Irrational Eigenvalues of the Discrete Laplacian: A Study of Simplicial Complexes

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IRRATIONAL EIGENVALUES OF THE DISCRETE LAPLACIAN

A STUDY OF SIMPLICIAL COMPLEXES

by

Brian Bollen

A Dissertation
Submitted to the University at Albany, State University of New York
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To Mimi and Jim
ABSTRACT

We study the behavior of eigenvalues of the discrete Laplacian of an abstract simplicial complex $\mathcal{K}$ when subdividing a single face of $\mathcal{K}$. We show that if $\mathcal{K}$ is a simplex, performing this kind of restricted subdivision twice on a single face produces irrational eigenvalues for the discrete Laplacian.
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“sort of smart - 3.8 gpa - work 5 days a week - not going to MIT”

– kind of guy without you.

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\textsuperscript{1}Even though all my friends think I’m a huge nerd.
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CHAPTER 1

Introduction

The discrete Laplacian matrix has been used in many fields of mathematics, electrical engineering, and physics. Also known as a Kirchoff matrix, Gustav Kirchoff used this operator to relate the current, voltage, charge and node potential to each other. Along with Kirchoff’s Theorem, it was also used to find the number of spanning trees in a graph. Since Kirchoff’s work, the discrete Laplacian has been generalized in many different ways, and the properties of its spectrum are the subject of active current research in diverse areas such as image processing, physics, combinatorics, and topology. See [1], [7], [10], [6], [5], and [8] for reference.

In recent years, Art M. Duval has been studying spectral recursion of the discrete Laplacian of simplicial complexes with respect to alterations of the complex [3]. Furthermore, Duval and Victor Reiner has shown in [4] that shifted simplicial complexes always have an integer valued spectrum. In this thesis, we investigate the behavior of the spectrum of the Laplacian of simplicial complexes after the topological operation of subdivison. We prove in our main result, Theorem 4.1, that performing a special case of subdivison called restricted subdivison on a simplex twice produces irrational eigenvalues of the discrete Laplacian.
CHAPTER 2
Preliminaries

2.1 Simplicial Complexes

We review some basic facts about simplicial complexes. For more details, the reader is referred to [9].

Definition 2.1. Let $V = \{v_0, v_1, \ldots, v_n\}$ be a set. Now, let $\sigma$ be the collection of all subsets of the set $V$. Then, $V$ is a vertex set and the collection $\sigma$ is an $n$-simplex. The integer $n$ is known as the dimension of the simplex.

Definition 2.2. An abstract simplicial complex is a family $K$ of finite subsets of a set $V$ such that for all $X \in K$, every subset of $X$ is also in $K$. $V$ is known as the vertex set of $K$.

So an abstract simplicial complex can be viewed as the union of simplices of varying dimension. The sets in the abstract simplicial complex $K$ – each by construction being a $d$-simplex for some $d \in \mathbb{N}$ – are referred to as faces. Let $S$ be the smallest collection of sets of $K$ such that for all $\sigma \in K$, $\sigma$ is a subset of an element in $S$. Then, $S$ is referred to as the facets of $K$.

Remark 2.1.1. Without loss of generality, we can denote the vertices of a vertex set $V$ by the natural numbers that index them. For example, a vertex set $V = \{v_0, v_1, \ldots, v_n\}$ can be referred to alternatively by $V = \{0, 1, \ldots, n\}$.

Example 2.1.1. Let $\mathcal{K}$ be the collection of sets $\{\{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 3\}, \{0, 1, 2\}\}$. This is an abstract simplicial complex with vertex set $V = \{0, 1, 2, 3\}$. The facets of $\mathcal{K}$ is the collection $\mathcal{S} = \{\{0, 3\}, \{0, 1, 2\}\}$.

Definition 2.3. Let $\mathcal{K}$ be an abstract simplicial complex as before. Suppose $\sigma$ is a simplex in $\mathcal{K}$. The star of $\sigma$ is the set of simplices in $\mathcal{K}$ that have $\sigma$ as a face. We denote this as $St(\sigma)$. In general, the star of a simplex is not an abstract simplicial complex.
Definition 2.4. With $K$ as before, we define the **closure** of a face $\sigma$ in $K$ as being the smallest simplicial subcomplex of $K$ that contains $\sigma$. We denote the closure of $\sigma$ as $\text{Cl}(\sigma)$.

Definition 2.5. The link of $\sigma$ is defined as $\text{Cl}(\text{St}(\sigma)) - \text{St}(\text{Cl}(\sigma))$.

### 2.2 Chain Groups and Boundary Operators

**Definition 2.6.** Let $\sigma$ be a simplex. We can define an **orientation** on $\sigma$ by saying that two orderings of the vertex set of $\sigma$ are equivalent if and only if they differ from one another by an even permutation. Each of these equivalence classes created are referred to as an orientation of the simplex $\sigma$.

Given the vertex set of a simplex $V = \{0, 1, \ldots, n\}$, we denote a particular ordering of the vertices as $[0, 1, \ldots, n]$.

Given that $\sigma$ is an oriented $n$-simplex, an **elementary chain** $c$ corresponding to $\sigma$ is a function defined as follows:

$$
c(\sigma) = 1, \\
c(\sigma') = -1, \quad \text{if } \sigma' \text{ and } \sigma \text{ have opposite orientations} \\
c(\tau) = 0, \quad \text{for all other oriented simplices } \tau
$$

**Definition 2.7.** Let $K$ be an abstract simplicial complex. A **$n$-chain** on $K$ is a function $c$ from the set of oriented $n$-simplices of $K$ to the integers, such that $c(\sigma) = -c(\sigma')$ if $\sigma$ and $\sigma'$ are opposite orientations of the same simplex and $c(\tau) = 0$ for all but finitely many oriented $n$-simplices $\tau$.

So, an $n$-chain is a finite linear combination of elementary chains. The group of oriented $n$-chains is denoted by $C_n(K)$, where $n \geq 0$. This group is free abelian, and a basis can be formed as follows: orient each $n$-simplex in $K$ and then take the elementary chain on each of these simplices.

**Definition 2.8.** We can now define a homomorphism

$$
\partial_d : C_d(K) \rightarrow C_{d-1}(K)
$$
called the **boundary operator**. We define our mapping on a basis element $\sigma = [0, \ldots, d]$ in $C_d(K)$ to $C_{d-1}(K)$ by the formula

$$\partial_d \sigma = \partial_d [0, \ldots, d] = \sum_{i=0}^{d} (-1)^i [0, \ldots, \hat{i}, \ldots, d]$$

where the symbol $\hat{i}$ means that the vertex $i$ is omitted. So, for example, given a 2-simplex $\sigma$ as $[0, 1, 2]$, then $\partial_1(\sigma) = [1, 2] - [0, 2] + [0, 1]$.

Using the definition of the boundary operator, we are ready to define the discrete Laplacian:

**Definition 2.9.** We define the $n$th discrete Laplacian $L_n$ as

$$L_n = \partial_n^T \partial_n + \partial_{n+1}^T \partial_{n+1}$$

So note, if we have a $n$-simplex, then $L_n = \partial_n^T \partial_n + \partial_{n+1}^T \partial_{n+1} = \partial_n^T \partial_n$ because $C_{n+1}(K)$ is the trivial group for a $n$-simplex. This defines a homomorphism from $C_n(K) \rightarrow C_n(K)$.
CHAPTER 3
Foundational Lemmas

3.1 Characteristic Polynomial of $\Gamma$

We begin with a proving a nice result concerning the characteristic polynomial of a specific matrix. Recall that the group $M_n(\mathbb{R})$ is the group of $n \times n$ matrices with entries in $\mathbb{R}$. The reader is referred to [2] by Dummit and Foote for more information on characteristic polynomials and eigenvalues of these matrices.

**Proposition 3.1.** Let $\Gamma \in M_n(\mathbb{R})$ such that

$$
\Gamma = \Gamma(\alpha) = \begin{bmatrix}
\alpha & 1 & -1 & \cdots & \cdots & (-1)^n \\
1 & \alpha & 1 & -1 & \cdots & (-1)^{n-1} \\
-1 & 1 & \alpha & & & \\
\vdots & -1 & \ddots & \ddots & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
(-1)^n & (-1)^{n-1} & \cdots & \cdots & 1 & \alpha
\end{bmatrix}
$$

Then $\Gamma$ has a determinant equal to $(\alpha + 1)^{n-1}(\alpha - n + 1)$.

**Proof.** Our goal is to reduce this matrix to upper triangular form using elementary row operations. It is easy to see that we can make the entries below the diagonal in rows 2 through $n - 1$ vanish by alternatingly adding and subtracting the $n$-th row to rows above it.

$$
\Gamma(\alpha) = \begin{bmatrix}
\alpha & 1 & -1 & \cdots & \cdots & (-1)^n \\
0 & \alpha + 1 & 0 & 0 & \cdots & (-1)^n(\alpha + 1) \\
0 & 0 & \alpha + 1 & 0 & & \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
(-1)^n & (-1)^{n-1} & \cdots & \cdots & 1 & \alpha
\end{bmatrix}
$$
The next (and last) step is to reduce the $a_{n,j}$ entries where $1 \leq j \leq n - 1$. To do this, we use the $a_{1,j}$ entry to eliminate the $a_{n,j}$ entry.

$$\Gamma(\alpha) = \begin{bmatrix}
\alpha & 1 & -1 & \cdots & \cdots & (-1)^n \\
0 & \alpha + 1 & 0 & 0 & \cdots & (-1)^n(\alpha + 1) \\
0 & 0 & \alpha + 1 & & & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \alpha + 1 & \alpha + 1 & \vdots \\
0 & 0 & \cdots & \cdots & 0 & \frac{(\alpha^2 - 1) - (n - 2)(\alpha + 1)}{\alpha}
\end{bmatrix}$$

Our matrix is now reduced to upper triangular form and provides the convenient closed-form equation of our characteristic polynomial:

$$|\Gamma(\alpha)| = \alpha(\alpha + 1)^{n-2}\frac{(\alpha^2 - 1) - (n - 2)(\alpha + 1)}{\alpha} = \alpha\alpha^{n-2}\frac{(\alpha + 1)((\alpha - 1) - (n - 2))}{\alpha} = (\alpha + 1)^{n-1}(\alpha - n + 1)$$

This completes our proof.

\[\square\]

### 3.2 Transpose Boundary Operator

**Lemma 3.1.** Let $\sigma$ be an $n$-simplex with vertex set $V = \{0, \ldots, n\}$. Recall that the $d^{th}$ chain group $C_d(\sigma)$ is generated by all the $d$ faces of $\sigma$. Thus, the transpose of the $d^{th}$ boundary operator

$$\partial_d^T : C_{d-1}(\sigma) \rightarrow C_d(\sigma)$$

acts on a basis element $\tau = [v_0, \ldots, v_{d-1}]$ of $C_{d-1}(\sigma)$ as follows:

$$\partial_d^T(\tau) = \sum_{\gamma \in V \setminus \{v_0, \ldots, v_{d-1}\}} [\gamma, v_0, \ldots, v_{d-1}].$$

**Proof.** Let $\sigma$ be an $n$-simplex with vertex set $V = \{0, \ldots, n\}$. Let $\tau = [v_0, \ldots, v_{d-1}] \in C_{d-1}(\sigma)$ be a $(d - 1)$-face. The transpose of this boundary operator needs to map $\tau$
to a finite linear combination of $d$-faces – say $\tau_0^d, \ldots, \tau_l^d$ – such that for each $\tau_p^d$, the face $\tau$ should show up in the linear combination $\partial_d(\tau_p^d)$. In other words, $\tau$ maps to a linear combination of $d$-faces $\tau_0^d, \ldots, \tau_l^d$ such that $\tau$ is a face of $\tau_p^d$ for all $0 \leq p \leq l$. By definition, since the faces of $\sigma$ are exactly all subsets of $V = \{0, \ldots, n\}$, any $d$-th order subset of $V$ is a $d$-face. Let $A := V \setminus \{v_0, \ldots, v_{d-1}\}$. Furthermore, suppose $\gamma \in A$. So the set $\{\gamma, v_0, \ldots, v_{d-1}\}$ is an unoriented $d$-face of $\sigma$. Thus, the set of faces $\tau_0^d, \ldots, \tau_l^d$ is exactly the collection of $d$-faces $\{\{\gamma, v_0, \ldots, v_{d-1}\}\}_{\gamma \in A}$, up to orientation. Define a homomorphism $\Delta_d$ as follows:

$$\Delta_d(\tau) = \sum_{\gamma \in V \setminus \{v_0, \ldots, v_{d-1}\}} [\gamma, v_0, \ldots, v_{d-1}]$$

Let $\delta = [k_0, \ldots, k_d]$ be a $d$-face of $\sigma$ with the following property: The $i^{th}$ summand in $\partial_d(\delta) = \partial_d([k_0, \ldots, k_d])$ is $(-1)^i[k_0, \ldots, k_i-1, k_{i+1}, \ldots k_d]$ where the vertices can be renamed as $[v_0, \ldots, v_{d-1}] = \tau$ by the association $k_j = v_j$ for $0 \leq j \leq i - 1$ and we have $k_j = v_{j-1}$ for $i + 1 \leq j \leq d$. Now, our $i^{th}$ summand in $\partial_d(\delta)$ is exactly $(-1)^i\tau$.

Consider the linear combination of $d$-faces created by applying $\Delta_d$ to $(-1)^i\tau$:

$$\Delta_d((-1)^i\tau) = (-1)^i\Delta_d([v_0, \ldots, v_{d-1}]) = (-1)^i \sum_{\gamma \in V \setminus V(\sigma)} [\gamma, v_0, \ldots, v_{d-1}]$$

Consider now the summand in $\Delta_d(\tau)$ with $\gamma = k_i$ (since, of course, $k_i$ is some vertex in $V$ but is obviously not in $\tau$). That summand is exactly $(-1)^i[\gamma, v_0, \ldots, v_{d-1}] = (-1)^i[k_i, v_0, \ldots, v_{d-1}]$. We now revert back to the original naming of the vertices:

$$(-1)^i \cdot [k_i, k_0, \ldots, k_{i-1}, k_{i+1}, \ldots, k_d]$$

Notice that $[k_i, k_0, \ldots, k_{i-1}, k_{i+1}, \ldots, k_d]$ is $\tau$ but not necessarily with the same orientation. But, if we want to get our $k_i$ element back to its original place, as in $\tau$, we have to make successive transpositions of $k_i$. This is exactly $i$ transpositions. Thus,

$$(-1)^i[k_0, \ldots, k_d, k_i] = (-1)^i((-1)^i[k_0, \ldots, k_{i-1}, k_i, k_{i+1}, \ldots, k_d]) = [k_0, \ldots, k_d] = \sigma_q^d.$$ 

Therefore, given any $(d - 1)$-face $\tau$, our defined homomorphism $\Delta_d$ maps the $(d - 1)$-
face to a linear combination of $d$-faces $\tau_0^d, \ldots, \tau_l^d$ such that $\partial_d(\tau_p^d)$ for all $0 \leq p \leq l$ has $\tau$ as a summand with the correct orientation. Thus, $\Delta_d = \partial_d^T$.

3.3 Sub-maximal Boundary Homomorphism of the $n$-simplex

Lemma 3.2. Let $\sigma$ be an $n$-simplex. The matrix representing the composition of the boundary homomorphism

$$ \partial_{n-1} : C_{n-1}(\sigma) \rightarrow C_{n-2}(\sigma) $$

and its transpose

$$ \partial_{n-1}^T : C_{n-2}(\sigma) \rightarrow C_{n-1}(\sigma) $$

is an $(n+1) \times (n+1)$ matrix with form

$$ \partial_{n-1}^T \partial_{n-1}(\sigma) = \begin{bmatrix}
    n & 1 & -1 & \cdots & \cdots & (-1)^{n+1} \\
    1 & n & 1 & -1 & \cdots & (-1)^n \\
    -1 & 1 & n & \ddots & \ddots & \ddots \\
    \vdots & -1 & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \vdots & \ddots & n & 1 & \ddots \\
    (-1)^{n+1} & (-1)^n & \cdots & \cdots & 1 & n
\end{bmatrix}. $$

Proof. The proof simply comes with careful ordering of the basis elements in $C_{n-1}$. Let $V = \{0, 1, \ldots, n\}$ be the vertex set of an $n$-simplex $\sigma$. Notice that the basis elements of the chain group $C_{n-1}(\sigma)$ are exactly the subsets of cardinality $n-1$ of $V$. One can easily see that this makes an $n$-dimensional basis. Let the basis elements of $C_{n-1}$ be indexed by $i = 0, 1, \ldots, n$. Now, let the $i^{th}$ basis element be the oriented face $F^i = [0, \ldots, (n-i), \ldots, n]$, where $(n-i)$ means that the $(n-i)^{th}$ term is ommitted. We rename the vertices of $F^i$ as $F^i = [0, \ldots, (n-i-1), (n-i+1), \ldots n] = [v_0^i, \ldots, v_{n-1}^i]$ for each basis element of $C_{n-1}(\sigma)$ by the association $p = v_p^i$ for $0 \leq p \leq (n-i-1)$ and $p = v_{p-1}^i$ for $(n-i+1) \leq p \leq n$. Recall from Lemma 3.1 our convenient equation for $\partial_{n-1}^T$: 

8
\[ \partial_{n-1}^{T}(\Delta) = \sum_{\gamma \in V \setminus \{k_0, \ldots, k_{n-2}\}}^{2} [\gamma, k_0, \ldots, k_{n-2}], \quad (3.1) \]

where \( \Delta = [k_0, \ldots, k_{n-2}] \) is an \( n-2 \)-face, \( V \) is the vertex set of \( \sigma \), and \( \gamma \) is a vertex in \( V \setminus \{k_0, \ldots, k_{n-2}\} \). Notice that the set \( V \setminus \{k_0, \ldots, k_{n-2}\} \) only has two elements. So (3.1) only has two two summands. Now, we continue by direct computation on a given basis element of \( C_{n-1}(\sigma) \):

\[ \partial_{n-1}^{T}(\partial_{n-1}(F^i)) \]

\[ = \partial_{n-1}^{T} \left( \sum_{j=0}^{n-1} (-1)^{j} [v_0^i, \ldots, \hat{v}_j^i, \ldots, v_{n-1}^i] \right) \]

\[ = \sum_{j=0}^{n-1} (-1)^{j} \left( \partial_{n-1}^{T}[v_0^i, \ldots, \hat{v}_j^i, \ldots, v_{n-1}^i] \right) \]

\[ = \sum_{j=0}^{n-1} (-1)^{j} \left( [(n-i), v_0^i, \ldots, \hat{v}_j^i, \ldots, v_{n-1}^i] + [v_j^i, v_0^i, \ldots, \hat{v}_j^i, \ldots, v_{n-1}^i] \right) \]

\[ = \sum_{j=0}^{n-1} (-1)^{j} \left( [(n-i), v_0^i, \ldots, \hat{v}_j^i, \ldots, v_{n-1}^i] + (-1)^{j} [v_0^i, \ldots, \hat{v}_j^i, \ldots, v_{n-1}^i] \right) \]

\[ = \sum_{j=0}^{n-1} \left( (-1)^{j} [(n-i), v_0^i, \ldots, \hat{v}_j^i, \ldots, v_{n-1}^i] + [v_0^i, \ldots, v_j^i, \ldots, v_{n-1}^i] \right) \]

\[ = \sum_{j=0}^{n-1} \left( (-1)^{j} [(n-i), v_0^i, \ldots, \hat{v}_j^i, \ldots, v_{n-1}^i] \right) + n \cdot [v_0^i, \ldots, v_j^i, \ldots, v_{n-1}^i]. \quad (*) \]

Consider the summation in (*). If we want to permute the element \((n-i)\) into its correct position, we must consider two cases: \( j < (n-i) \) and \( j \geq (n-i) \). If \( j < (n-i) \), then we need only make \((n-i-1)\) consecutive transpositions to move \((n-i)\) into its correct position. If \( j \geq (n-i) \), we must make exactly \((n-i)\) consecutive transpositions. We will disregard the term

\[ n \cdot [v_0^i, \ldots, v_j^i, \ldots, v_{n-1}^i] \]

in (*) for now. If we consider solely the summation in (*), we can see that it can be
written as
\[
\sum_{j<(n-i)} \left( (-1)^j [(n-i), v_0^j, \ldots, v_{n-1}^j] \right)
+ \sum_{j \geq (n-i)} \left( (-1)^j [(n-i), v_0^j, \ldots, v_{n-1}^j] \right)
= \sum_{j<(n-i)} \left( (-1)^{j+n-i-1} [v_0^i, \ldots, v_j^i, (n-i), \ldots, v_{n-1}^i] \right)
+ \sum_{j \geq (n-i)} \left( (-1)^{j+n-i} [v_0^i, \ldots, v_j^i, (n-i), \ldots, v_{n-1}^i] \right).
\]

We now revert back to our original labeling of the vertices.

\[
\sum_{j<(n-i)} \left( (-1)^{j+n-i-1} [0, \ldots, \hat{j}, \ldots, (n-i), \ldots, n] \right)
+ \sum_{j \geq (n-i)} \left( (-1)^{j+n-i} [0, \ldots, (n-i), \ldots, (j+1), \ldots, n] \right). \tag{**}
\]

Notice that each face in each sum is unique due to us adding back in our originally removed vertex and proceeding to remove \(n-1\)-distinct vertices. Consider the first summation in (**). We can reorder this as

\[
\sum_{h=0}^{n-i-1} \left( (-1)^{2(n-i)-1-h} [0, \ldots, (n-i-1-h), (n-i), \ldots, n] \right)
\tag{3.2}
\]

Next, consider the second sum in (**). We can reorder the sum as

\[
\sum_{h=0}^{i-1} \left( (-1)^{2(n-i)+h} [0, \ldots, (n-i), (n-i+1+h), \ldots, n] \right)
\tag{3.3}
\]

Recall that \([v_0^i, \ldots, v_{n-1}^i]\) in (*) is exactly \([0, \ldots, (n-i), \ldots, n]\). Combining this fact with equations (3.2) and (3.3), we can produce a column vector for the operator
\( \partial^T_{n-1} \partial_{n-1} \) on a basis element \( F^i \):

\[
\partial^T_{n-1} \left( \partial_{n-1}(F^i) \right) = \begin{bmatrix}
\{n\} & (-1)^{n-i-1} \\
\{n-1\} & (-1)^{n-i-2} \\
\vdots & \vdots \\
\{n-i\} & n \\
\{i\} & (-1)^{i-2} \\
\{0\} & (-1)^{i-1}
\end{bmatrix},
\]

where our rows are labeled as the vertices we remove from \( F^i \). Note that this is exactly what our basis elements are defined as. Therefore, we have

\[
\partial^T_{n-1} \partial_{n-1}(\sigma) = \begin{bmatrix}
n & 1 & -1 & \cdots & \cdots & (-1)^{n+1} \\
1 & n & 1 & -1 & \cdots & (-1)^{n} \\
(-1) & 1 & n & \ddots & \vdots & \vdots \\
\vdots & -1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & n & 1 \\
(-1)^{n+1} & (-1)^{n} & \cdots & \cdots & 1 & n
\end{bmatrix},
\]

where the columns and rows are each indexed by the basis elements \( F^i \) as defined.
3.4 Irrational Roots

Lemma 3.3. Any polynomial of degree 3 of the form

\[(n + 3) - (3n + 7)x + (n + 5)x^2 - x^3\]

has only irrational roots.

Proof. We will quickly show that this polynomial must have all real roots. First off, we know that if this has any non-real roots, there must be 2 of them since non-real complex numbers come in pairs. Thus, we need only show that this polynomial crosses the x-axis two times to conclude it will cross it a third time. So, notice first that if \(x < 0\), this polynomial is strictly positive (note that \(n \in \mathbb{Z} > 0\)). Now we look what happens when we let \(x = 1\):

\[
(n + 3) - (3n + 7)x + (n + 5)x^2 - x^3 = n + 3 - 3n - 7 + n + 5 - 1 \\
= 2n - 3n + 8 - 8 \\
= -n.
\]

Thus, for all \(n > 0, n \in \mathbb{Z}\), if \(x = 1\), this polynomial is negative. Thus we must have at least one real root. We now let \(x = 3\):

\[
(n + 3) - (3n + 7)x + (n + 5)x^2 - x^3 = (n + 3) - 3(3n) - 3(7) + 9(n) + 9(45) - 27 \\
= n - 9n + 9n + 3 - 21 + 45 - 27 \\
= n + 48 - 48 \\
= n.
\]

Therefore, for all \(n > 0, n \in \mathbb{Z}\), if \(x = 3\), this polynomial is positive. Thus, it crosses the x-axis yet again, meaning it has another real root. Since it has at least 2 real
roots, it must have 3 real roots because it is of degree 3. So, this polynomial has only real roots.

If this polynomial has rational roots, the roots must be of the form \( \frac{a}{b} \), \( a, b \in \mathbb{Z} \), where \( b \) divides \(-1\) and \( a \) divides \((n + 3)\). So our rational roots, if any, are integer roots that must divide \( n + 3 \). Thus, \( n + 3 = xk \) for some integer \( k \). We will now rearrange our polynomial:

\[
(n + 3) - (3n + 7)x + (n + 5)x - x^3 = (n + 3) - (3(n + 3) - 2)x + ((n + 3) + 2)x^2 - x^3 = 0
\]

\[
\Rightarrow (n + 3) - 3x(n + 3) + x^2(n + 3) + 2x + 2x^2 - x^3 = 0
\]

\[
\Rightarrow (n + 3)(1 - 3x + x^2) + x(2 + 2x - x^2) = 0
\]

\[
\Rightarrow xk(1 - 3x + x^2) + x(2 + 2x - x^2) = 0
\]

\[
\Rightarrow k(1 - 3x + x^2) + (2 + 2x - x^2) = 0.
\]

Now we are going to distribute \( k \) and reorder our equation:

\[
\Rightarrow k(1 - 3x + x^2) + (2 + 2x - x^2) = k - 3kx + kx^2 + 2 + 2x - x^2 = 0
\]

\[
\Rightarrow (k + 2) - (3k - 2)x + (k - 1)x^2 = 0
\]

\[
\Rightarrow ((k - 1) + 3) - (3(k - 1) + 1)x + (k - 1)x^2 = 0
\]

\[
\Rightarrow (k - 1) - 3x(k - 1) + x^2(k - 1) + 3 - x = 0
\]

\[
\Rightarrow (k - 1)(1 - 3x + x^2) + (3 - x) = 0
\]

\[
\Rightarrow (k - 1)(1 - 3x + x^2) = x - 3.
\]

Since we know that \( x \in \mathbb{Z} \), then \((x - 3)\) and \((1 - 3x + x^2)\) must be integers as well. Thus, \((k - 1)\) divides \((x - 3)\) evenly. Let \(-l = (x - 3)/(k - 1)\). Thus,

\[
(1 - 3x + x^2) = -l \Rightarrow 1 - 3x + x^2 + l = 0 \Rightarrow -3x + x^2 + (l + 1) = 0
\]

\[
\Rightarrow x(x - 3) + (l + 1) = 0 \Rightarrow l + 1 = x(3 - x).
\]

Thus, \((3 - x)\) divides \(l + 1\) and therefore \(|(3 - x)| < |(l + 1)|\). However, from our equation earlier, we have \(-l = (x - 3)/(k - 1)\), meaning \(l\) divides \((3 - x)\), so \(l < |3 - x|\).

So, we now have \(|l| < |x - 3| < |l + 1| \Rightarrow |l + 3| < |x| < |l + 4|\). But with our assumption of rational roots came the fact that \( x \in \mathbb{Z} \). However, there is no integer between \(|l + 3|\) and \(|l + 4|\). Thus, our original assumption of rational roots produced a contradiction and therefore this polynomial has only irrational roots.
CHAPTER 4
Subdivision

4.1 Structure

Definition 4.1. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two simplicial complexes with vertex sets $V_1$ and $V_2$, and sets of facets $F_1$ and $F_2$, respectively. We define $\mathcal{K} := \mathcal{K}_1 \ast \mathcal{K}_2$ as having vertex set $V_1 \cup V_2$ and whose facets are $[v^1_0, \ldots, v^1_n, v^2_0, \ldots v^2_m]$ where $[v^1_0, \ldots, v^1_n] \in F_1$ and $[v^2_0, \ldots, v^2_m] \in F_2$. Then, we refer to $\mathcal{K}$ as the join of simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$.

If we are given a single vertex $\omega$, the join of this vertex and a simplicial complex $\mathcal{K}$ is known as the cone of $\mathcal{K}$ with respect to $\omega$.

Definition 4.2. Let $\mathcal{K}$ be an abstract simplicial complex. Choose a face $\sigma = [v_0, \ldots v_m]$. Let $\Omega$ denote the link of $\sigma$ and let $\omega$ be a new vertex not in the vertex set of $\mathcal{K}$. Let $\mathcal{K}_\sigma$ be an abstract simplicial complex with vertex set $V' = \{v_0, \ldots v_m\} \cup \{\omega\}$ whose facets are all order $n$ subsets of $V'$ except the facet $[v_0, \ldots, v_m]$. Then we say that $\mathcal{K}' := \mathcal{K}_\sigma \ast \Omega$ is the restricted subdivision of $\mathcal{K}$ with respect to $\sigma$.

Example 4.1.1. Let $\sigma = [0, 1, 2]$ be the 2-simplex. Subdividing $\sigma$ creates an abstract simplicial complex $\mathcal{K}$ defined as follows:

$$\mathcal{K} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\}$$

4.2 The Discrete Laplacian and Restricted Subdivision

Theorem 4.1. Let $\sigma$ be an $n$-simplex. Let $\mathcal{K}$ be the corresponding restricted subdivision of $\sigma$ with respect to itself. Fix an $n$-simplex $\sigma'$ of $\mathcal{K}$ and let $\mathcal{K}'$ be the restricted subdivision...
subdivision of \( K \) with respect to \( \sigma' \).

Then, the top dimensional discrete Laplacian \( L_n(K') \) has irrational eigenvalues. Furthermore, the characteristic polynomial of \( L_n(K') \) is exactly

\[
(n + 1 - x)^{n-1}(n + 3 - x)^{n-1}(n + 3 - (3n + 7)x + (n + 5)x^2 - x^3).
\]

**Remark 4.2.1.** For any given abstract simplicial complex \( K \), a rearrangement of the vertices only affects the orientation of the simplices within the simplicial complex. Thus, carrying out a restricted subdivision on an arbitrary face is sufficient enough to prove the result. Furthermore, we are free to rearrange the vertices as we like as well.

**Proof.** Let \( \sigma = [0, \ldots, n] \) be an \( n \)-simplex and let \( K \) be the corresponding restricted subdivision of \( \sigma \) with respect to itself. Thus, the set of \( n \)-facets of \( K \) is exactly \( \{[0, \ldots, (n - i), \ldots, n, n + 1]\}_{i \in \{0, \ldots, n\}} \) due to the fact that each of these faces have dimension \( n \) and none are exactly the original simplex \( \sigma \). Note that

\[
\{[0, \ldots, (n - i), \ldots, n, n + 1]\}_{i \in \{0, \ldots, n\}} = [0, \ldots, n - 1, n + 1] \cup \{[0, \ldots, (n - i), \ldots, n, n + 1]\}_{i \in \{1, \ldots, n\}}.
\]

We rename the vertices of \( \{[0, \ldots, (n - i), \ldots, n, n + 1]\}_{i \in \{1, \ldots, n\}} \cup [0, \ldots, n - 1, n + 1] \) by associating \( i = v_i \) for all \( i \neq n, (n + 1) \) and \( (n + 1) = v_n \). We disregard \( n \) for a moment. Replacing this back into equation (4.1), we get

\[
[v_0, \ldots, v_{n-1}, v_n] \cup \{[v_0, \ldots, (v_{n-i}), \ldots, v_{n-1}, n, v_n]\}_{i \in \{1, \ldots, n\}}.
\]

We carry out restricted subdivision of \( K \) with respect to the facet \( \alpha = [v_0, \ldots, v_n] \). Deonte the subdivision of \( \alpha \) as \( K_\alpha \) and \( K' \) as the restricted subdivision of \( K \) with respect to \( \alpha \). Thus, \( K' = K \cup K_\alpha \setminus \{\alpha\} \). As before, \( K_\alpha \) has facets

\[
\{[v_0, \ldots, (v_{n-j}), \ldots, v_n, v_{n+1}]\}_{j \in \{0, \ldots, n\}}.
\]
In addition, we note that
\[
\{[v_0, \ldots, (v_{n-j}^\wedge), \ldots, v_n, v_{n+1}]\}_{j \in \{0, \ldots, n\}}
= [v_0, \ldots, v_{n-1}, v_{n+1}] \cup \{[v_0, \ldots, (v_{n-j})^\wedge, \ldots, v_n, v_{n+1}]\}_{j \in \{1, \ldots, n\}}.
\]

(4.3)

Replacing equation (4.3) into equation (4.2), we obtain the set of facets for $K'$:
\[
[v_0, \ldots, v_{n-1}, v_{n+1}] \cup \{[v_0, \ldots, (v_{n-j})^\wedge, \ldots, v_n, v_{n+1}]\}_{j \in \{1, \ldots, n\}}
\cup \{[v_0, \ldots, (v_{n-i})^\wedge, \ldots, v_{n-1}, n, v_n]\}_{i \in \{1, \ldots, n\}}.
\]

Again, we rename the vertices with a new association. Let this association be $v_p = k_p$ for $p \neq n, n+1$, $v_{n+1} = k_n$, $v_n = k_{n+1}$, and $n = k_{n+2}$. We then have
\[
[k_0, \ldots, k_{n-1}, k_n] \cup \{[k_0, \ldots, (k_{n-j})^\wedge, \ldots, k_n, k_{n+1}]\}_{j \in \{1, \ldots, n\}}
\cup \{[k_0, \ldots, (k_{n-i})^\wedge, \ldots, k_{n-1}, k_{n+2}, k_{n+1}]\}_{i \in \{1, \ldots, n\}}.
\]

Recall that the orientation of our faces does not have any bearing on the characteristic polynomial of the discrete Laplacian. So, we change the orientation of these faces as so:
\[
[k_0, \ldots, k_{n-1}, k_n] \cup \{[k_0, \ldots, (k_{n-j})^\wedge, \ldots, k_n, k_{n+1}]\}_{j \in \{1, \ldots, n\}}
\cup \{[k_0, \ldots, (k_{n-i})^\wedge, \ldots, k_{n-1}, k_{n+1}, k_{n+2}]\}_{i \in \{1, \ldots, n\}}.
\]

We define the order of basis elements of $C_n(K')$ by choosing the $p^{th}$ basis element as follows:
\[
\begin{align*}
[k_0, \ldots, k_n] & \quad \text{if } p = 0; \\
[k_0, \ldots, (k_{n-p})^\wedge, \ldots, k_{n+1}] & \quad \text{if } p \in \{1, \ldots, n\}; \\
[k_0, \ldots, (k_{n-p-n})^\wedge, \ldots, k_{n-1}, k_{n+1}, k_{n+2}] & \quad \text{if } p \in \{n+1, \ldots, 2n\}.
\end{align*}
\]

We have now constructed an ordered basis for $C_n(K')$. Notice that there is
a total of $2n + 1$ basis elements of $C_n(K')$. To show the general form of our matrix representing the discrete Laplacian $L_n(K') = \partial^T_n \partial_n(K')$, we describe first how $\partial_n$ maps distinct basis elements of $C_n(K')$ to basis elements of $C_{n-1}(K')$. Recall that in Section 2.1 we stated that without loss of generality, we are free to refer to the faces of an abstract simplicial complex exactly by the indexes of the vertices that make up the face. We continue this trend for the rest of the proof. Let $\tau_r$ denote the $r^{th}$ basis element of $C_n(K')$. We first look at the $\tau_0$, the face $[0,\ldots,n]$. Consider the boundary homomorphism acting on this face:

$$\partial_n(\tau_0) = \partial_n[0,\ldots,n] = \sum_{i=0}^{n} (-1)^i[0,\ldots,\hat{i},\ldots,n].$$

As stated earlier, for $q = 0,\ldots,n$, the $q^{th}$ basis element of $C_{n-1}$ is created by removing the vertex $(n-q)$ from the set $[0,\ldots,n]$. So, the first summand in $\partial_n(\tau_0)$ is actually the last $n^{th}$ basis element of $C_{n-1}(K')$. Furthermore, the second summand is the $(n-1)^{th}$ basis, and so on. Thus, $\partial_n(\tau_0)$ maps to the first $n$ basis elements of $C_{n-1}(K')$ like so:

$$\begin{bmatrix}
[0,\ldots,n] & (-1)^n \\
[0,\ldots,n-1] & (-1)^{n-1} \\
[0,\ldots,n-2,\ldots,n] & \vdots \\
[0,\ldots,(n-\hat{i}),\ldots,n] & (-1)^i \\
\vdots & \vdots \\
[0,2,\ldots,n] & -1 \\
[1,2,\ldots,n] & 1
\end{bmatrix}.$$  \quad (4.4)

Refer to the first entry in this column as a $1 \times 1$ block matrix $A_1$. Furthermore, refer to the next $n$ entries as the $n \times 1$ block matrix $A_2$. Consider now the next $n$ basis elements of $C_n(K')$. Note that the $p^{th}$ basis element $\tau_p$ for $p \in \{1,\ldots,n\}$ is $\tau_p = [0,\ldots,(n-p),\ldots,n,n+1]$. Rename these vertices of $\tau_p$ with the association $i = v^p_i$ for $0 \leq i \leq n-p-1$ and $i = v^p_{i-1}$ for $n-p+1 \leq i \leq n-1$. So
\[ \tau_p = [v_0^p, \ldots, v_{n-p}^p, v_{n-p}^p, \ldots, v_{n-2}^p, n, n + 1]. \] Then,

\[
\partial_n(\tau_p) = \partial_n[v_0^p, \ldots, v_{n-2}^p, n, n + 1] \\
= \left( \sum_{i=0}^{n-2} (-1)^i [v_0^p, \ldots, \hat{v}_i^p, \ldots, v_{n-2}^p, n, n + 1] \right) \\
+ (-1)^{n-1} [v_0^p, \ldots, v_{n-2}^p, n + 1] + (-1)^n [v_0^p, \ldots, v_{n-2}^p, n]. \tag{4.5} \]
Let $\Phi_{n+1} := \langle \{[v^p_0, \ldots, v^p_{n-2}, n, n+1] \}_{p \in \phi} \rangle$, the subspace of $C_n(K')$ generated by the indicated oriented faces, where $\phi = \{1, \ldots, n\}$. Define the map $\zeta$ as

$$\zeta : \Phi_{n+1} \rightarrow C_{n-2}(K'),$$

$$[v^p_0, \ldots, v^p_{n-2}, n, n+1] \mapsto [v^p_0, \ldots, v^p_{n-2}].$$

Observe that for

$$z = \sum_{i=0}^{n-2} (-1)^i[v^p_0, \ldots, \hat{v}^p_i, \ldots, v^p_{n-2}, n, n+1]$$

– which is the first summation in (4.5) – we have

$$\zeta(z) = \sum_{i=0}^{n-2} (-1)^i[v^p_0, \ldots, \hat{v}^p_i, \ldots, v^p_{n-2}] = \partial_{n-2}[v^p_0, \ldots, v^p_{n-2}].$$

Thus, $\partial_n(\tau_p)$ maps to a basis element $\iota_k = [v^p_0, \ldots, v^p_i, \ldots, v^p_{n-2}, n, n+1]$ in $C_{n-1}(K')$ in the same fashion that $\partial_{n-2}([v^p_0, \ldots, v^p_{n-2}])$ maps to $\iota'_k = [v^p_0, \ldots, \hat{v}^p, \ldots, v^p_{n-2}]$ in $C_{n-3}(K')$, where $k \in \{1, \ldots, m = \binom{n}{2}\}$. But notice, that the collection of $n-2$-faces $\{[v^p_0, \ldots, v^p_{n-2}]\}_{p \in \phi}$ spans the $(n-2)$-faces of the $(n-1)$-simplex. Thus, we are guaranteed to be able to reorder the basis elements $\iota_1, \ldots, \iota_m$ such that the matrix

\[
\begin{bmatrix}
0, \ldots, n-2, n, n+1 & \cdots & 0, \ldots, (n-p), \ldots, n, n+1 & \cdots & 1, \ldots, n, n+1 \\
\iota_1 & * & \cdots & * & \cdots & * \\
\iota_2 & * & \cdots & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\iota_{m-1} & * & \cdots & * & \cdots & * \\
\iota_m & * & \cdots & * & \cdots & * \\
\end{bmatrix}
\]

is the same as the matrix representing the boundary operator $\partial_{n-2} : C_{n-2}(\sigma') \rightarrow C_{n-3}(\sigma')$, where $\sigma'$ is the $(n-1)$ simplex.

Let $\iota'_\alpha = [v^\alpha_0, \ldots, v^\alpha_{n-2}, n]$ be a basis element of $C_{n-1}(K')$. Consider now the following matrix that represents how the basis elements $\tau_1, \ldots, \tau_n$ map to the basis
elements $\iota'_1, \ldots, \iota'_n$ of $C_{n-1}(K')$ by $\partial_n$.

First note that the rows are indexed by basis elements of $C_{n-1}(K')$ that are not in the set $\iota_1, \ldots, \iota_m$ because none have $n + 1$ as a vertex. Also notice that for all $p, q \in \{1, \ldots, n\}$, where $p \neq q$, $\{v^p_0, \ldots, v^p_{n-2}\} \neq \{v^q_0, \ldots, v^q_{n-2}\}$. Note that in equation (4.5), $\iota'_p$ has a coefficient of $(-1)^n$. Thus, for each column $p$, there is a $(-1)^n$ in the $p^{th}$ row. Therefore, the matrix has form as follows:

$$
\begin{bmatrix}
\begin{array}{cccc}
[v^1_0, \ldots, v^{1\ast}_{n-2}, n, n+1] & \cdots & [v^p_0, \ldots, v^{p\ast}_{n-2}, n, n+1] & \cdots & [v^n_0, \ldots, v^{n\ast}_{n-2}, n, n+1] \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
\end{array}
\end{bmatrix}
$$

This matrix is the $n \times n$ identity matrix with $n$ is even, and the negative when $n$ odd. Thus we will refer to this matrix as having form $(-I)^n_{n \times n}$. Continuing, notice that the basis elements $\iota'_1, \ldots, \iota'_n$ are exactly the same basis elements $[0, \ldots, n - 2, n], \ldots, [0, \ldots, (n - i), \ldots, n - 1, n], \ldots, [1, 2, \ldots, n]$. So if we order the basis as $[0, \ldots, n - 1], \iota'_1, \ldots, \iota'_n$, then we can the combine matrix
We can refer to the matrix above as

$$
\begin{pmatrix}
A_1 & 0 \\
A_2 & (-I)^n_{n \times n}
\end{pmatrix}
$$

(4.7)

Our goal is to build upon this matrix to get a closed form for $L_n(K')$. Let $\iota'' = [v_0^a, \ldots, v_{n-2}^a, n + 1]$. Notice that in equation (4.5), the second to last summand is actually $\iota''_p$ with a coefficient of $(-1)^{n-1}$. Thus, as in (4.6), we have

$$
\begin{pmatrix}
[0, \ldots, n-1] & [v_0^1, \ldots, v_{n-2}^1, n, n+1] & \ldots & [v_0^p, \ldots, v_{n-2}^p, n, n+1] & \ldots & [v_0^n, \ldots, v_{n-2}^n, n, n+1] \\
[0, \ldots, n-1] & (-1)^{n-1} & \ldots & 0 & \ldots & 0 \\
[v_0^1, \ldots, v_{n-2}^1, n-2] & (-1)^{n-1} & 0 & \ldots & 0 \\
[v_0^2, \ldots, v_{n-2}^2, n-2] & (-1)^n & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
[v_0^p, \ldots, v_{n-2}^p, n-2] & (-1)^n & 0 & \ldots & 0 \\
[v_0^{n-1}, \ldots, v_{n-2}^{n-1}, n-2] & (-1)^n & 0 & \ldots & 0 \\
[v_0^n, \ldots, v_{n-2}^n, n-1] & 0 & \ldots & 0 & \ldots & (-1)^{n-1}
\end{pmatrix}
$$

(4.8)

We refer to matrix (4.8) as $(-I)^{n-1}_{n \times n}$. We can order the previous basis elements of $C_{n-1}(K')$ as $[0, \ldots, n-1], \iota'_1, \ldots, \iota'_m, \iota''_n, \iota''_1, \ldots, \iota''_m$. Adjoining this to matrix (4.7) gives us

$$
\begin{pmatrix}
A_1 & 0 \\
A_2 & (-I)^n_{n \times n} \\
0 & (-I)^{n-1}_{n \times n} \\
0 & \partial_{n-2}^{n-1}
\end{pmatrix}
$$

where $\partial_{n-2}^{n-1}$ refers to the matrix representing $\partial_n - 2 : C_{n-2}(\sigma') \to C_{n-3}(\sigma')$, with $\sigma'$
being the $n - 1$-simplex.

We now consider the next set of basis elements of $C_n(K')$, the set

$$\{[0, \ldots, (n - p), \ldots, n - 1, n + 1, n + 2]_{p \in \phi}\},$$

where $\phi = \{1, \ldots, n\}$ as before. Let $\tau'_p = [0, \ldots, (n - p), \ldots, n - 1, n + 1, n + 2]$. Notice that $\tau'_p$ is very similar to $\tau_p$. In fact, to get from $\tau_p$ to $\tau'_p$, we transpose $n$ and $n + 1$ and replace the vertex $n$ with $n + 2$. Thus, the $n - 2$ vertices are unchanged. So, we rename these $n - 2$ vertices with the same association as earlier: $i = v^p_i$ for $0 \leq i \leq n - p - 1$ and $i = v^p_{i-1}$ for $n - p + 1 \leq i \leq n - 1$. Then

$$\partial_n(\tau'_p) = \partial_n([v^p_0, \ldots, v^p_{n-2}, n + 1, n + 2])$$

$$= \left(\sum_{i=0}^{n-2} (-1)^i [v^p_0, \ldots, \hat{v}^p_i, \ldots, v^p_{n-2}, n + 1, n + 2]\right)$$

$$+ (-1)^{n-1}[v^p_0, \ldots, v^p_{n-2}, n + 2] + (-1)^n[v^p_0, \ldots, v^p_{n-2}, n + 1].$$

(4.9)

Then, define $\Phi_{n+2} := \langle \{[v^p_0, \ldots, v^p_{n-2}, n + 1, n + 2]_{p \in \phi'}\rangle$, the subspace of $C_n(K')$ generated by the indicated orientated faces. Define a map $\zeta'$ as follows:

$$\zeta': \Phi_{n+2} \to C_{n-2}(K'),$$

$$[v^p_0, \ldots, v^p_{n-2}, n + 1, n + 2] \mapsto [v^p_0, \ldots, v^p_{n-2}].$$

Observe that if

$$z = \sum_{i=0}^{n-2} (-1)^i [v^p_0, \ldots, \hat{v}^p_i, \ldots, v^p_{n-2}, n + 1, n + 2]$$

– which is the first summation in (4.9) – we have

$$\zeta'(z) = \sum_{i=0}^{n-2} (-1)^i [v^p_0, \ldots, v^p_i, \ldots, v^p_{n-2}] = \partial_{n-2}([v^p_0, \ldots, v^p_{n-2}]).$$

Then, $\partial_n(\tau'_p)$ maps to a basis element $\eta_k = [v^p_0, \ldots, v^p_i, \ldots, v^p_{n-2}, n+1, n+2]$ in $C_{n-1}(K')$ in the same fashion that $\partial_{n-2}([v^p_0, \ldots, v^p_{n-2}])$ maps to $\eta'_k = [v^p_0, \ldots, v^p_i, \ldots, v^p_{n-2}]$ in
$C_{n-3}(K')$, where $k = \{1, \ldots, m = \binom{n}{2}\}$. Just as before, we are guaranteed to be able to reorder the basis elements $\eta_1, \ldots, \eta_m$ such that the matrix

$$
\begin{bmatrix}
[0, \ldots, n-2, n+1, n+2] & \ldots & [0, \ldots, (n-p), \ldots, n+1, n+2] & \ldots & [1, \ldots, n+1, n+2]
\end{bmatrix}
$$

is the same as $\partial_{n-2}^\Delta$. Next, we consider how the basis element $\tau'_p$ maps to the set of basis elements $\eta'_1, \ldots, \eta'_n \in C_{n-1}(K)$, where $\eta'_\beta = [v^\beta_0, \ldots, v^\beta_{n-2}, n+2]$. These basis elements have not been introduced before, as they are the only set of elements of $C_{n-1}(K)$ that have $n+2$ as a vertex but not $n+1$. Thus, we obtain the matrix

$$
\begin{bmatrix}
\begin{array}{cccc}
\vdots \\
0 & \ldots & 0 \\
\vdots & \vdots & \vdots \\
0 & \ldots & 0 \\
0 & \ldots & 0 \\
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\end{array}
\end{bmatrix}
$$

due to the fact that in equation (4.9) the face $[v^p_0, \ldots, v^p_{n-2}, n+2]$ has a coefficient of $(-1)^{n-1}$ in $\partial_n(\tau'_p)$ and 0 as a coefficient for all $\tau'_q, q \neq p$.

Recall how we defined the basis elements $\iota''_1, \ldots, \iota''_m$ as $\iota''_\alpha = [v^\alpha_0, \ldots, v^\alpha_{n-2}, n+1]$. The coefficient of $\iota''_p$ in $\partial_n(\tau'_p)$ is $(-1)^n$, and 0 for all $\tau'_q, q \neq p$. Just as we did before, we can construct the matrix mapping the basis elements $\tau'_1, \ldots, \tau'_n$ to the basis elements...
\( i_1'', \ldots, i_m'' \) as
\[
\begin{bmatrix}
[0, \ldots, n-1, i_1', \ldots, i_n']
& [0, \ldots, n-1, i_1', \ldots, i_n']
& [0, \ldots, n-1, i_1', \ldots, i_n']
\end{bmatrix}
\]

As you can see, we have now shown how each basis element in \( C_n(K) \) maps to a basis element in \( C_{n-1}(K) \). Order the basis elements of \( C_{n-1}(K) \) as follows:
\[
[0, \ldots, n-1, i_1', \ldots, i_n'], i_1', \ldots, i_n', i_1', \ldots, i_m', \eta_1', \ldots, \eta_m', \eta_1, \ldots, \eta_m,
\]
where the \( i^{th} \) index in this set refers to the \( i^{th} \) row in the following matrix:
\[
\partial_n(K') = \begin{bmatrix}
A_1 & 0 & 0 \\
(-I)^n_{n \times n} & 0 \\
0 & (-I)^{n-1}_{n \times n} & (-I)^n_{n \times n} \\
0 & \partial\Delta^{-1}_{n-2} & 0 \\
0 & 0 & (-I)^{n-1}_{n \times n} \\
0 & 0 & \partial\Delta^{-1}_{n-2}
\end{bmatrix}.
\]

With this closed form for \( \partial_n(K') \), we obtain
\[
\mathcal{L}_n(K') = \begin{bmatrix}
A_T^1 A_1 + A_T^2 A_2 & A_T^1 (-I)^n \\
A_2 (-I)^n & (-I)^{2n} + (-I)^{2(n-1)} + (\partial\Delta^{-1}_{n-2})^T \partial\Delta^{-1}_{n-2} & (-I)^{n-1} (-I)^n \\
0 & (-I)^{n-1} (-I)^n & (-I)^{2n} + (-I)^{2(n-1)} + (\partial\Delta^{-1}_{n-2})^T \partial\Delta^{-1}_{n-2}
\end{bmatrix}.
\]

Note that this is a block matrix, where each entry is a matrix of varying size. Denote the blocks as \( B_{i,j} \), where \( i \) is its row and \( j \) its column. We have omitted \( n \times n \) as the subscript for the matrices \((-I)^n_{n \times n}\) and \((-I)^n_{n \times n}\) to avoid redundancy.
Our goal now is to compute the characteristic polynomial of this matrix. To do this, we begin by describing what each of these block matrix entries in $L_n(K')$ are. First, recall the forms of the block matrices $A_1$ and $A_2$:

\[ A_1 = \begin{bmatrix} (-1)^n \end{bmatrix}, \quad A_2 = \begin{bmatrix} (-1)^{n-1} \\ (-1)^{n-2} \\ \vdots \\ -1 \\ 1 \end{bmatrix}. \]

Therefore, $A_1^T A_1 + A_2^T A_2$ is the $1 \times 1$ block entry with $1 + n$ as its only entry. We can refer to this block entry as $(n + 1)$. Furthermore, note that

\[ A_2(-I)^n = \begin{bmatrix} (-1)^{n-1}(-1)^n \\ (-1)^{n-2}(-1)^n \\ \vdots \\ -1(-1)^n \\ 1(-1)^n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ \vdots \\ (-1)^{n-1} \\ (-1)^n \end{bmatrix}. \]

Similarly, $A_2^T(-I)^n$ is a row with $n$-entries where its $i^{th}$ entry is equal to the $i^{th}$ entry of $A_2(-I)^n$. We will refer to $A_2(-I)^n$ and $A_2^T(-I)^n$ as $A'_2$ and $A''_2$, respectively. Consider now the entry

\[ B_{2,2} = B_{3,3} = (-I)^{2n} + (-I)^{2(n-1)} + (\partial_{n-2}^\Delta)^T \partial_{n-2}^\Delta = 2(I) + (\partial_{n-2}^\Delta)^T \partial_{n-2}^\Delta. \]

Recall from Lemma 3.2 from Section 3.3. Since $\partial_{n-2}^\Delta$ refers to the boundary operator $\partial_{n-2} : C_{n-2}(\Delta) \to C_{n-3}(\Delta)$ where $\Delta$ is the $n - 1$-simplex, $(\partial_{n-2}^\Delta)^T \partial_{n-2}^\Delta$ is a $n \times n$
matrix of the form
\[
\begin{pmatrix}
    n-1 & 1 & -1 & \cdots & \cdots & (-1)^n \\
    1 & n-1 & 1 & -1 & \cdots & (-1)^{n-1} \\
    -1 & 1 & n-1 & 1 & \cdots & \vdots \\
    \vdots & -1 & \ddots & \ddots & \vdots \\
    \vdots & \vdots & n-1 & 1 & \vdots \\
    (-1)^n & (-1)^{n-1} & \cdots & \cdots & 1 & n-1
\end{pmatrix}.
\]

Therefore, we obtain
\[
2(I) + (\partial_{n-2}^{\Delta-1})^T \partial_{n-2}^{\Delta-1} =
\begin{pmatrix}
    n+1 & 1 & -1 & \cdots & \cdots & (-1)^n \\
    1 & n+1 & 1 & -1 & \cdots & (-1)^{n-1} \\
    -1 & 1 & n+1 & 1 & \cdots & \vdots \\
    \vdots & -1 & \ddots & \ddots & \vdots \\
    \vdots & \vdots & n+1 & 1 & \vdots \\
    (-1)^n & (-1)^{n-1} & \cdots & \cdots & 1 & n+1
\end{pmatrix}
\]
as our $B_{2,2}$ and $B_{3,3}$ entry. We will refer to this matrix as $\Psi$. In addition, entries $B_{3,2}$ and $B_{2,3}$ are simply negative identity matrices. With the renaming of these entries, we have
\[
\mathcal{L}_n(\mathcal{K}') =
\begin{pmatrix}
    (n+1) & A_2^T & 0 \\
    A_2' & \Psi & -I \\
    0 & -I & \Psi
\end{pmatrix}.
\]
So, to begin computing the characteristic polynomial, we have
\[
(\mathcal{L}_n(\mathcal{K}') - xI) =
\begin{pmatrix}
    (n+1) - x & A_2^T & 0 \\
    A_2' & \Psi - xI & -I \\
    0 & -I & \Psi - xI
\end{pmatrix}.
\]

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Our goal is to row reduce this matrix to be an upper triangular block matrix by using elementary row operations, where each block on the diagonal has a computable determinant. For ease of notation, let \((n + 1) - x = \theta\). We begin with the following:

\[
R_2 \leftrightarrow R_3 \Rightarrow \begin{bmatrix}
\theta & A_2^T & 0 \\
0 & -I & \Psi - xI \\
A_2' & \Psi - xI & -I
\end{bmatrix},
\]

\[
R_3 + \theta R_2 \Rightarrow \begin{bmatrix}
\theta & A_2^T & 0 \\
0 & -I & \Psi - xI \\
A_2' & (\Psi - xI) - \theta I & \theta(\Psi - xI) - I
\end{bmatrix}.
\]

Notice that our entry \(B_{3,2} = (\Psi - xI) - \theta I\) is actually \(-\Gamma(\alpha)\) from Lemma 3.1 where \(\alpha = 0\) and \((\Psi - xI) = -\Gamma(-\theta)\). So we have

\[
(L_n(K') - xI) = \begin{bmatrix}
\theta & A_2^T & 0 \\
0 & -I & -\Gamma(-\theta) \\
A_2' & -\Gamma(0) & -\theta \Gamma(-\theta) - I
\end{bmatrix}.
\]

We now reduce all entries in \(B_{3,2} = -\Gamma(0)\) to 0 by adding various rows of \(B_{2,2}\). We ignore how this affects our \(B_{3,3}\) entry for the moment. Notice that for each entry \(d_{i,k}\) of \(B_{3,2} = -\Gamma(0)\) where \(i \neq j\), \(d_{i,k} = (-1)^{i+j+1}\). Furthermore, note that the \(j^{th}\) row of \(B_{2,2}\) has a \(-1\) in its \(j^{th}\) column. So to cancel out an entry \(d_{i,k}\) in \(B_{3,2}\), we add \((-1)^{i+j+1}\) multiplied by the \(j^{th}\) row of \(B_{2,2}\) for all \(j \in \{1, \ldots, n\}\) so that we are guaranteed to have at some point \(j = k\). In that case, the entries cancel out. In every other case where \(j \neq k\), we add a value of 0, not affecting the matrix. Refer to the \(i^{th}\) row of matrix \(B_{a,b}\) as \(R_{a,b}^i\). For all \(i, j \in \{1, \ldots, n\}\) where \(i \neq j\) we carry out the
following elementary row operation:

\[ R_i^{3,2} + (-1)^{i+j+1} R_j^{2,2}. \]

With this, we obtain

\[
(\mathcal{L}_n(K^c) - xI) = \begin{bmatrix}
\theta & A_2^T & 0 \\
0 & -I & -\Gamma(-\theta) \\
A_2' & 0 & (*)
\end{bmatrix}.
\]

The next set of row operations is

\[ R_1^{1,2} + (-1)^j R_j^{2,2} \]

for all \( j \in \{1, \ldots, n\} \). Then we are left with

\[
(\mathcal{L}_n(K^c) - xI) = \begin{bmatrix}
\theta & 0 & C \\
0 & -I & -\Gamma(-\theta) \\
A_2' & 0 & (*)
\end{bmatrix},
\]

where \( C \) is a row of size \( n \) with \((-1)^i(\theta - n + 1)\) as its \( i^{\text{th}} \) entry. Our final step of row reduction is to eliminate \( B_{3,1} = A_2' \) with

\[
R_i^{3,1} + \frac{(-1)^{i+1}}{\theta} R_1^{1,1}
\]

for all \( i \in \{1, \ldots, n\} \), which will complete our row reduction of our discrete Laplacian.
Our final matrix has form
\[
\begin{bmatrix}
\theta & 0 & C \\
0 & -I & -\Gamma(-\theta) \\
0 & 0 & (**) 
\end{bmatrix}.
\]

Our last step is to simply compute (**) and compute the determinant. To do this, we must first compute \(-\theta\Gamma(-\theta) - I\), (\*), and (**).

\[
-\theta\Gamma(-\theta) = -\theta \cdot \begin{bmatrix}
-\theta & -1 & 1 & \cdots & \cdots & (-1)^n-1 \\
-1 & -\theta & -1 & 1 & \cdots & (-1)^n-2 \\
1 & -1 & -\theta & \ddots & & \\
0 & 0 & \ddots & \ddots & & \\
0 & 0 & \ddots & \ddots & \ddots & \\
(-1)^n-1 & (-1)^n-2 & \cdots & -1 & -\theta 
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\theta^2 & \theta & -\theta & \cdots & \cdots & (-1)^n\theta \\
\theta^2 & \theta^2 & \theta & -\theta & \cdots & (-1)^{n-1}\theta \\
-\theta & \theta & \theta^2 & \ddots & & \\
\vdots & \theta & \ddots & \ddots & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
(-1)^n\theta & (-1)^{n-1}\theta & \cdots & \cdots & \theta & \theta^2 
\end{bmatrix}
\]

\[
\Rightarrow -\theta\Gamma(-\theta) - I = \begin{bmatrix}
\theta^2 - 1 & \theta & -\theta & \cdots & \cdots & (-1)^n\theta \\
\theta & \theta^2 - 1 & \theta & -\theta & \cdots & (-1)^{n-1}\theta \\
-\theta & \theta & \theta^2 - 1 & \ddots & & \\
\vdots & \theta & \ddots & \ddots & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
(-1)^n\theta & (-1)^{n-1}\theta & \cdots & \cdots & \theta & \theta^2 - 1 
\end{bmatrix}
\]
Now to compute (\(\ast\)), first notice that \(n - 1\) rows from \(B_{2,3}\) are getting added to each row in \(B_{3,3}\). Furthermore, note that for each entry \(a_{j,k}\) of \(B_{2,3} = -\Gamma(-\theta)\) where \(i \neq j\), \(a_{j,k} = (-1)^{j+k+1}\), just as in \(B_{3,2}\). Consider the entries \(b_{i,k}\) of \(B_{3,3}\). Following the row operations

\[
R_i^{3,2} + (-1)^{i+j+1} R_j^{2,2},
\]

we can see that each entry \(b_{k,j}\) is summed \(n - 1\) times with \((-1)^{i+j+1} a_{j,k}\). If \(i \neq k\), then we have

\[
b_{i,k} + (n - 2)(-1)^{i+j+1}(-1)^{j+k+1} + (-1)^{i+j}(-\theta) = (-1)^{i+k+1}\theta - (n - 2)(-1)^{2j+i+k+2} + (-1)^{i+j+1}\theta.
\]

But notice that when we sum \((-1)^{i+j+1}\theta\) to \(b_{i,k}\), \(j\) must be the same as \(k\). So then we have

\[
(-1)^{i+k+1}\theta - (n - 2)(-1)^{i+k+1} + (-1)^{i+k+1}\theta = (-1)^{i+k+1}(2\theta - (n - 2)).
\]

Thus, each entry \(b_{i,k}\) in \((\ast)\) where \(i \neq k\) is exactly \((-1)^{i+k+1}(2\theta - (n - 2))\). Otherwise, if \(i = k\), we never have the case of adding entry \(a_{j,k}\) where \(j = k\). So instead we have

\[
b_{i,i} = \theta^2 - 1 + (-1)^{i+k+1}(-1)^{i+k+1}(n - 1) \Rightarrow b_{i,i} = \theta^2 - 1 + (n - 1).
\]

Thus,

\[
(\ast) = \begin{bmatrix}
\theta^2 + n - 2 & (2\theta - (n - 2)) & \cdots & (-1)^{n-1}(2\theta - (n - 2)) & (-1)^n(2\theta - (n - 2)) \\
(2\theta - (n - 2)) & \theta^2 + n - 2 & \cdots & (-1)^{n-1}(2\theta - (n - 2)) & (-1)^n(2\theta - (n - 2)) \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
(-1)^{n-1}(2\theta - (n - 2)) & (2\theta - (n - 2)) & \theta^2 + n - 2 & (2\theta - (n - 2)) & \theta^2 + n - 2 \\
(-1)^n(2\theta - (n - 2)) & (2\theta - (n - 2)) & (-1)^{n-1}(2\theta - (n - 2)) & (2\theta - (n - 2)) & \theta^2 + n - 2
\end{bmatrix}
\]

To compute (\(\ast\*\)), we note that the row operations

\[
R_i^{3,1} + \frac{(-1)^{i+1}}{\theta} R_1^{1,1}
\]

imply that the \(b_{i,j}\) entry of (\(\ast\*)) is summed with \((\frac{(-1)^{i+j+1} \theta^n}{\theta - n + 1})\). So, for all \(i \neq j\), we
have

\[ b_{i,j} = (-1)^{i+j+1}(2\theta - (n - 2)) + \frac{(-1)^{i+j+1}(\theta - n + 1)}{\theta} \]

\[ = (-1)^{i+j+1}\left(\frac{\theta(2\theta - (n - 2))(\theta - n + 1)}{\theta}\right) \]

\[ = (-1)^{i+j+1}\left(2\theta^2 - \theta(n - 3) - n + 1\right) \]

\[ = (-1)^{i+j+1}\left(2\theta - n + 3 - \frac{(n - 1)}{\theta}\right). \]

Furthermore, when \( i = j \), we obtain

\[ b_{i,i} = \theta^2 + n - 2 + \frac{(-1)^{i+i+1}(-\theta + n - 11)}{\theta} \]

\[ = \theta^2 + n - 2 + \frac{-\theta + n - 1}{\theta} \]

\[ = \theta^2 + (n - 2) - 1 + \frac{(n - 1)}{\theta} \]

\[ = \theta^2 + (n - 3) + \frac{(n - 1)}{\theta}. \]

Let \( B_n = (n - 3) + \frac{(n-1)}{\theta} \). Then, our final matrix has a \( B_{3,3} \) entry of

\[
\begin{bmatrix}
\theta^2 + B_n & 2\theta - B_n & \cdots & (-1)^{n-1}(2\theta - B_n) & (-1)^n(2\theta - B_n) \\
2\theta - B_n & \theta^2 + B_n & \cdots & (-1)^{n-1}(2\theta - B_n) \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{n-1}(2\theta - B_n) & (-1)^{n-1}(2\theta - B_n) & \cdots & 2\theta - B_n & \theta^2 + B_n
\end{bmatrix}.
\]

We can now divide each row by \( 2\theta - B_n \) to get this into a familiar form:

\[
B_{3,3} = \begin{bmatrix}
\frac{\theta^2 + B_n}{2\theta - B_n} & 1 & \cdots & (-1)^{n-1} & (-1)^n \\
1 & \frac{\theta^2 + B_n}{2\theta - B_n} & \cdots & (-1)^{n-1} & (-1)^n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^{n-1} & \frac{\theta^2 + B_n}{2\theta - B_n} & \cdots & 1 & \frac{\theta^2 + B_n}{2\theta - B_n}
\end{bmatrix}.
\]
Thus, $B_{3,3}$ is exactly $\Gamma\left(\frac{\theta^2 + B_n}{2\theta - B_n}\right)$. Recall from our review of determinants that if $x \in \mathbb{R}$, $A$ is an $n \times n$ matrix and $B = \frac{A}{x}$, then

$$det(A) = x^n(det(B)).$$

Furthermore, recall that the determinant of an upper triangular block matrix is the product of the determinants along the diagonal. If we look back at the form of our final matrix, we can see that the entry $B_{2,2}$ has no bearing on our determinant since it is just $(-I)^n$. Also, in Lemma 3.1, we computed the determinant of $\Gamma(\alpha)$ to be $(\alpha + 1)^n(\alpha - n + 1)$. Using these facts, we compute our determinant of $(\mathcal{L}_n(\mathcal{K'}) - xI)$ to be

$$\theta\left(\frac{2\theta^2 - \theta(n-3) - n + 1}{\theta}\right)^n\left(\frac{\theta^3 + \theta(n-3) + (n-1)}{2\theta^2 - \theta(n-3) - (n-1) + 1}\right)^{n-1}\left(\frac{\theta^3 + \theta(n-3) + (n-1)}{2\theta^2 - \theta(n-3) - (n-1)} - n + 1\right).$$

After a bit of algebra, the expression can be reduced to

$$|\mathcal{L}_n(\mathcal{K'}) - xI| = (\theta^2 + 2\theta)^{n-1}(\theta^3 + (2\theta^2 - n)(n-1) + n\theta(n-3)).$$

If we substitute $(n + 1) - x$ in for $\theta$, after a generous amount of simplifying the expression, we get

$$|\mathcal{L}_n(\mathcal{K'}) - xI| = (n + 1 - x)^{n-1}(n + 3 - x)^{n-1}(n + 3 - (3n + 7)x + (n + 5)x^2 - x^3).$$

We have shown in Lemma 3.3 that $(n + 3 - (3n + 7)x + (n + 5)x^2 - x^3)$ has only irrational roots. Thus, our proof is done.

The previous theorem only gives us insight into the eigenvalues of the top dimensional Laplacian of a very specific simplicial complex. Consider now an $n$-simplex $\sigma$. We will show how irrationality occurs if we subdivide any lower-dimensional face of $\sigma$ twice.

**Lemma 4.1.** Let $\mathcal{K}$ be an $n$-dimensional abstract simplicial complex and let $\omega$ be a
vertex not in the vertex set of $K$. Then

$$L_{n+1}(\omega \ast K) = L_n(K) + I_{m \times m},$$

where $m$ is the size of $L_n(K)$.

**Remark 4.2.2.** For any given simplicial complex, we are not guaranteed that the facets of $K$ are all of the same dimension. In the following proof, we differ between *facets* and $n$-dimensional *faces*. While any $n$-dimensional face is indeed a facet, the converse is not always true.

**Proof.** Let $K$ be an $n$-dimensional abstract simplicial complex with vertex set $V = \{0, \ldots, p\}$ and $n$-dimensional faces $\{\tau_0, \ldots, \tau_m\}$. Let $V(\tau_i)$ be the vertex set of the facet $\tau_i$. If $\omega \notin V$, then $w \ast K$ has vertex set $V' = \{0, \ldots, n, \omega\}$ and a set of facets $\{\tau'_0, \ldots, \tau'_m\}$ where each $\tau'_i$ is spanned by the set of vertices $V(\tau'_i) = V(\tau_i) \cup \{\omega\}$. Suppose $V(\tau_i) = \{\nu^n_0, \ldots, \nu^n_i\}$. Then, the $i^{th}$ basis element of $C_{n-1}(w \ast K)$ is $[\nu^n_0, \ldots, \nu^n_i]$. Furthermore, the basis elements of $C_n(w \ast K)$ are all faces of size $n$ that are faces of at least one of $\{\tau'_0, \ldots, \tau'_m\}$. Define a collection of maps

$$\Theta^d_\omega : C_d(K) \to C_{d+1}(\omega \ast K)$$

by

$$[k_0, \ldots, k_d] \mapsto [k_0, \ldots, k_d, \omega].$$

By construction, this is a collection of homomorphisms. We define $\Theta^d_\omega$ acting on a chain in $C_{d+1}(\omega \ast K)$ by appending the vertex $\omega$ to each face in the chain. So the basis elements of $C_{n+1}(\omega \ast K)$ are precisely constructed by $\Theta^n_\omega(\tau_i)$ for all $i \in \{0, \ldots m\}$. Therefore, $\Theta^n_\omega$ is actually an isomorphism in this case, since we can see that it is both injective and surjective. So there is an inverse map

$$(\Theta^n_\omega)^{-1} : C_{n+1}(\omega \ast K) \to C_n(K).$$
Now, we can construct the following diagram:

\[
\begin{array}{ccc}
\partial_n : C_n(K) & \longrightarrow & C_{n-1}(K) \\
\Theta^n \downarrow & & \downarrow \Theta^n_{\omega} \\
\partial_{n+1} : C_{n+1}(\omega \ast K) & \longrightarrow & C_n(\omega \ast K).
\end{array}
\]

This diagram shows that the matrix representing the operator \( \partial_{n+1} : C_{n+1}(\omega \ast K) \rightarrow C_n(\omega \ast K) \) where we restrict our basis elements of \( C_n(\omega \ast K) \) to those constructed by the map \( \omega_{\omega}^{n-1} \) is exactly the same as \( \partial_n : C_n(K) \rightarrow C_{n-1}(K) \). Furthermore, notice that the coefficient of \( \tau_i = \left[ \nu^i_0, \ldots, \nu^i_n \right] \) in the equation

\[
\partial_{n+1}(\tau_i) = \sum_{j=0}^{n+1} (-1)^j [\nu^i_0, \ldots, \nu^i_j, \ldots, \nu^i_n, \omega]
\]

is \((-1)^{n+1}\). Thus, if we be sure to order the basis elements of \( \partial_{n+1}(\omega \ast K) \) as first the ones generated by \( \Theta_{\omega}^n(\tau_i) \) for all \( i \in \{0, \ldots, m\} \) and then the basis elements \( \tau_i \), we obtain

\[
\partial_{n+1}(\omega \ast K) = \left[ \begin{array}{c}
\partial_n(K) \\
(-I)^{n+1}_{m \times m}
\end{array} \right] \Rightarrow \mathcal{L}_{n+1}(\omega \ast K) = \left[ \partial_n^T(K) \partial_n(K) + I_{m \times m} \right].
\]

Thus, \( \mathcal{L}_{n+1}(\omega \ast K) = \mathcal{L}_n(K) + I_{m \times m} \) and our proof is done. \( \square \)

**Lemma 4.2.** Let \( K \) be an abstract simplicial complex of dimension \( n \) and let \( \omega \) be a vertex not in the vertex set of \( K \). If \( \mathcal{L}_n(K) \) has irrational eigenvalues, then \( \mathcal{L}_{n+1}(\omega \ast K) \) has irrational eigenvalues as well.

**Proof.** Let \( \chi_A(x) \) denote characteristic polynomial of \( A \), where \( A \) is a square matrix. Thus, \( \chi_{\mathcal{L}_{n+1}(\omega \ast K)}(x) = |\mathcal{L}_{n+1}(\omega \ast K) - xI| \). By Lemma 4.1, we know that \( \mathcal{L}_{n+1}(\omega \ast K) = \mathcal{L}_n(K) + I_{m \times m} \). Then, we have

\[
\chi_{\mathcal{L}_{n+1}(\omega \ast K)}(x) = |\mathcal{L}_{n+1}(\omega \ast K) - xI| = |\mathcal{L}_n(K) + I - xI| = |\mathcal{L}_n(K) - (x - 1)I|
\]

So then the \( \chi_{\mathcal{L}_{n+1}(\omega \ast K)}(x) = \chi_{\mathcal{L}_n(K)}(x - 1) \). Then, since we assumed \( \chi_{\mathcal{L}_n(K)}(x) \) has
irrational roots, then the polynomial $\mathcal{X}_{L_1(K)}(x-1)$ must have irrational roots as well. Precisely, if $\alpha$ is a root of $\mathcal{X}_{L_1(K)}(x)$, then $\alpha + 1$ is a root of $\mathcal{X}_{L_1(K)}(x-1)$. Thus, $\mathcal{X}_{L_{n+1}\omega K}(x)$ must have irrational roots.


