1^r a_2^s a_4^r - a_1^r a_3^t a_2^s a_4^r + a_2^s a_3^t a_1^r a_4^r) - (a_2^s a_3^t a_4^r a_1^t - a_3^t a_2^s a_4^r a_1^t + a_4^s a_3^t a_4^r a_1^t - a_3^t a_4^s a_2^t a_1^t + a_4^s a_3^t a_2^t a_1^t - a_4^s a_3^t a_2^t a_1^t)) z_r dz_r \land dz_s \land dz_t
\]

Hence we can write the sum to be over all \( j \neq r, s, t \) and \( r < s < t \):

\[
w_4 dw_1 \land dw_2 \land dw_3 - w_3 dw_1 \land dw_2 \land dw_4 + w_2 dw_1 \land dw_3 \land dw_4 - w_1 dw_2 \land dw_3 \land dw_4
\]

\[
= \sum_{r<s<t,j \neq r,s,t} (a_1^r a_2^s a_3^t a_4^j - a_2^s a_1^r a_3^t a_4^j + a_3^t a_1^r a_2^s a_4^j - a_1^r a_3^t a_2^s a_4^j + a_2^s a_3^t a_1^r a_4^j - a_3^t a_2^s a_1^r a_4^j) - a_1^r a_2^s a_3^t a_4^j + a_2^s a_1^r a_3^t a_4^j - a_3^t a_1^r a_2^s a_4^j - a_1^r a_3^t a_2^s a_4^j + a_2^s a_3^t a_1^r a_4^j + a_3^t a_2^s a_1^r a_4^j)
\]

\[-a_1^r a_2^s a_3^t a_4^j + a_2^s a_1^r a_3^t a_4^j - a_3^t a_1^r a_2^s a_4^j - a_1^r a_3^t a_2^s a_4^j + a_2^s a_3^t a_1^r a_4^j + a_3^t a_2^s a_1^r a_4^j)
\]

\[-a_3^t a_2^s a_3^t a_1^4 + a_2^s a_3^t a_2^t a_1^4 + a_3^t a_2^s a_3^t a_1^4 - a_3^t a_2^s a_3^t a_1^4 + a_2^s a_3^t a_2^t a_1^4 + a_3^t a_2^s a_3^t a_1^4)
\]

We can write it in a more simplified form if we notice that:

\[
c_{j,r,s,t} = \det \begin{pmatrix}
 a_1^j & a_1^r & a_1^s & a_1^t \\
 a_2^j & a_2^r & a_2^s & a_2^t \\
 a_3^j & a_3^r & a_3^s & a_3^t \\
 a_4^j & a_4^r & a_4^s & a_4^t
\end{pmatrix}
\]

\[
=(a_1^r a_2^s a_3^t a_4^j - a_5^r a_2^s a_3^t a_4^j + a_6^s a_5 a_2^t a_4^j - a_1^s a_5 a_2^t a_4^j + a_7^s a_5 a_2^t a_4^j - a_3^s a_5 a_2^t a_4^j) - a_1^r a_2^s a_3^t a_4^j + a_5^r a_2^s a_3^t a_4^j - a_6^s a_5 a_2^t a_4^j - a_1^s a_5 a_2^t a_4^j + a_7^s a_5 a_2^t a_4^j + a_3^s a_5 a_2^t a_4^j)
\]

\[-a_3^s a_5 a_2^t a_4^j + a_6^s a_5 a_2^t a_4^j - a_1^s a_5 a_2^t a_4^j + a_7^s a_5 a_2^t a_4^j + a_3^s a_5 a_2^t a_4^j - a_4^s a_5 a_2^t a_4^j)
\]

Now, we have:
\[ w_4 dw_1 \wedge dw_2 \wedge dw_3 - w_3 dw_1 \wedge dw_2 \wedge dw_4 + w_2 dw_1 \wedge dw_2 \wedge dw_4 - w_1 dw_2 \wedge dw_3 \wedge dw_4 \]
\[ = \sum_{r<s<t, j \neq r, s, t} c_{j,r,s,t} z_j z_r \wedge dz_s \wedge dz_t. \]

Each collection of \( j, r, s, t \) such that \( j \neq r, s, t \) and \( r < s < t \) defines a unique element in \( S \), moreover each element \( \sigma \in S \) defines a unique term in our sum. Hence we have the formula

\[ w_4 dw_1 \wedge dw_2 \wedge dw_3 - w_3 dw_1 \wedge dw_2 \wedge dw_4 + w_2 dw_1 \wedge dw_2 \wedge dw_4 - w_1 dw_2 \wedge dw_3 \wedge dw_4 \]
\[ = \sum_{\sigma \in S} c_\sigma z_{\sigma(1)} dz_{\sigma(2)} \wedge dz_{\sigma(3)} \wedge dz_{\sigma(4)}. \]

Hence, since our substitution led to \( W = A(z) \) we have:

\[ tr(\omega^3_A(z)) = \frac{3}{(\det A(z))^2} \left( \sum_{\sigma \in S} c_\sigma z_{\sigma(1)} dz_{\sigma(2)} \wedge dz_{\sigma(3)} \wedge dz_{\sigma(4)} \right). \]

**Example 4.3.6.** For example, when \( n=4 \), there are only four elements in \( S \). The elements are as follows \( \sigma_1 : 1 \to 4, 2 \to 1, 3 \to 2, 4 \to 3, \sigma_2 : 1 \to 3, 2 \to 1, 3 \to 2, 4 \to 3, \sigma_3 : 1 \to 2, 2 \to 1, 3 \to 3, 4 \to 4 \) and \( \sigma_4 : 1 \to 1, 2 \to 2, 3 \to 3, 4 \to 4 \). Hence \( c_{\sigma_1} = -c_{\sigma_2}, c_{\sigma_2} = -c_{\sigma_3}, c_{\sigma_3} = -c_{\sigma_4}, c_{\sigma_4} = -c_{\sigma_1} \).

Thus we have a familiar equation

\[ tr(\omega^3_A(z)) = \frac{3 \cdot c_{\sigma_1}}{(\det A(z))^2} (s(z)). \]

where \( s(z) = -z_4 dz_1 \wedge dz_2 \wedge dz_3 + z_3 dz_1 \wedge dz_2 \wedge dz_4 - z_2 dz_1 \wedge dz_3 \wedge dz_4 + z_1 dz_2 \wedge dz_3 \wedge dz_4 \)

and

\[ c_{\sigma_1} = \det \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 & a_4^1 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ a_1^4 & a_2^4 & a_3^4 & a_4^4 \end{pmatrix}. \]

**Example 4.3.7.** Finally things become clearer for specific matrices. Let \( A_j \) be the Pauli matrices with \( A_1 = I \) so that we have a basis for \( M_2(\mathbb{C}) \). Hence \( A(z) = z_1 + z_2 + z_3 + z_4 \).
Note \( \det(A(z)) = (z_1^2 - z_2^2 - z_3^2 - z_4^2) \). Now all that is left is to calculate the \( c_{\sigma_l}, l = 1, 2, 3, 4 \). It follows that:

\[
c_{\sigma_1} = \det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \\ -1 & 1 & 0 & 0 \end{pmatrix} = 4i, \quad c_{\sigma_2} = \det \begin{pmatrix} 0 & 1 & 0 & 1 \\ -i & 0 & 1 & 0 \\ i & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = -4i,
\]

\[
c_{\sigma_3} = \det \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -i & 0 \\ 1 & 0 & i & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = 4i, \quad c_{\sigma_4} = \det \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} = -4i.
\]

Thus we have \( \text{tr}\omega^3_{A(z)} = \frac{-12i}{(z_1^2 - z_2^2 - z_3^2 - z_4^2)}(z_4dz_1 \wedge dz_2 \wedge dz_3 - z_3dz_1 \wedge dz_2 \wedge dz_4 + z_2dz_1 \wedge dz_3 \wedge dz_4 - z_1dz_2 \wedge dz_3 \wedge dz_4) \). □

We can see Jacobi’s formula is can be recovered from this method. As presented this provides a proof of the fact \( \text{tr}\omega_A = \frac{d\det(A(z))}{d\det(A(z))} \). From the above calculation we see

\[
\text{tr}\omega_W = \frac{w_4dw_1 - w_2dw_3 + w_1dw_4 - w_3dw_2}{w_4w_1 - w_2w_3} = \frac{d(w_4w_1 - w_2w_3)}{w_4w_1 - w_2w_3} = \frac{d\det(W)}{\det(W)}.
\]

The simple change of variable used in the Theorem will provide the final formula.

One can see, by continuing the calculation from the above, that each entry of \( \omega^5_W \) will be zero. Hence regardless of the number of variables, the size of the matrix restricts the non-zero traces of \( \omega^k_{A(z)} \). Thus we have that the structure of the Banach algebra also restricts the growth of the non-zero powers of \( \phi(\omega^m_A) \).

We have the following Corollary to 4.1.11.

**Corollary 4.3.8.** Let \( A \) be an \( n \)-tuple of elements from the \( m^2 \)-dim algebra \( M_m(\mathbb{C}) \). Then \( \text{tr}(\omega^j_A) = 0 \) for \( j > m^2 \).

**Proof.** It is clear that if \( m > n \) then \( \omega^m_A = 0 \). Thus when \( j < n \) we are done. But if \( j \geq n \) then \( n \geq m^2 \). Thus since \( M_m(\mathbb{C}) \) is \( m^2 \)-dimensional we have that \( A \) must have a term that is linearly dependent on some subset of \( A \). Hence by the 4.1.11 we have that \( \text{tr}(\omega^j_A) = 0 \). □
Chapter 5

Cyclic Cochains

5.1 A Cohomology Map

Invariant linear functionals are not the only objects that give rise to an interesting study of $P^c(A)$. We can extend the given map $\tau$, evaluation of invariant $k$-linear functionals at the form $\omega_A$, to include another important class of linear functionals. Recall that we are using $\mathcal{B}_A$ to denote the smallest inversion-closed sub-algebra of the global Banach algebra $\mathcal{B}$ that contains the elements $A_1, A_2, \ldots, A_n$ of the tuple $A = (A_1, A_2, \ldots, A_n)$. Then $\mathcal{B}_A$ is a topological algebra over $\mathbb{C}$, where it is important to note that $\mathcal{B}_A$ is not typically commutative. Now we are able to create the cyclic cohomology of $\mathcal{B}_A$, $HC^*(\mathcal{B}_A)$.

Furthermore, we can look at the manifold $P^c(A)$ corresponding to the same tuple. As seen before we can construct the de Rham cohomology, $H^*_d(P^c(A))$.

The goal here is to use the operator valued 1-form $\omega_A$ to construct a map between these two cohomology theories. This map will be generated much the same way as the map $\tau$ studied previously.

**Definition 5.1.1.** Let $A = (A_1, A_2, \ldots, A_m)$ be an $m$-tuple of operators from the Banach algebra $\mathcal{B}$. Then we define the linear map $\kappa : C^n_\Lambda(\mathcal{B}_A) \to \Omega^{n+1}(P^c(A))$ by

$$\kappa(\phi) = \phi(\omega_A, \omega_A, \ldots, \omega_A)$$
Example 5.1.2. Let $A = (A_1, A_2, \ldots, A_m)$ be an $m$-tuple of elements from the Banach algebra $\mathcal{B}$. When $\phi \in HH^0(\mathcal{B}_A)$, i.e. $\phi$ is a trace, then

$$\kappa(\phi) = \phi(\omega_A) = \phi\left( \sum_{i=1}^{m} A(z)^{-1} A_i dz_i \right) = \sum_{i=1}^{m} \phi(A(z)^{-1} A_i) dz_i$$

We are looking to establish that the map $\kappa$ will descend to the cohomology level, i.e.;

$$\kappa : HC^n(\mathcal{B}_A) \rightarrow H^{n+1}_d(P^c(A)).$$

In order to establish this map we make the following observation.

Lemma 5.1.3. Let $A = (A_1, A_2, \ldots, A_m)$ be a $m$-tuple of elements from the Banach algebra $\mathcal{B}$. If $\phi$ is a cyclic $(n-1)$-linear functional on $\mathcal{B}_A$ then

$$\kappa(b\phi) = d(\kappa(\phi)) + (-1)^{n-1} \phi(\omega_A, \omega_A, \ldots, \omega_A)$$

Proof. For simplicity let $B_i = A^{-1}(z)A_i$ for $i = 1 \ldots m$.

Recall that the group of permutation, $S_m$ of $m$ objects can be decomposed into $\binom{m}{n}$ subsets where each subset $S^i_m$ consists of the $n!$ permutations of some subset of size $n$ of the $m$ objects.

This fact is proven in the same procedure are the trace case 4.1.6.

In this case

$$\kappa(\phi) = \phi(\omega_A, \omega_A, \ldots, \omega_A)$$

$$= \sum_{i=1}^{\binom{m}{n}} \sum_{\pi \in S^i_m} (-1)^{\text{sgn}(\pi)} \phi(B_{\pi(1)}, B_{\pi(2)}, \ldots, B_{\pi(n)}) dz_{\pi},$$

where $dz_{\pi}$ is the ordered $n$-form of the elements that define $S^i_m$.

Now $(b\phi)$ is a $n$-cochain defined by

$$(b\phi)(a_1, a_2, \ldots, a_{n+1})$$

$$= \sum_{j=1}^{n} (-1)^j \phi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1} \phi(a_{n+1}, a_1, \ldots, a_n)$$

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Thus we have that:

\[
\kappa(b\phi) = b\phi(\omega_A, \omega_A, \ldots, \omega_A)
\]

\[
= \sum_{i=1}^{n} \sum_{\pi \in S_n^i} (-1)^{sgn(\pi)} b\phi(B_{\pi(1)}, B_{\pi(2)}, \ldots, B_{\pi(n+1)}) dz_\pi
\]

\[
= \sum_{i=1}^{n} \sum_{\pi \in S_n^i} (-1)^{sgn(\pi)} \left( \sum_{j=1}^{n} (-1)^j \phi(B_{\pi(1)}, \ldots, B_{\pi(j)}B_{\pi(j+1)}, \ldots, B_{\pi(n+1)}) \right)
\]

\[
+ (-1)^{n+1} \phi(B_{\pi(n+1)}B_{\pi(1)}, \ldots, B_{\pi(n)}) dz_\pi.
\]

Now we make the following observation, if we replace \(\omega_A\) in the \(j\)th coordinate by \(\omega_A \wedge \omega_A\) we get the following:

\[
\phi(\omega_A, \ldots, \omega_A \wedge \omega_A, \ldots, \omega_A)
\]

\[
= \sum_{i=1}^{n} \sum_{\pi \in S_n^i} (-1)^{sgn(\pi)} \phi(B_{\pi(1)}, \ldots, B_{\pi(j)}B_{\pi(j+1)}, \ldots, B_{\pi(n+1)}) dz_\pi
\]

Here is the first time we use the structure of the 1-form \(\omega_A\). Recall that \(d\omega_A = -\omega_A \wedge \omega_A\). Recall as well that for any \(n\)-linear functional \(\theta\) we have;

\[
d\theta(\omega_A, \omega_A, \ldots, \omega_A) = \sum_{i=1}^{n} (-1)^i \theta(\omega_A, \ldots, d\omega_A, \ldots, \omega_A)
\]

Now, all the sums are finite, so they can be interchanged. Hence we have that

\[
\kappa(b\phi) = d(\kappa(\phi)) + (-1)^{n+1} \phi(d\omega_A, \omega_A, \ldots, \omega_A).
\]

\[
\square
\]

This lemma tells us that we do not have a chain homomorphism because the following diagram does not commute;

\[
\begin{array}{ccc}
C^{n}(\mathcal{B}_A) & \xrightarrow{\kappa} & \Omega^{n+1}(P^c(A)) \\
\downarrow b & & \downarrow d \\
C^{n+1}(\mathcal{B}_A) & \xrightarrow{\kappa} & \Omega^{n+2}(P^c(A)).
\end{array}
\]
**Example 5.1.4.** It is enlightening to have the previous Lemma worked out in a concrete case. Let $A = (A_1, A_2, A_3, A_4)$ and let $\phi$ be a 2-cochain. Let $B_i := (A(z))^{-1} A_i$. Then we have that $\omega_A = B_1 dz_1 + B_2 dz_2 + B_3 dz_3 + B_4 dz_4$. For the 2-cochain case the coboundary operator is the following

$$b\phi(a_1, a_2, a_3) = \phi(a_1 a_2, a_3) - \phi(a_1, a_2 a_3) + \phi(a_3 a_1, a_2).$$

Now if we apply $\kappa$ we get

$$\kappa(b\phi) = b\phi(B_1 dz_1 + B_2 dz_2 + B_3 dz_3 + B_4 dz_4, B_1 dz_1 + B_2 dz_2 + B_3 dz_3 + B_4 dz_4)$$

$$= (b\phi(B_1, B_2, B_3) - b\phi(B_1, B_3, B_2) - b\phi(B_2, B_1, B_3) + b\phi(B_2, B_3, B_1) + b\phi(B_3, B_1, B_2) - b\phi(B_3, B_2, B_1)) dz_1 \wedge dz_2 \wedge dz_3$$

$$+ (b\phi(B_1, B_2, B_4) - b\phi(B_1, B_4, B_2) - b\phi(B_2, B_1, B_4) + b\phi(B_2, B_4, B_1) + b\phi(B_4, B_1, B_2) - b\phi(B_4, B_2, B_1)) dz_1 \wedge dz_2 \wedge dz_4$$

$$+ (b\phi(B_1, B_3, B_4) - b\phi(B_1, B_4, B_3) - b\phi(B_3, B_1, B_4) + b\phi(B_3, B_4, B_1) + b\phi(B_4, B_1, B_3) - b\phi(B_4, B_3, B_1)) dz_1 \wedge dz_2 \wedge dz_3$$

$$+ (b\phi(B_2, B_3, B_4) - b\phi(B_2, B_4, B_3) - b\phi(B_3, B_2, B_4) + b\phi(B_3, B_4, B_2) + b\phi(B_4, B_2, B_3) - b\phi(B_4, B_3, B_2)) dz_2 \wedge dz_3 \wedge dz_4.$$
\(= (\phi(B_1 B_2, B_3) - \phi(B_1, B_2 B_3) + \phi(B_3 B_1, B_3) - (\phi(B_1 B_3, B_2) - \phi(B_1, B_3 B_2) + \phi(B_2 B_1, B_3)) - (\phi(B_2 B_1, B_3) - \phi(B_2, B_1 B_3) + \phi(B_3 B_2, B_1)) + (\phi(B_2 B_3, B_1) - \phi(B_2, B_3 B_1) + \phi(B_1 B_2, B_3)) + (\phi(B_3 B_1, B_2) - \phi(B_3, B_1 B_2) + \phi(B_2 B_3, B_1)) - (\phi(B_3 B_2, B_1) - \phi(B_3, B_2 B_1) + \phi(B_1 B_3, B_2))) dz_1 \wedge dz_2 \wedge dz_3
\)

Using the linearity of \(\phi\) we can add in the three other 3-forms and get the following:

\[b(\phi(\omega_A, \omega_A, \omega_A)) = \phi(\omega_A \wedge \omega_A, \omega_A) - \phi(\omega_A, \omega_A \wedge \omega_A) + \phi(\omega_A \wedge \omega_A, \omega_A)
\]

\[= d(\phi(\omega_A, \omega_A)) - \phi(d\omega_A, \omega_A)
\]

**Lemma 5.1.5.** If \(\phi\) is a closed cyclic \(n\)-cochain then:

\[\phi(d\omega_A, \omega_A, \ldots, \omega_A) = 0.
\]

**Proof.** We first need a property of the group \(S_m\). As described above there is a subgroup \(S^i_m\) that consists of the permutations of a subset of size \(n + 1\) of the set of \(m\) objects. WLOG fix \(i\) and denote that subset \(m_i := \{1, 2, 3, \ldots, n + 1\}\) and a permutation \(\pi_0 \in S^i_m\). Then there exists \(\pi_k \in S^1_m\) such that \(\pi_k(k) = \pi_0(n + 1), \pi_k(k + j) = \pi_0(k + j - 1)\) for \(1 \leq j \leq n + 1\). Note that \(\text{sgn}(\pi_j) \equiv \text{sgn}(\pi_0) + j\). We have that \(\pi_k \neq \pi_l\) when \(k \neq l\), call this set \(G_{\pi_0}\) and \(|G_{\pi_0}| = n + 1\). Since the size of \(S^i_m\) is divisible by \(n + 1\), we can partition the set \(S^i_m\) into sets of type \(G_{\pi_0}\). Hence, by the cyclicity of \(\phi\)
we have the following:

\[
\phi(d\omega_A, \omega_A, \ldots, \omega_A)
= \sum_j \sum_{\pi_j^i \in G_{\pi_j}} (-1)^{sgn\pi^j} \phi(B_{\pi^j_1(1)} B_{\pi^j_2(1)} \cdots B_{\pi^j_{n+1}(1)}) dz_{\pi^j_1(n+1)}
\]

\[
= \sum_j \sum_i (-1)^{sgn\pi^j} (-1)^{i(n+1)} \phi(B_{\pi^j_1(i+1)} \cdots B_{\pi^j_{n+1}(i)}) dz_{\pi^j_1(n+1)}
\]

\[
= \sum_j (-1)^{sgn\pi^j} \sum_i ((-1)^{in+2i} \phi(B_{\pi^j_1(1)} \cdots B_{\pi^j_{n+1}(1)} B_{\pi^j_1(i+1)} \cdots B_{\pi^j_{n+1}(i)})
\]

\[
\quad + (-1)^{(n+1)n} \phi(B_{\pi^j_1(1)} \cdots B_{\pi^j_{n+1}(1)} B_{\pi^j_1(n+1)} d\pi_{\pi^j_1(n+1)})
\]

\[
= \sum_j (-1)^{sgn\pi^j} b\phi(B_{\pi^j_1(1)} \cdots B_{\pi^j_{n+1}(1)})
\]

\[
= \sum_j (-1)^{sgn\pi^j} . 0 = 0.
\]

\[\square\]

**Example 5.1.6.** Let us continue the previous example and calculate the result of the previous lemma. Recall, we are working with $A$ a 4-tuple and $\phi$ a cyclic 2-cochain. If we further assume that $\phi$ is closed we have the following:

\[
- \phi(d\omega_A, \omega_A) = \phi(\omega_A \wedge \omega_A, \omega_A) = \phi((\sum_{i=1}^4 B_i dz_i) \wedge (\sum_{i=1}^4 B_i dz_i), (\sum_{i=1}^4 B_i dz_i))
\]

\[
= (\phi(B_1 B_2 - B_2 B_1, B_3) + \phi(B_3 B_1 - B_1 B_3, B_2) + \phi(B_2 B_3 - B_3 B_2, B_1)) dz_1 \wedge dz_2 \wedge dz_3
\]

\[
+ (\phi(B_1 B_2 - B_2 B_1, B_4) + \phi(B_4 B_1 - B_1 B_4, B_2) + \phi(B_2 B_4 - B_4 B_2, B_1)) dz_1 \wedge dz_2 \wedge dz_4
\]

\[
+ (\phi(B_1 B_3 - B_3 B_1, B_4) + \phi(B_4 B_1 - B_1 B_4, B_3) + \phi(B_3 B_4 - B_4 B_3, B_1)) dz_1 \wedge dz_3 \wedge dz_4
\]

\[
+ (\phi(B_2 B_3 - B_3 B_2, B_4) + \phi(B_4 B_2 - B_2 B_4, B_3) + \phi(B_3 B_4 - B_4 B_3, B_2)) dz_2 \wedge dz_3 \wedge dz_4.
\]

Once again I will simplify the calculations and only work with the first term, the $dz_1 \wedge dz_2 \wedge dz_3$ term. Then:

\[
(\phi(B_1 B_2 - B_2 B_1, B_3) + \phi(B_3 B_1 - B_1 B_3, B_2) + \phi(B_2 B_3 - B_3 B_2, B_1))
\]

\[
= \phi(B_1 B_2, B_3) - \phi(B_1, B_2 B_3) + \phi(B_3 B_1, B_2)
\]

\[
+ \phi(B_1 B_3, B_2) - \phi(B_1, B_3 B_2) + \phi(B_2 B_1, B_3)
\]

\[
= b\phi(B_1, B_2, B_3) + b\phi(B_1, B_2, B_3) = 0.
\]

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**Corollary 5.1.7.** The map $\kappa$ takes closed cyclic cochains to closed forms and exact cyclic cochains to exact forms.

**Proof.** As before let $A$ be an $n$-tuple. The previous two lemmas give us the fact that closed cyclic cochains go to closed forms. We have if $\phi$ is a closed cochain then $b\phi = 0$. Also we have that:

$$d(\kappa(\phi)) = \sum((-1)^i \phi(\omega_A, \ldots, d(\omega_A), \ldots, \omega_A)).$$

Because $\phi$ is cyclic we have that $\phi(\omega_A, \ldots, d\omega_A, \ldots, \omega_A) = (-1)^s \phi(d\omega_A, \omega_A, \ldots, \omega_A) = 0$ for some value of $s$. Hence, if $b\phi = 0$ then $d(\kappa(\phi)) = 0$. Thus we have shown that $\kappa$ maps closed cochains to closed forms.

What is left is to show that $\kappa$ maps exact cochains to exact forms. Let $\theta$ be an exact $m$-cochain. Thus there exists a $\phi \in C^{m-1}_A(A)$ such that $b\phi = \theta$. We have that $\kappa(\theta) = d(\kappa(\phi)) + (-1)^{m-1} \phi(d\omega_A, \omega_A, \ldots, \omega_A)$. Since $d(\kappa(\phi))$ is an exact element of $\Omega^m(M)$ all that is left is to show that $(-1)^{m-1} \phi(d\omega_A, \omega_A, \ldots, \omega_A)$ is exact. Keeping the notation from above, $B_i = (A(z))^{-1}A_i$. Let $R$ be the collection of all $n-1$ tuples of one forms $(B_{i_1}dz_{i_1}, \ldots, B_{i_m}dz_{i_m})$ such that $i_1 \leq i_j \neq i_k$ for all $1 \leq j, k \leq n$.

Let

$$\beta = \sum_{\alpha \in R} \phi(\alpha) = \sum_{\alpha \in R} \phi(B_{i_1}dz_{i_1}, \ldots, B_{i_m}dz_{i_m}).$$

Now we have that $\beta \in \Omega^m(P^s(A))$. Furthermore we have that:

$$d\beta = \sum_{R} \left( \sum_{j} (-1)^{j+1} \phi(B_{i_1}dz_{i_1}, \ldots, dB_{i_j}dz_{i_j}, \ldots, B_{i_m}dz_{i_m}) \right)$$

$$= \sum_{R} \left( \sum_{j} (-1)^{j+1} \phi(B_{i_1}dz_{i_1}, \ldots, (\sum_k B_k dz_k) \wedge B_{i_j}dz_{i_j}, \ldots, B_{i_m}dz_{i_m}) \right)$$

$$= \sum_{R} \left( \sum_{j} (-1)^{j+1}(-1)^{m-1}(-1)^{j-1} \phi((\sum_k B_k dz_k) \wedge B_{i_j}dz_{i_j}, \ldots, B_{i_m}dz_{i_m}, \ldots, B_{i_1}dz_{i_1}) \right)$$

$$= \sum_{R} \left( \sum_{j} (-1)^{m-1} \phi((\omega_A) \wedge B_{i_j}dz_{i_j}, \ldots, B_{i_m}dz_{i_m}, \ldots, B_{i_1}dz_{i_1}) \right).$$

We are allowed to insert any $B_k dz_k$ if they are already in the form.
Thus;

\[
d\beta = \sum_{R} \left( (-1)^{m-1} \phi((\omega_A) \wedge (\sum_{k} B_k dz_k), \ldots, B_i dz_i, \ldots, B_{i_k} dz_{i_k}) \right)
\]

\[
= \sum_{R} \left( (-1)^{m-1} \phi((\omega_A) \wedge (\omega_A), \ldots, B_i dz_i, \ldots, B_{i_k} dz_{i_k}) \right)
\]

\[
= (-1)^{m-1} \phi((\omega_A) \wedge (\omega_A), \sum_{k} B_k dz_k, \ldots, \sum_{k} B_k dz_k) \]

\[
= (-1)^m \phi(-\omega_A \wedge \omega_A, \omega_A, \ldots, \omega_A)
\]

\[
= (-1)^m \phi(d\omega_A, \omega_A, \ldots, \omega_A).
\]

Thus we have that if \( \theta \) is an exact cochain then \( \kappa(\theta) \) is an exact form. Furthermore, if \( b\phi = \theta \) then \( d(\kappa(\phi) + (-1)^m \beta) = \kappa(\theta) \)

\[\Box\]

**Example 5.1.8.** Let us see a low dimensional example that will illustrate why \( \beta \) is an exact form. Let \( A = \{A_1, A_2, A_3\} \) be a 3-tuple of operators and \( \psi \) an exact cyclic 2-cochain. Then following the notation above \( B_1 = A(z)^{-1}A_1 \), \( B_2 = A(z)^{-1}A_2 \) and \( B_3 = A(z)^{-1}A_3 \). Because \( \psi \) is exact there exists a cyclic 1-cochain \( \phi \) such that \( b\phi = \psi \). We define the 2-form \( \beta := \phi(B_1 dz_1, B_2 dz_2) + \phi(B_1 dz_1, B_3 dz_3) + \phi(B_2 dz_2, B_3 dz_3) \). We have that

\[
\beta = \phi(B_1, B_2)dz_1 \wedge dz_2 + \phi(B_1, B_3)dz_1 \wedge dz_3 + \phi(B_2, B_3)dz_2 \wedge dz_3.
\]

Since \( \phi \) is cyclic we have that \( \phi(a_0, a_1) = -\phi(a_1, a_0) \). From this we have the following:

\[
d\beta = \phi(B_1, B_2)dz_1 \wedge dz_2 + \phi(B_1, B_3)dz_1 \wedge dz_3 + \phi(B_2, B_3)dz_2 \wedge dz_3
\]

\[
= \phi(d(B_1), B_2)dz_1 \wedge dz_2 + \phi(d(B_1), B_3)dz_1 \wedge dz_3 + \phi(d(B_2), B_3)dz_2 \wedge dz_3
\]

\[
- \phi(B_1, d(B_2))dz_1 \wedge dz_2 - \phi(B_1, d(B_3))dz_1 \wedge dz_3 - \phi(B_2, d(B_3))dz_2 \wedge dz_3
\]

\[
= \phi((B_3 dz_3)B_1, B_2)dz_1 \wedge dz_2 + \phi((B_2 dz_2)B_1, B_3)dz_1 \wedge dz_3
\]

\[
+ \phi((B_1 dz_1)B_2, B_3)dz_2 \wedge dz_3 - \phi(B_1, (B_3 dz_3)B_2)dz_1 \wedge dz_2
\]

\[
- \phi(B_1, (B_2 dz_2)B_3)dz_1 \wedge dz_3 - \phi(B_2, (B_1 dz_1)B_3)dz_2 \wedge dz_3
\]

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\[ =\phi(B_3B_1, B_2) - \phi(B_2B_1, B_3) + \phi(B_1B_2, B_3) \]
\[ + \phi(B_1, B_3B_2) - \phi(B_1, B_2B_3) + \phi(B_2, B_1B_3) \]
\[ dz_1 \wedge dz_2 \wedge dz_3 \]
\[ = (\phi(B_3B_1, B_2) - \phi(B_2B_1, B_3) + \phi(B_1B_2, B_3) \]
\[ - \phi(B_3B_2, B_1) + \phi(B_2B_3, B_1) - \phi(B_1B_3, B_2) \]
\[ dz_1 \wedge dz_2 \wedge dz_3 \]
\[ = (\phi(B_3B_1 - B_1B_3, B_2) + \phi(B_1B_2 - B_2B_1, B_3) \]
\[ + \phi(B_2B_3 - B_3B_2, B_1) \]
\[ dz_1 \wedge dz_2 \wedge dz_3 \]
\[ = \phi(\omega_A \wedge \omega_A, \omega_A) = -\phi(d\omega_A, \omega_A). \]

The last step is identical to the step in 5.1.4.

What this gives us is a homomorphism of \( HC^n(B_A) \rightarrow H^{n+1}_d(P^c(A)) \), that is not induced from a chain homomorphism.

What we have is a map that takes a purely algebraically defined object related to the tuple \( A \) and identifies it with an analytic geometric property of the set \( P^c(A) \) that is related to the tuple.

**Example 5.1.9.** Let \( A = A_\theta \) be the algebra of which the generic element is a formal sum \( \sum a_{n,m} U^n V^m \) where \( (a_{n,m}) \in S(\mathbb{Z}^2) \) and the product is specified by the equality \( UV = \lambda VU \). We let the canonical trace, \( tr \), on \( A_\theta \) be given by
\[
tr(\sum a_{n,m} U^n V^m) = a_{0,0}.
\]

[Connes] showed that \( HC^1(A_\theta) \) is generated by the cyclic cocycles \( \phi_1 \) and \( \phi_2 \) given by \( \phi_j(x_0, x_1) = tr(x_0 \delta_j(x_1)) \) for all \( x_i \in A_\theta \), for the derivations \( \delta_1(U^mV^n) = mU^mV^n \) and \( \delta_2(U^mV^n) = nU^mV^n \).

For these cyclic cochains we have that \( \phi_j(I, C) = 0 \) and \( \phi_j(C, C) = 0 \) for all \( C \) in \( A_\theta \).

Let \( A = (A_1, A_2, A_3) \) be any 3-tuple of elements in \( A_\theta \). Thus we have that;
\[
\kappa(\phi_j) = \phi_j(\omega_A, \omega_A)
\]
\[ = 2\phi_j(W_1, W_2)dz_1 \wedge dz_2 + 2\phi_j(W_1, W_3)dz_1 \wedge dz_3 + 2\phi_j(W_2, W_3)dz_2 \wedge dz_3. \]
Now we have that;

\[
2z_1\phi_j(W_1, W_2) = 2z_1\phi_j(W_1, W_2) + 2z_2\phi_j(W_2, W_2) + 2z_3\phi_j(W_3, W_2) - 2z_3\phi_j(W_3, W_2)
\]

\[
= 2\phi_j(I, W_2) - 2z_3\phi_jW_3, W_2)
\]

\[
= 2z_3\phi_j(W_2, W_3).
\]

Hence, we can write

\[
\kappa(\phi_j) = g(z)\cdot s(z),
\]

where \( g(z) = 2\phi_j(W_1, W_2) \) and \( s(z) = z_1dz_2 \wedge dz_3 - z_2dz_1 \wedge dz_3 + z_3dz_1 \wedge dz_2. \)

We can see that any cochain with the property that \( \phi(I, C) = 0 \) will have the form 4.1.9.

Cyclic cohomology is a useful tool in the noncommutative case, but it can say some nontrivial things in the commutative case. We have seen that when we have a trace the map \( \tau \) is trivial. But in the case of \( \kappa \) we get something a bit different.

**Lemma 5.1.10.** Let \( A \) be a commuting \( n \)-tuple of elements from a Banach algebra \( B \). If \( \phi \) is in \( HC^m(B_A) \) then \( \phi(d\omega_A, \omega_A, \ldots, \omega_A) = 0 \), for \( m \geq 2 \).

**Proof.** In the commutative case we have that \( d(\omega_A) = 0 \). Hence we have that

\[
\phi(d\omega_A, \omega_A, \ldots, \omega_A) = \phi(0, \omega_A, \ldots, \omega_A) = 0.
\]

\( \square \)

### 5.2 Operator-Valued Entire Functions

There is much more work to be done in this direction for example we should note that the only property of the form used in the proceeding proofs was the fact that \( d(\omega_A) = -\omega_A \wedge \omega_A \). Thus we can make some generalizations. This is from my paper with Dr. Bannon and Dr. Yang. [?] 

**Definition 5.2.1.** Let \( f \) be an entire function with values in an unital Banach algebra \( B \). Then we define \( \omega_f(z) := f(z)^{-1}df(z) \) on the set \( \sigma^c(f) \).
Proposition 5.2.2 (Bannon, C, Yang). Let $f$ be an entire function with values in an unital Banach algebra $\mathcal{B}$. Then,

$$d\omega_f(z) = -\omega_f(z) \wedge \omega_f(z).$$

It seems, that the entirety of the previous section will follow from the previous proposition. The hope is that there are a whole family of homomorphisms $\kappa_f : HC^n(\mathcal{B}_f) \to H_d^{n+1}(\sigma_f)$ defined by

$$\kappa_f(\phi) = \phi(\omega_f(z), \omega_f(z), \ldots, \omega_f(z))$$

for all entire functions $f$ with values in a unital Banach algebra $\mathcal{B}$ and $\phi \in HC^n(\mathcal{B}_f)$. Here, $\mathcal{B}_f$ is the smallest inversion closed sub-algebra that contains $f(z)^{-1}$ for $z \in \sigma_f$.

Example 5.2.3. It has been noted in [Bannon, C, Yang] that if $U_i$ are $*$-free Haar unitary elements then the entire function $f(z) = \sum_{i=1}^n z_i U_i$ yields a calculable $\phi(\omega_f)$ for $\phi$ a faithful normal trace. It is shown that $\phi(\omega_f) = \frac{dz_i}{z_i}$ on $\Omega_i := \{ z \in \mathbb{C} | 2|z_i|^2 > |z|^2 \}$.

If we denote $\sum_{i=1}^{n-1} \frac{z_{i+1}}{z_i} U^* U_{i+1}$ by $W(\xi)$ on $\Omega_1$ we have that $\omega_f = (I + W(\xi))^{-1} dW(\xi) + \frac{dz_1}{z_1}$. Hence on $\Omega_1$

$$\omega_f^2 = ((I + W(\xi))^{-1} dW(\xi))^2 + c \cdot ((I + W(\xi))^{-1} dW(\xi))^2 \wedge \frac{dz_1}{z_1}$$

where $c = 1, 3$ depending on the parity of $n$.

We have $\phi((I + W(\xi))^{-1} dW(\xi)) = 0$ for $|\xi|$ small, and since $\phi((I + W(\xi))^{-1} dW(\xi))$ is holomorphic on $\Omega_1$ we have that $\phi((I + W(\xi))^{-1} dW(\xi))^2 = 0$ on $\Omega_1$ for all $j \in \mathbb{N}$. Hence $\phi(\omega_f^2) = 0$ on $\Omega_1$. 

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