Spring 5-2019

Weyl Groups and the Nil-Hecke Algebra

Arta Holaj

Weyl Groups and the Nil-Hecke Algebra

An honors thesis presented to the
Department of Mathematics and Statistics
University at Albany, State University of New York
in partial fulfillment of the requirements
for graduation with Honors in Mathematics and Statistics
and
graduation from The Honors College

Arta Holaj

Research Advisor: Changlong Zhong, Ph.D.

May 2019

Abstract

We begin this paper with a short survey on finite reflection groups. First we establish what a reflection in Euclidean space is. Then we introduce a root system, which is then partitioned into two sets: one of positive roots and one with negative roots. This articulates our understanding of groups generated by simple reflections. Furthermore, we develop our insight to Weyl groups and crystallographic groups before exploring crystallographic root systems. The section section of this paper examines the twisted group algebra along with the Demazure element X_i and the Demazure-Lusztig element T_i. Lastly, the third section of this paper computes such X_i and T_i in the case of S_3 where S_3 is the symmetric group S_n, $n=3$.

Acknowledgements

I would like to express my gratitude for my advisor, Professor Changlong Zhong, for his patience and expertise throughout these last two years of study. I would like to thank him for providing multiple opportunities to study abstract algebra in much more detail than the classroom alone can. The independent studies have made clear those ideas from AMAT 327 (that I found most fascinating) are even more nuanced than I originally thought. I also want to thank him for constantly encouraging my research and reminding me that even the most tedious of calculations are still worthwhile. After all, many profound mathematical ideas were first observed by repetitive, routine computations.

I would also like to thank my family for their continuous support of my education. Words cannot express how grateful I am to my parents for all the sacrifices they have made on my behalf. I would also like to express my gratitude to my sister for constantly listening to me discuss my thesis (even though she did not understand a single idea). Thank you to all my friends for being a great source of comfort whenever I felt overwhelmed throughout this process. I am deeply indebted to you all.

Table of Contents

Chapter 1

Introduction

We recall related concepts of reflections, root systems, Weyl groups, and Crystallographic root systems for the convenience for the readers. We need these concepts in order to define the twisted group algebra, the Demazure elements, and the Demazure-Lusztig elements. Nothing is new in this section. We also follow the structure of [3]. This provides contextual information in understanding future commutations outlined in later sections.

1.1 Reflections

We begin this introduction by recalling finite groups generated by reflections in (real) Euclidean space. Before we define what a reflection in Euclidean space is, we wish to define the following:

Definition 1.1.1. *[6] A symmetric bilinear form on a vector space V is a bilinear function $Q : V \times V \to \mathbb{R}$.*

Definition 1.1.2. *[6] A symmetric bilinear form with $Q(v, v) > 0$ for all $v \neq 0 \in V$ is called positive definite.*

Definition 1.1.3. *[2] A hyperplane is a subspace whose dimension is one less than that of its ambient space.*

Now we can define what it means to be a reflection in Euclidean space.

Definition 1.1.4. *[3] A reflection in (real) Euclidean space V endowed with a positive definite symmetric bilinear form (λ, μ) is a linear operator s on*

V which sends some nonzero vector α to its negative while fixing point-wise the hyperplane H_α orthogonal to α. We may denote $s = s_\alpha$. The formula of reflection is given by: $s_\alpha\lambda = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \lambda \in V$.

One would expect s_α to preserve the length of vectors and the angles between them. Thus, we wish to show that such reflections are orthogonal transformations (i.e. $(s_\alpha\lambda, s_\alpha\mu) = (\lambda, \mu), \lambda, \mu \in V$) as seen in the following verification:

$$(s_\alpha\lambda, s_\alpha\mu) = (\lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \mu - \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha)$$

$$= (\lambda,\mu) - (\lambda, \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha) - (\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \mu) + (\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha) = (\lambda,\mu)$$

We also claim that $s_\alpha^2 = 1$. Suppose $v \in V$. Reflecting v across a hyperplane before reflecting it across the same hyperplane should yield the same point. Therefore $|s_\alpha| = 2$ in the group of all orthogonal transformations of V, which is denoted by $O(V)$.

1.2 Roots

In this section, we establish how selecting a well-chosen set of vectors, called roots, orthogonal to reflecting hyperplanes simplify our understanding of reflection groups.

Denote W as a finite reflection group acting on Euclidean space V. We wish to more thoroughly investigate how W acts on V. For each reflection $s_\alpha \in W$ determines a reflecting hyperplane H_α. Thus, there must exist some line segment $L_\alpha \perp H_\alpha$ connecting the original vector to its reflection.

Proposition 1.2.1. *[3] If $t \in O(V)$ and α is any nonzero vector in V, then $ts_\alpha t^{-1} = s_{t\alpha}$. In particular, if $w \in W$, then $s_{w\alpha} \in W$ when $s_\alpha \in W$.*

Proof. The proof comes from [3]. We claim that $ts_\alpha t^{-1}$ sends $t\alpha$ to its negative (this follows from the fact that $s_\alpha(\alpha) = -\alpha$). Now, we wish to show that $ts_\alpha t^{-1}$ fixes $H_{t\alpha}$ point-wise. Note that $\lambda \in H_\alpha \iff t\lambda \in H_{t\alpha}$ because $(\lambda,\alpha) = (t\lambda, t\alpha)$ since it is an orthogonal transformation. In turn, $(ts_\alpha t^{-1})(t\lambda) = ts_\alpha\lambda = t\lambda$ whenever $\lambda \in H_\alpha$. $\square$

Therefore, clearly W permutes the lines L_α corresponding to $s_\alpha \in W$ such that $W(L_\alpha) = L_{w\alpha}$. All such L_α are determined by the reflections so W determines these lines.

Now we wish to further define W. Let Φ be a finite set of vectors of V. Let W be the group generated by reflections $s_\alpha, \alpha \in \Phi$. We wish to call Φ to be the root system with associated reflection group W in which $\Phi = \{\text{nonzero vectors in } V\}$ satisfying:

$$\text{I } \Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}.$$

$$\text{II } s_\alpha \Phi = \Phi, \forall \alpha \in \Phi.$$

Elements of α of V are called roots. Any finite reflection group can be determined with a root system $\Phi \subset V$ subject to the two axioms listed above. The choice of Φ is determined case wise, as different reflection groups require different root systems.

Definition 1.2.2. *[5] Let group G act on a set X. The group action is said to be faithful if $gx = x \ \forall x \in X \Rightarrow g = 1$.*

Since W is a group, it contains the identity element $w = 1$, which would fix all elements of the selected root system. The action of S_n on Φ is also faithful, which implies that W is finitely generated.

1.3 Positive and Simple Systems

Let Φ be a fixed root system in V so that $W =< s_\alpha >, \forall s_\alpha(\alpha \in \Phi)$. Since our root system is designed from the geometry of our finite reflection group, we may experience situations in which Φ is much larger than the dimension of the vector space V. Thus, we wish to partition Φ in such a way to reduce the number of elements of our selected root system. Therefore, we introduce the notion of simple systems, which is a linear independent subset of Φ. Before we formally define simple systems, we which to define the total ordering of V.

Definition 1.3.1. *[3] A total ordering of a real-vector space V is a transitive relation on V, denoted $<$, satisfying the following axioms:*

1. *For each pair $\lambda, \mu \in V$, only one of $\lambda < \mu$, $\lambda > \mu$, or $\lambda = \mu$ holds.*

2. *$\forall \lambda, \mu, v \in V$, if $\mu < v$, then $\lambda + \mu < \lambda + v$.*

3. *If $\mu < v$ and $c \neq 0 \in \mathbb{R}$, then $c\mu < cv$ if $c > 0$ and $cv < c\mu$ if $c < 0$.*

In order to design such a partition, we wish to define that $\lambda \in V$ is positive if $0 < \lambda$. Let $x_1, ..., x_n$ be a basis of V with all x_i positive and let a_i, b_i be coefficients for $i = 1, 2, ..., n$. This basis is then organized alphabetically where where $\sum a_i x_i < \sum b_i x_i$ means that $a_k < b_k$ if k is the least index i for which $a_i \neq b_i$. Ergo, it becomes apparent that V is now totally ordered. Consequently, all elements of this basis is positive for all i.

According to I from section 1.2, all roots come in pairs $\{\alpha, -\alpha\}$. The total ordering of V will now partition Φ into two subsets: one of positive roots and one of negative negative. Therefore, call $\Pi \subset \Phi$ a positive system if it consists of all those roots which are positive relative to some total ordering of V. On the contrary, $-\Pi \subset \Phi$ denotes a negative system. Clearly Φ is the disjoint union of $\Pi, -\Pi$. We now will denote $\alpha > 0$ instead of $\alpha \in \Pi$ because all elements of Π are positive. We now proceed to the following definition:

Definition 1.3.2. *[3] Let $\Delta \subset \Phi$ be a simple system (and its elements simple roots) if Δ is a basis for the vector space for the $\mathbb{R}$-span if Φ in V and if each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign (i.e. all nonnegative or all non-positive).*

The subsequent theorem proves the existence of such simple systems.

Theorem 1.3.3. *[3] The following statements hold:*

1. *If Δ is a simple system in Φ, then there exists a unique positive system containing Δ.*

2. *Every positive system $\Pi \in \Phi$ contains a unique simple system.*
   In particular, simple systems exist.

Proof. Proof will be skipped. Reference section 1.3 of [3]. □

1.4 Conjugacy of Positive and Simple Systems

We wish to explore the relationship between different systems. Consider the special case of when $w = s_\alpha, \alpha \in \Delta$.

Proposition 1.4.1. *[3] Let Δ be a simple system contained in the positive system Π. If $\alpha \in \Delta$, then $s_\alpha(\Pi \backslash \{\alpha\}) = \Pi \backslash \{\alpha\}$.*

Proof. The proof comes from [3]. Let $\beta \in \Pi$, $\beta \neq \alpha$, and write $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$ with all $c_\gamma \geq 0$. But the only multiples of $\alpha \in \Phi$ are $\pm\alpha$, some $c_\gamma > 0$ for $\gamma \neq \alpha$. Now apply s_α to both sides: $s_\alpha \beta = \beta - c\alpha$ is a linear combination of Δ involving γ with some coefficient c_γ. Since all coefficients in such an expression have like sign, $s_\alpha \beta$ must be positive. It cannot be α, for then we reach the following contradiction: $\beta = s_\alpha s_\alpha \beta = s_\alpha \alpha = -\alpha \notin \Pi$. Therefore, s_α maps $\Pi \backslash \{\alpha\}$ to itself (injectively) hence onto itself. □

This result is useful when we wish to check if an arbitrary root is a simple root. In our positive system, it is apparent that $\alpha > 0$. Therefore, s_α maps α to $-\alpha$ (i.e. the negative root of α).

Theorem 1.4.2. *[3] Any two positive (respectively simple) systems in Φ are conjugate under W.*

Proof. The proof comes from [3]. Let Π, Π' be positive systems such that each contains precisely half of all the roots. We proceed inductively on $r = Card(\Pi \cap \Pi')$. If $r = 0$, then $\Pi = \Pi'$. and we are done. If $r > 0$, then clearly the simple system Δ in Π cannot entirely be contained in Π'. Choose $\alpha \in \Delta$ with $\alpha \in \Pi'$. By Proposition 1.4.1, we have that $Card(s_\alpha \Pi \cap \Pi') = r - 1$. Induction, applied to the positive systems, $s_\alpha \Pi, \Pi'$ furnishes an element $w \in W$ for which $w(s_\alpha \Pi) = \Pi'$. □

1.5 Generation by Simple Reflections

We wish to explore how our construction of positive and simple systems describe how reflection groups are generated. Let Δ be a fixed simple system with corresponding positive system $\Pi \in \Phi$. The choice of Δ is unimportant as all simple systems are conjugate to each other, so we can alter one simple system to another. If W is a finite reflection group, one would expect this group to be generated by reflections. We proceed to the following definition before arriving to the conclusion that W is generated by simple reflections.

Definition 1.5.1. *[3] If $\beta \in \Phi$, we uniquely write $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ and call $\sum c_\alpha$ the height of β (relative to Δ). This is denoted by $ht(\beta)$.*

For instance, we have that $ht(\beta) = 1$ if $\beta \in \Delta$.

Theorem 1.5.2. *[3] For a fixed simple system Δ, W is generated by the reflections s_α for $\alpha \in \Delta$.*

Proof. The proof comes from [3]. Denote W' to be the subgroup of W generated by Δ. We proceed in steps to show $W' = W$.

1. If $\beta \in \Phi$, consider $W'\beta \cap \Pi$. This is a nonempty set of positive roots (containing at least β) and we can choose from it an element γ of the smallest possible height. We claim $\gamma \in \Delta$. Write $\gamma = \sum_{\alpha\in\Delta}$ and note $0 < (\gamma, \gamma) = \sum c_\alpha(\gamma, \alpha)$ forcing $0 < (\gamma, \alpha)$ for some $\alpha \in \Delta$. If $\gamma = \alpha$, we are satisfied. Otherwise, consider the root $s_\alpha\gamma$ which is positive according to Proposition 1.4.1. Since $s_\alpha\gamma$ is obtained from γ by subtracting a positive multiple of α, we have $ht(s_\alpha\gamma) < ht(\gamma)$. But $s_\alpha\gamma \in W'\beta$ because $s_\alpha \in W'$, which contradicts our original choice of γ. Therefore, $\gamma = \alpha$ must be simple.

2. We wish to show that $W'\Delta = \Phi$. We showed that the W'-orbit of any positive root β meets Δ so that $\Pi \subset W'\Delta$. On the other-hand, if β is negative, then $-\beta \in \Pi$ is conjugate by some $w \in W$ and $\alpha \in \Delta$. So we have that $-\beta = w\alpha \Rightarrow \beta = (ws_\alpha)\alpha$ with some $ws_\alpha \in W'$. Therefore, $-\Pi \subset W'\Delta$.

3. Finally, take any generator s_β of W. By (2), we can write $\beta = w\alpha$ for some $w \in W'$ and $\alpha \in \Delta$. Then Proposition 1.2.1 implies that $s_\beta = ws_\alpha w^{-1} \in W \Rightarrow W = W'$.

□

Remark 1.5.3. *[3] Relative to some fixed positive system, any arbitrary root can attain the status of a simple root.*

Corollary 1.5.4. *[3] Given a simple system Δ, for every $\beta \in \Phi$, $\exists w \in W$ such that $w\beta \in \Delta$.*

1.6 The Length Function

We wish to explore how an arbitrary $w \in W$ can be written as a product of simple reflections, say $w = s_1 ... s_r$ with $s_i = s_{\alpha i}, \alpha_i \in \Delta$.

Definition 1.6.1. *[3] The length $l(w)$ of w, relative to Δ, is the smallest r for such an expression exists and call the expression reduced. By convention, $l(1) = 0$.*

Definition 1.6.2. *[3] Let $n(w) := Card(\Pi \cap w^{-1}(-\Pi))$, which is the number of positive roots sent to negative roots by w.*

Lemma 1.6.3. *[3] Let $\alpha \in \Delta, w \in W$. Then we have the following:*

1. $0 < w\alpha \Rightarrow n(ws_\alpha) = n(w) + 1$.
2. $w\alpha < 0 \Rightarrow n(ws_\alpha) = n(w) - 1$.
3. $0 < w^{-1}\alpha \Rightarrow n(s_\alpha w) = n(w) + 1$.
4. $w^{-1}\alpha < 0 \Rightarrow n(s_\alpha w) = n(w) - 1$.

Proof. The proof comes from [3]. Set $\Pi(W) := \Pi \cap W^{-1}(-\Pi)$ so that $n(w) = Card\Pi(W)$. If $w\alpha > 0$, observe that $\Pi(ws_\alpha)$ is the disjoint union of $s_\alpha\Pi(w)$ and $\{\alpha\}$ by Proposition 1.7. If $w\alpha < 0$, the same result implies that $s_\alpha\Pi(ws_\alpha) = \Pi(W)\backslash\{\alpha\}$ where α does lie in $\Pi(w)$. This establishes (1) and (2). For (3) and (4), replace w by w^{-1} and use the fact $n(w^{-1}s_\alpha) = (s_\alpha w)$. $\square$

Proposition 1.6.4. *[3] If $w \in W$ is written in any way as a product of simple reflections, say $w = s_1...s_r$, then $n(w) \leq r$. In particular, $n(w) \leq l(w)$.*

Proof. The proof comes from [3]. As we build up the expression for w in r steps, the value of the n function (initially 0) can increase by at most one at each step, according to Lemma 1.11. $\square$

1.7 The Deletion and Exchange Conditions

This section introduces both the deletion and exchange conditions, which are useful in simplifying elements $w \in W$ when each w is a product of simple reflections. The following result is vital in simplifying products of simple reflections if it is not irreducible.

Theorem 1.7.1. *[3] Fix a simple system Δ and let $w = s_1...s_r$ be any expression of $w \in W$ as a product of simple reflections (say $s_i = s_{\alpha i}$ with repetitions allowed). Suppose $n(w) < r$. Then there exists indices $1 \leq i < j \leq r$ satisfying the following:nm*

1. $\alpha_i = (s_{i+1}...s_{j-1})\alpha_j$.

2. $s_{i+1}s_{i+2}...s_j = s_i s_{i+1}...s_{j-1}$.

3. $w = s_1...\hat{s_i}...\hat{s_j}...s_r$ *(where the hat denotes omission).*

Proof. Proof will be skipped. Reference section 1.7 of [3]. □

We call condition (3) of the pervious theorem to be the *Deletion Condition* in which w will eventually yield a reduced expression due to the consecutive omissions of factors (assuming w is not already a reduced expression). For instance, if $w = s_1s_2s_2s_3s_4s_4$, then w can be reduced to $w = s_1s_3$ as $(s_i)^2 = 1$. As a formal consequence of the deletion condition, we have the following theorem, called the *Exchange Condition*:

Theorem 1.7.2. *[3] Let $w = s_1...s_r$ where s_i are simple reflections (note that w may not be a reduced expression). If $l(ws) < l(w)$ for some simple reflection $s = s_\alpha$, then there exists an index i for which $ws = s_1...\hat{s_1}...s_r$ (therefore $w = s_1...\hat{s_i}...s_rs$ with a factor s exchanged for a factor of s_i). In particular, w has a reduced expression ending in $s \iff l(ws) < l(w)$.*

Proof. Proof will be skipped. Reference section 1.7 of [3]. □

1.8 Crystallographic Groups

In the analysis of finite reflection groups, it becomes apparent that many of these groups are in fact crystallographic. We proceed to define the following types of lattices in order for us to define crystallographic subgroups. We proceed to define coroots:

Definition 1.8.1. *[3] Let $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. The set $\Phi^\vee$ of all coroots $\alpha^\vee, \alpha \in \Phi$ is a root system in V with simple system $\Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\}$. This is also called the inverse or dual root system.*

Now, let L denote the lattice (a free abelian group) generated by the set $\Phi^\vee$. It follows:

Definition 1.8.2. *[3] The $\mathbb{Z}$ - span $L(\Phi)$ of Φ in V is called the root lattice. It is a lattice in the subspace of V spanned by Φ, which we can usually assume to be V itself.*

Similarly, we can define the coroot lattice $L(\Phi^\vee)$. In other words, the coroot lattice is generated by the set of coroots

Definition 1.8.3. *[3] Let the weight lattice be defined as:*

$$\hat{L}(\Phi) := \{\lambda \in V | (\lambda, \alpha) \in \mathbb{Z}, \forall \alpha \in \Phi\}.$$

Definition 1.8.4. *[3] Let the coweight lattice be defined as:*

$$\hat{L}(\Phi^v) := \{\lambda \in \Phi | (\lambda, \alpha) \in \mathbb{Z}, \forall \alpha \in \Phi\}.$$

Definition 1.8.5. *[3] A subgroup G of $GL(V)$ is said to be crystallographic if it stabilizes a lattice L in V (the $\mathbb{Z}$-span of a basis of V): $gL \subset L \forall g \in G$ (since G is a group, it is automatic that $gL = L$).*

The following theorem determines the conditions for a finite reflection group to be crystallographic.

Proposition 1.8.6. *[3] If W is crystallographic, then each integer $m(\alpha, \beta) \in \{2, 3, 4, 6\}$.*

Proof. The proof comes from [3]. If $\alpha \neq \beta$, we know that $s_\alpha s_\beta \neq 1$ acts on the plane spanned by α and β as a rotation through the angle $\theta := \frac{2\pi}{m(\alpha,\beta)}$ while fixing the orthogonal complement point-wise. Thus its trace relative to a compatible choice of basis for V is $(n-2) + 2\cos(\theta)$ where $n = dimV$. It follows that $cos\theta$ must be a half-integer, while $0 < \theta \leq \pi$. The only possibilities are $cos\theta = -1, \frac{-1}{2}, 0, \frac{1}{2}$ corresponding to the cases $m(\alpha, \beta) = 2, 3, 4, 6$. □

1.9 Crystallographic Root Systems and Weyl Groups

Now that crystallographic groups have been established, we wish to examine crystallographic root systems before defining what it means for a group to be a Weyl group.

Definition 1.9.1. *[3] A root system Φ is crystallographic if it satisfies the equation: $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}, \forall \alpha, \beta \in \Phi$. These are called Cartan integers*

Definition 1.9.2. *[3] The group W generated by all reflections s_α for $\alpha \in \Phi$ is the Weyl group of Φ.*

Consequently, Weyl groups are synonymous to the reflection group of *crystallographic root systems* of various types, which are examined more closely in section 1.10. For our convenience, let us suppose that Φ or W is irreducible so each root can be one of two prescribed lengths. Since the length of one root is longer than the other, we will denote these root lengths by *long* and *short*. According to [3], if there are both long and short roots, then the ratio of the length squared can only be 2 or 3.

Now that we can classify a group as being crystallographic or not, we will now discuss crystallographic groups in more detail. Let $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, then the set $\Phi^\vee = \{\text{all coroots } \alpha^\vee | \alpha \in \Phi\}$ is also a crystallographic root system in V with simple system $\Delta^\vee := \{\alpha^\vee | \alpha \in \Delta\}$. It is called the inverse (or dual) root system. The Weyl group of $\Phi^\vee$ is W with $w\alpha^\vee = w(\alpha)^\vee$. Most of the time $\Phi^\vee \cong \Phi$, however, root types B_n and C_n are actually dual to each other.

We conclude this section by describing how one can construct (crystallographic) root systems of all possible types. According to Humphreys, crystallographic root systems are fashioned in the following steps:

1. In a suitably chosen lattice L in $\mathbb{R}^n$, define:

$$\Phi := \{\text{all vectors having 1 or 2 prescribed lengths}\}$$

2. Verify that the following scalars $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.

3. It follows by 1.9.1 that the vectors with respect to vectors in Φ stabilize L and hence permute Φ as required.

1.10 Types of Reflection Groups

This section outlines finite reflection groups of types A_n, B_n, C_n, and D_n.

Example 1.10.1. *[3] The name of this type is A_n. Let S_n denote the symmetric group, which is a subgroup of $O(n, \mathbb{R})$ of $n \times n$ orthogonal matrices. Let V' be the hyperplane in $\mathbb{R}^{n+1}$ consisting of vectors whose coordinates sum to zero. Let $V = \mathbb{R}^n$ be acted on by permuting the subscripts of the standard basis vectors $x_1, ..., x_n$. Recall that S_n is generated by transpositions. In this type, each transposition (ij) in S_n is a reflection $x_i - x_j$ mapped to $-(x_i - x_j)$ while fixing the orthogonal complement point-wise. The orthogonal complement is represented by the set:*

$$\{\textit{all vectors in } \mathbb{R}^n \textit{ where the i-th and j-th components are equal}\}.$$

Let Φ be the set of vectors of squared length 2 in the intersection of V with standard lattice vectors $a_1x_1 + ... + a_nx_n$ with $a_i \in \mathbb{Z}, \forall i$. Then root system, Φ is composed by the $n(n+1)$ vectors $x_i - x_j$ with $1 \leq i \neq j \leq n+1$. For our simple system, we take Δ to be $\alpha_1 = x_1 - x_2, \alpha_2 = x_2 - x_3, ..., \alpha_n = x_n - x_{n-1}$. The Weyl group W of A_n is a reflection group generated by transpositions $(i, i+1), 1 \leq i \leq n-1$, which is isomorphic to S_n.

Example 1.10.2. *[3] The name of this type is B_n. Let $V = \mathbb{R}^n$ and let S_n act on V as described in example 1.10.1. In this case, we fix all x_j while sending x_i to $-x_i$ for $i \neq j$. These reflections generates a group $G \cong (\mathbb{Z}/2\mathbb{Z})^n$ where $|G| = 2^n$. Additionally, we have that $S_n \cap B_n = \{1\}$ within $O(V)$ and S_n normalizes $(\mathbb{Z}/2\mathbb{Z})^n$.*

Let Φ be the set of vectors of squared length 1 or 2 in the standard lattice with cardinality $2n^2$. Moreover, Φ consists of the $2n$ short roots $\pm x_i$ and the $2n(n-1)$ long roots $\pm x_i \pm x_j$ for $i < j$. For Δ, take $\alpha_1 = x_1 - x_2, ..., \alpha_{n-1} = \alpha_{n-1} - \alpha_n, \alpha_n = x_n$. Therefore, define W to be this semi-direct product $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n = W$ where W is a reflection group of order $2^n n!$ because the order of W is the product of the orders of S_n and $(\mathbb{Z}/2\mathbb{Z})^n$. Furthermore, the $(\mathbb{Z}/2\mathbb{Z})^n$ is normal in W.

Example 1.10.3. *[3] This example is of type C_n. Starting with type B_n, one can define C_n to be its inverse root system. We wish to state that $B_2 \cong C_2$. Type C_n consists of the $2n$ long roots $\pm e_i$ and $2n(n-1)$ short roots $\pm x_i \pm x_j$ for $i < j$. For Δ, take $\alpha_1 = x_1 - x_2, ..., \alpha_{n-1} = \alpha_{n-1} - \alpha_n, \alpha_n = 2x_n$.*

Example 1.10.4. *[3] The name of this type is D_n. Let $V = \mathbb{R}^n$. In this case, we take a subgroup H of index 2 in $(\mathbb{Z}/2\mathbb{Z})^n$, (i.e., the subgroup consisting of even number of sign changes) such that H acts on V. It is obvious that S_n normalizes the subgroup consisting of even number of sign changes that is generated by reflections $x_i + x_j \mapsto -(x_i + x_j), i \neq j$. We then define W as $W = S_n \ltimes H$ where W is also a finite reflection group.*

Let Φ be the set of vectors of squared length 2 in the standard lattice. Thus, Φ consists of the $2n(n-1)$ roots $\pm x_i \pm x_j$ for $1 \leq i < j \leq n$. Furthermore, W is the semi-direct product S_n (permuting the x_i) and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ (acting by an even number of sign changes). The latter is normal in W.

Moreover, there also exist reflection groups of types G_2, F_4, E_6, E_7, and E_8. However, these groups will not be referenced in this paper. See section 2.10 of [3] book for more details.

Chapter 2

Twisted Group Algebra

In this section, we recall the definition of twisted group algebra and Demazure elements, while also recalling proofs of certain properties. Nothing in this section is new.The computations in section 2.2 and section 3 are specific towards reflection groups of type A_{n-1}.

2.1 Twisted group algebra and the Demazure elements

Let $\Lambda = L(\Phi)$ be the root lattice. Then it is a free abelian group with rank n, where $n = |\Pi|$. Let $S = Sym_{\mathbb{Z}}\Lambda$ be the symmetric algebra, which is a polynomial ring with coefficients in $\mathbb{Z}$. Indeed, $S = \mathbb{Z}[\alpha_1, \alpha_2, ..., \alpha_n]$. Let $\{\alpha_i, ..., \alpha_n\}$ be simple roots. Then the action of W on Λ extends to an action on S.

Let Q denote the field of fractions of S. Define Q_W to be the Q-vector space with basis $\{\delta_w : w \in W\}$ in which such Q_W is called the twisted group algebra. Such Q_W is a non-commutative ring. We define the *twisted formal group algebra* to be the R-module $Q_W := Q \otimes_R R[W]$. The product is defined as follows: $(q\delta_w)(q'\delta_{w'}) = qw(q')\delta_{ww'}, \forall w, w' \in W$and $q, q' \in Q$. [1]

Note that for Q-algebras, Q should be central (i.e. Q commutes with all other elements). But this fails in the definition of Q_W, so Q_W is not a Q-algebra.

For $i = 1, ..., n-1$, we can define the Demazure element $X_i \in Q_W$ as $X_i := \frac{1}{\alpha_i} - \frac{1}{\alpha_i}\delta_i$. For any reduced decomposition for $w = s_{i_1} \cdots s_{i_k}$, define $X_w = X_{i_1}...X_{i_k}$. According to the Proposition 4.2 of [4], this does not depend on the choice of reduced sequence. We proceed to the following definition:

Definition 2.1.1. *[1] The nil-Hecke algebra is the sub-algebra of the twisted group algebra Q_W generated by X_i for $i = 1, 2, ..., n$.*

2.2 Type A_n

In the following, we will derive some relations between elements X_i. Let s_{α_i}, for $i = 1, ..., n$, act on S such that s_{α_i} switches x_i and x_{i+1}. For simplicity, we will denote $s_{\alpha_i} = s_i$ as corresponding (simple) reflections for α_i for all i. For example, $s_1(x_1^3x_2x_3^2) = x_2^3x_1x_3^2$. We now denote $\alpha_i = x_i - x_{i+1}$ in which $\alpha_1, ..., \alpha_n$ is a basis of $L(\Phi)$.

Let p be a polynomial and let $i = 1, 2, ..., n-1$. The group W generated by such s_i is precisely the symmetric group S_n. Furthermore, δ_i commutes with rational functions as defined in the following relation: $\delta_i p = s_i(p)\delta_i$. For example, $\delta_2(x_1^3x_2x_3^2) = s_2(x_1^3x_2x_3^2)\delta_2 = (x_1^3x_3x_2^2)\delta_2$. In addition, $\delta_i^2 = 1$. Since any element $w \in W$ can be decomposed into a product of simple reflections, we can write $w = s_{i_1}...s_{i_k}$ for $1 \leq i_j \leq n-1$ for any j. Thus, we can define $\delta_w = \delta_{i_1...i_k}$. For a simpler notation, we can write $\delta_w = \delta_{i_1...i_k}$. For example, $\delta_{123} = \delta_1\delta_2\delta_3$.

Lemma 2.2.1. *For X_i, the following three relations hold:*

1. $X_i^2 = 0$.
2. $X_iX_jX_i = X_jX_iX_j$ *if* $|i-j| = 1$.
3. $X_iX_j = X_jX_i$ *if* $|i-j| \geq 2$.

Proof. The verification of each identity is shown below:

1. $X_i^2 = 0$:

$$X_i^2 = (\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_i}\delta_i)(\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_i}\delta_i)$$
$$= \tfrac{1}{\alpha_i^2} - \tfrac{1}{\alpha_i^2}\delta_i - \tfrac{1}{\alpha_i}\delta_i\tfrac{1}{\alpha_i} + \tfrac{1}{\alpha_i}\delta_i\tfrac{1}{\alpha_i}\delta_i$$
$$= \tfrac{1}{\alpha_i^2} - \tfrac{1}{\alpha_i^2}\delta_i - \tfrac{1}{\alpha_i}s_i(\tfrac{1}{\alpha_i})\delta_i + \tfrac{1}{\alpha_i}s_i(\tfrac{1}{\alpha_i})\delta_i\delta_i$$
$$= \tfrac{1}{\alpha_i^2} - \tfrac{1}{\alpha_i^2}\delta_i - \tfrac{1}{\alpha_i}(\tfrac{1}{-\alpha_i})\delta_i + \tfrac{1}{\alpha_i}(\tfrac{1}{-\alpha_i}) = 0.$$

2. $X_iX_jX_i = X_jX_iX_j$:

We verify this identity by expanding the left hand side and the right hand side separately.

LHS:

$$X_iX_jX_i = (\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_i}\delta_i)(\tfrac{1}{\alpha_j} - \tfrac{1}{\alpha_j}\delta_j)(\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_i}\delta_i)$$
$$= (\tfrac{1}{\alpha_i\alpha_j} - \tfrac{1}{\alpha_i\alpha_j}\delta_j - \tfrac{1}{\alpha_i}\delta_i\tfrac{1}{\alpha_j} + \tfrac{1}{\alpha_i}\delta_i\tfrac{1}{\alpha_j}\delta_j)(\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_i}\delta_i)$$
$$= (\tfrac{1}{\alpha_i\alpha_j} - \tfrac{1}{\alpha_i\alpha_j}\delta_j - \tfrac{1}{\alpha_i}s_i(\tfrac{1}{\alpha_j})\delta_i + \tfrac{1}{\alpha_i}s_i(\tfrac{1}{\alpha_j})\delta_{ij})(\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_i}\delta_i)$$
$$= (\tfrac{1}{\alpha_i\alpha_j} - \tfrac{1}{\alpha_i\alpha_j}\delta_j - \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}\delta_i + \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}\delta_{ij})(\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_i}\delta_i)$$
$$= \tfrac{1}{\alpha_i\alpha_j\alpha_i} - \tfrac{1}{\alpha_i\alpha_j}\delta_j\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}\delta_i\tfrac{1}{\alpha_i} + \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}\delta_{ij}\tfrac{1}{\alpha_i}$$
$$- \tfrac{1}{\alpha_i\alpha_j\alpha_i}\delta_i + \tfrac{1}{\alpha_i\alpha_j}\delta_j\tfrac{1}{\alpha_i}\delta_i + \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}\delta_i\tfrac{1}{\alpha_i}\delta_i - \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}\delta_{ij}\tfrac{1}{\alpha_i}\delta_i$$
$$= \tfrac{1}{\alpha_i\alpha_j\alpha_i} - \tfrac{1}{\alpha_i\alpha_j}s_j(\tfrac{1}{\alpha_i})\delta_j - \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}s_i(\tfrac{1}{\alpha_i})\delta_i + \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}s_{ij}(\tfrac{1}{\alpha_i})\delta_{ij}$$
$$- \tfrac{1}{\alpha_i\alpha_j\alpha_i}\delta_i + \tfrac{1}{\alpha_i\alpha_j}s_j(\tfrac{1}{\alpha_i})\delta_{ji} + \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}s_i(\tfrac{1}{\alpha_i})\delta_i^2 - \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)}s_{ij}(\tfrac{1}{\alpha_i})\delta_{iji}$$
$$= \tfrac{1}{\alpha_i\alpha_j\alpha_i} - \tfrac{1}{\alpha_i\alpha_j(\alpha_i-\alpha_j)}\delta_j - \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)(-\alpha_i)}\delta_i + \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)(\alpha_j)}\delta_{ij}$$
$$- \tfrac{1}{\alpha_i\alpha_j\alpha_i}\delta_i + \tfrac{1}{\alpha_i\alpha_j(\alpha_i-\alpha_j)}\delta_{ji} + \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)(-\alpha_i)} - \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)(\alpha_j)}\delta_{iji}$$
$$= \tfrac{(\alpha_i-\alpha_j)(-\alpha_i)+\alpha_j\alpha_i}{\alpha_i\alpha_j\alpha_i(\alpha_i-\alpha_j)(-\alpha_i)} + \tfrac{\alpha_j\alpha_i+(\alpha_i-\alpha_j)(\alpha_i)}{\alpha_i\alpha_j\alpha_i(\alpha_i-\alpha_j)(-\alpha_i)}\delta_i - \tfrac{1}{\alpha_i\alpha_j(\alpha_i-\alpha_j)}\delta_j + \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)(\alpha_j)}\delta_{ij}$$
$$+ \tfrac{1}{\alpha_i\alpha_j(\alpha_i-\alpha_j)}\delta_{ji} - \tfrac{1}{\alpha_i(\alpha_i-\alpha_j)(\alpha_j)}\delta_{iji}.$$

RHS:
$X_jX_iX_j = (\frac{1}{\alpha_j} - \frac{1}{\alpha_j}\delta_j)(\frac{1}{\alpha_i} - \frac{1}{\alpha_i}\delta_i)(\frac{1}{\alpha_j} - \frac{1}{\alpha_j}\delta_j)$
$= (\frac{1}{\alpha_j\alpha_i} - \frac{1}{\alpha_j\alpha_i}\delta_i - \frac{1}{\alpha_j}\delta_j\frac{1}{\alpha_i} + \frac{1}{\alpha_j}\delta_j\frac{1}{\alpha_i}\delta_i)(\frac{1}{\alpha_j} - \frac{1}{\alpha_j}\delta_j)$
$= (\frac{1}{\alpha_j\alpha_i} - \frac{1}{\alpha_j\alpha_i}\delta_i - \frac{1}{\alpha_j}s_j(\frac{1}{\alpha_i})\delta_j + \frac{1}{\alpha_j}s_j(\frac{1}{\alpha_i})\delta_{ji})(\frac{1}{\alpha_j} - \frac{1}{\alpha_j}\delta_j)$
$= (\frac{1}{\alpha_j\alpha_i} - \frac{1}{\alpha_j\alpha_i}\delta_i - \frac{1}{\alpha_j(\alpha_i-\alpha_j)}\delta_j + \frac{1}{\alpha_j(\alpha_i-\alpha_j)}\delta_{ji})(\frac{1}{\alpha_j} - \frac{1}{\alpha_j}\delta_j)$
$= \frac{1}{\alpha_j\alpha_i\alpha_j} - \frac{1}{\alpha_j\alpha_i}\delta_i\frac{1}{\alpha_j} - \frac{1}{\alpha_j(\alpha_i-\alpha_j)}\delta_j\frac{1}{\alpha_j} + \frac{1}{\alpha_j(\alpha_i-\alpha_j)}\delta_{ji}\frac{1}{\alpha_j}$
$- \frac{1}{\alpha_j\alpha_i\alpha_j}\delta_j + \frac{1}{\alpha_j\alpha_i}\delta_i\frac{1}{\alpha_j}\delta_j + \frac{1}{\alpha_j(\alpha_i-\alpha_j)}\delta_j\frac{1}{\alpha_j}\delta_j - \frac{1}{\alpha_j(\alpha_i-\alpha_j)}\delta_{ji}\frac{1}{\alpha_j}\delta_j$
$= \frac{1}{\alpha_j\alpha_i\alpha_j} - \frac{1}{\alpha_j\alpha_i}s_i(\frac{1}{\alpha_j})\delta_i - \frac{1}{\alpha_j(\alpha_i-\alpha_j)}s_j(\frac{1}{\alpha_j})\delta_j + \frac{1}{\alpha_j(\alpha_i-\alpha_j)}s_{ji}(\frac{1}{\alpha_j})\delta_{ji}$
$- \frac{1}{\alpha_j\alpha_i\alpha_j}\delta_j + \frac{1}{\alpha_j\alpha_i}s_i(\frac{1}{\alpha_j})\delta_{ij} + \frac{1}{\alpha_j(\alpha_i-\alpha_j)}s_j(\frac{1}{\alpha_j})\delta_j^2 - \frac{1}{\alpha_j(\alpha_i-\alpha_j)}s_{ji}(\frac{1}{\alpha_j})\delta_{jij}$
$= \frac{1}{\alpha_j\alpha_i\alpha_j} - \frac{1}{\alpha_j\alpha_i(\alpha_i-\alpha_j)}\delta_i - \frac{1}{\alpha_j(\alpha_i-\alpha_j)(-\alpha_j)}\delta_j + \frac{1}{\alpha_j(\alpha_i-\alpha_j)(\alpha_i)}\delta_{ji}$
$- \frac{1}{\alpha_j\alpha_i\alpha_j}\delta_j + \frac{1}{\alpha_j\alpha_i(\alpha_i-\alpha_j)}\delta_{ij} + \frac{1}{\alpha_j(\alpha_i-\alpha_j)(-\alpha_j)} - \frac{1}{\alpha_j(\alpha_i-\alpha_j)(\alpha_i)}\delta_{jij}$
$= \frac{(\alpha_i-\alpha_j)(-\alpha_j)+\alpha_i\alpha_j}{\alpha_j\alpha_i\alpha_j(\alpha_i-\alpha_j)(-\alpha_j)} + \frac{\alpha_i\alpha_j+(\alpha_i-\alpha_j)(\alpha_i)}{\alpha_j\alpha_i\alpha_j(\alpha_i-\alpha_j)(-\alpha_i)}\delta_j - \frac{1}{\alpha_j\alpha_i(\alpha_i-\alpha_j)}\delta_i + \frac{1}{\alpha_j(\alpha_i-\alpha_j)(\alpha_i)}\delta_{ji} +$
$\frac{1}{\alpha_j\alpha_i(\alpha_i-\alpha_j)}\delta_{ij} + \frac{1}{\alpha_j(\alpha_i-\alpha_j)(-\alpha_j)} - \frac{1}{\alpha_j(\alpha_i-\alpha_j)(\alpha_i)}\delta_{jij}.$

Clearly the left-hand side and the right hand sides equate.

3. $X_iX_j = X_jX_i$ if $|i-j| \geq 2$:
$X_iX_j = (\frac{1}{x_i} - \frac{1}{x_i}\delta_i)(\frac{1}{x_j} - \frac{1}{x_j}\delta_j)$
$= \frac{1}{x_ix_j} - \frac{1}{x_i}\delta_i\frac{1}{x_j} - \frac{1}{x_ix_j}\delta_j + \frac{1}{x_i}\delta_i\frac{1}{x_j}\delta_j$
$= \frac{1}{x_ix_j} - \frac{1}{x_i}s_i(\frac{1}{x_j})\delta_i - \frac{1}{x_ix_j}\delta_j + \frac{1}{x_i}s_i(\frac{1}{x_j})\delta_{ij}$
$= \frac{1}{x_ix_j} - \frac{1}{x_i}(\frac{1}{x_j})\delta_i - \frac{1}{x_ix_j}\delta_j + \frac{1}{x_i}(\frac{1}{x_j})\delta_{ij} = X_jX_i.$

□

To properly define T_i, first we let h be a parameter that commutes with all symmetric action. We then replace S with $S' := Sym_{\mathbb{Z}[h]}\Lambda$, that is, it is the polynomial ring with coefficients in the polynomial ring $\mathbb{Z}[h]$. Define T_i the Demazure-Lusztig elements for $i = 1, ..., n-1$ such that $T_i = \frac{\alpha_i+h}{\alpha_i}\delta_i - \frac{h}{\alpha_i} = hX_i + \delta_i$. We will now determine some relations between the T-elements, which are all special cases of Lemma 2.1 and Theorem 2.13 of Zhao- Zhong paper [7].

For T_i, the following three relations hold:

Lemma 2.2.2. *[4] For the Demazure-Lusztig element T_i, we have the following identities:*

1. $T_i^2 = 1$.
2. $T_iT_j = T_jT_i$ *if* $|i-j| \geq 2$.
3. $T_iT_jT_i = T_jT_iT_j$ *where* $i < j, j = i-1$.

Proof.

1. $T_i^2 = 1$
$$\begin{aligned}
T_i^2 &= (\tfrac{\alpha_i+h}{\alpha_i}\delta_i - \tfrac{h}{\alpha_i})(\tfrac{\alpha_i+h}{\alpha_i}\delta_i - \tfrac{h}{\alpha_i}) \\
&= \tfrac{\alpha_i+h}{\alpha_i}\delta_i\tfrac{\alpha_i+h}{\alpha_i}\delta_i - \tfrac{\alpha_i+h}{\alpha_i}\delta_i\tfrac{h}{\alpha_i} - \tfrac{h(\alpha_i+h)}{\alpha_i^2}\delta_i + \tfrac{h^2}{\alpha_i^2} \\
&= \tfrac{\alpha_i+h}{\alpha_i}s_i(\tfrac{\alpha_i+h}{\alpha_i})\delta_i^2 - \tfrac{\alpha_i+h}{\alpha_i}s_i(\tfrac{h}{\alpha_i})\delta_i - \tfrac{h(\alpha_i+h)}{\alpha_i^2}\delta_i + \tfrac{h^2}{\alpha_i^2} \\
&= \tfrac{\alpha_i+h}{\alpha_i}(\tfrac{-\alpha_i+h}{-\alpha_i}) - \tfrac{\alpha_i+h}{\alpha_i}(\tfrac{h}{-\alpha_i})\delta_i - \tfrac{h(\alpha_i+h)}{\alpha_i^2}\delta_i + \tfrac{h^2}{\alpha_i^2} \\
&= \tfrac{-\alpha_i^2+\alpha_ih-h\alpha_i+h^2}{-\alpha_i^2} + \tfrac{h(\alpha_i+h)}{\alpha_i^2}\delta_i - \tfrac{h(\alpha_i+h)}{\alpha_i^2} + \tfrac{h^2}{\alpha_i^2} \\
&= \tfrac{-\alpha_i^2+h^2}{-\alpha_i^2} + \tfrac{h(\alpha_i+h)}{\alpha_i^2}\delta_i - \tfrac{h(\alpha_i+h)}{\alpha_i^2} + \tfrac{h^2}{\alpha_i^2} \\
&= \tfrac{-\alpha_i^2+h^2}{-\alpha_i^2} + \tfrac{h(\alpha_i+h)}{\alpha_i^2}\delta_i - \tfrac{h(\alpha_i+h)}{\alpha_i^2} + \tfrac{-h^2}{-\alpha_i^2} \\
&= \tfrac{-\alpha_i^2}{-\alpha_i^2} = 1.
\end{aligned}$$

2. $T_iT_j = T_jT_i$ if $|i-j| \geq 2$

$$\begin{aligned}
T_iT_j &= (\tfrac{\alpha_i+h}{\alpha_i}\delta_i - \tfrac{h}{\alpha_i})(\tfrac{\alpha_j+h}{\alpha_j}\delta_j - \tfrac{h}{\alpha_j}) \\
&= \tfrac{\alpha_i+h}{\alpha_i}\delta_i\tfrac{\alpha_j+h}{\alpha_j}\delta_j - \tfrac{\alpha_i+h}{\alpha_i}\delta_i\tfrac{h}{\alpha_j} - \tfrac{h(\alpha_j+h)}{\alpha_i\alpha_j}\delta_j + \tfrac{h^2}{\alpha_i\alpha_j} \\
&= \tfrac{\alpha_i+h}{\alpha_i}s_i(\tfrac{\alpha_j+h}{\alpha_j})\delta_{ij} - \tfrac{\alpha_i+h}{\alpha_i}s_i(\tfrac{h}{\alpha_j})\delta_i - \tfrac{h(\alpha_j+h)}{\alpha_i\alpha_j}\delta_j + \tfrac{h^2}{\alpha_i\alpha_j} \\
&= \tfrac{\alpha_i+h}{\alpha_i}(\tfrac{\alpha_i+\alpha_j+h}{\alpha_i+\alpha_j})\delta_{ij} - \tfrac{\alpha_i+h}{\alpha_i}(\tfrac{h}{\alpha_i+\alpha_j})\delta_i - \tfrac{h(\alpha_j+h)}{\alpha_i\alpha_j}\delta_j + \tfrac{h^2}{\alpha_i\alpha_j}.
\end{aligned}$$

$$\begin{aligned}
T_jT_i &= (\tfrac{\alpha_j+h}{\alpha_j}\delta_j - \tfrac{h}{\alpha_j})(\tfrac{\alpha_i+h}{\alpha_i}\delta_i - \tfrac{h}{\alpha_i}) \\
&= \tfrac{\alpha_j+h}{\alpha_j}\delta_j\tfrac{\alpha_i+h}{\alpha_i}\delta_i - \tfrac{\alpha_j+h}{\alpha_j}\delta_j\tfrac{h}{\alpha_i} - \tfrac{h(\alpha_i+h)}{\alpha_j\alpha_i}\delta_i + \tfrac{h^2}{\alpha_j\alpha_i} \\
&= \tfrac{\alpha_j+h}{\alpha_j}s_j(\tfrac{\alpha_i+h}{\alpha_i})\delta_{ji} - \tfrac{\alpha_j+h}{\alpha_j}s_j(\tfrac{h}{\alpha_i})\delta_j - \tfrac{h(\alpha_i+h)}{\alpha_j\alpha_i}\delta_i + \tfrac{h^2}{\alpha_j\alpha_i} \\
&= \tfrac{\alpha_j+h}{\alpha_j}(\tfrac{\alpha_i+\alpha_j+h}{\alpha_i+\alpha_j})\delta_{ji} - \tfrac{\alpha_j+h}{\alpha_j}(\tfrac{h}{\alpha_i+\alpha_j})\delta_j - \tfrac{h(\alpha_i+h)}{\alpha_j\alpha_i}\delta_i + \tfrac{h^2}{\alpha_j\alpha_i}.
\end{aligned}$$

3. $T_iT_jT_i = T_jT_iT_j$ where $i < j, j = i - 1$

We proceed by expanding the left hand side and right hand side separately before showing that they equate.

LHS: $T_iT_jT_i = (\frac{\alpha_i+h}{\alpha_i}\delta_i - \frac{h}{\alpha_i})(\frac{\alpha_j+h}{\alpha_j}\delta_j - \frac{h}{\alpha_j})(\frac{\alpha_i+h}{\alpha_i}\delta_i - \frac{h}{\alpha_i})$
$= (\frac{(\alpha_i+h)(\alpha_i+\alpha_j+h)}{\alpha_i(\alpha_i+\alpha_j)}\delta_{ij} - \frac{(\alpha_i+h)h}{\alpha_i(\alpha_i+\alpha_j)}\delta_i - \frac{h(\alpha_j+h)}{\alpha_i\alpha_j}\delta_j + \frac{h^2}{\alpha_i\alpha_j})(\frac{\alpha_i+h}{\alpha_i}\delta_i - \frac{h}{\alpha_i})$
$= \frac{(\alpha_i+h)(\alpha_i+\alpha_j+h)}{\alpha_i(\alpha_i+\alpha_j)}\delta_{ij}\frac{\alpha_i+h}{\alpha_i}\delta_i - \frac{(\alpha_i+h)h}{\alpha_i(\alpha_i+\alpha_j)}\delta_i\frac{\alpha_i+h}{\alpha_i}\delta_i - \frac{h(\alpha_j+h)}{\alpha_i\alpha_j}\delta_j\frac{\alpha_i+h}{\alpha_i}\delta_i + \frac{h^2}{\alpha_i\alpha_j}\frac{\alpha_i+h}{\alpha_i}\delta_i$
$- \frac{(\alpha_i+h)(\alpha_i+\alpha_j+h)}{\alpha_i(\alpha_i+\alpha_j)}\delta_{ij}\frac{h}{\alpha_i} + \frac{(\alpha_i+h)h}{\alpha_i(\alpha_i+\alpha_j)}\delta_i\frac{h}{\alpha_i} + \frac{h(\alpha_j+h)}{\alpha_i\alpha_j}\delta_j\frac{h}{\alpha_i} - \frac{h^2}{\alpha_i\alpha_j}\frac{h}{\alpha_i}$
$= \frac{(\alpha_i+h)(\alpha_i+\alpha_j+h)}{\alpha_i(\alpha_i+\alpha_j)}s_{ij}(\frac{\alpha_i+h}{\alpha_i})\delta_{iji} - \frac{(\alpha_i+h)h}{\alpha_i(\alpha_i+\alpha_j)}s_i(\frac{\alpha_i+h}{\alpha_i})\delta_i^2 - \frac{h(\alpha_j+h)}{\alpha_i\alpha_j}s_j(\frac{\alpha_i+h}{\alpha_i})\delta_{ji} +$
$\frac{h^2(\alpha_i+h)}{\alpha_i\alpha_j\alpha_i}\delta_i$
$- \frac{(\alpha_i+h)(\alpha_i+\alpha_j+h)}{\alpha_i(\alpha_i+\alpha_j)}s_{ij}(\frac{h}{\alpha_i})\delta_{ij} + \frac{(\alpha_i+h)h}{\alpha_i(\alpha_i+\alpha_j)}s_i(\frac{h}{\alpha_i})\delta_i + \frac{h(\alpha_j+h)}{\alpha_i\alpha_j}s_j(\frac{h}{\alpha_i})\delta_j - \frac{h^3}{\alpha_i\alpha_j\alpha_j}$
$= \frac{(\alpha_i+h)(\alpha_i+\alpha_j+h)}{\alpha_i(\alpha_i+\alpha_j)}(\frac{\alpha_j+h}{\alpha_j})\delta_{iji} - \frac{(\alpha_i+h)h}{\alpha_i(\alpha_i+\alpha_j)}(\frac{-\alpha_i+h}{-\alpha_i}) - \frac{h(\alpha_j+h)}{\alpha_i\alpha_j}(\frac{\alpha_i+\alpha_j+h}{\alpha_i+\alpha_j})\delta_{ji} +$
$\frac{h^2(\alpha_i+h)}{\alpha_i\alpha_j\alpha_i}\delta_i$
$- \frac{(\alpha_i+h)(\alpha_i+\alpha_j+h)}{\alpha_i(\alpha_i+\alpha_j)}(\frac{h}{\alpha_j})\delta_{ij} + \frac{(\alpha_i+h)h}{\alpha_i(\alpha_i+\alpha_j)}(\frac{h}{-\alpha_i})\delta_i + \frac{h(\alpha_j+h)}{\alpha_i\alpha_j}(\frac{h}{\alpha_i+\alpha_i})\delta_j - \frac{h^3}{\alpha_i\alpha_j\alpha_j}.$

RHS: $T_jT_iT_j = (\frac{\alpha_j+h}{\alpha_j}\delta_j - \frac{h}{\alpha_j})(\frac{\alpha_i+h}{\alpha_i}\delta_i - \frac{h}{\alpha_i})(\frac{\alpha_j+h}{\alpha_j}\delta_j - \frac{h}{\alpha_j})$
$= (\frac{(\alpha_j+h)(\alpha_i+\alpha_j+h)}{\alpha_j(\alpha_i+\alpha_j)}\delta_{ji} - \frac{(\alpha_j+h)h}{\alpha_j(\alpha_i+\alpha_j)}\delta_j - \frac{h(\alpha_i+h)}{\alpha_j\alpha_i}\delta_i + \frac{h^2}{\alpha_j\alpha_i})(\frac{\alpha_j+h}{\alpha_j}\delta_j - \frac{h}{\alpha_j})$
$= \frac{(\alpha_j+h)(\alpha_i+\alpha_j+h)}{\alpha_j(\alpha_i+\alpha_j)}\delta_{ji}\frac{\alpha_j+h}{\alpha_j}\delta_j - \frac{(\alpha_j+h)h}{\alpha_j(\alpha_i+\alpha_j)}\delta_j\frac{\alpha_j+h}{\alpha_j}\delta_j - \frac{h(\alpha_i+h)}{\alpha_j\alpha_i}\delta_i\frac{\alpha_j+h}{\alpha_j}\delta_j + \frac{h^2}{\alpha_j\alpha_i}\frac{\alpha_j+h}{\alpha_j}\delta_j$
$- \frac{(\alpha_j+h)(\alpha_i+\alpha_j+h)}{\alpha_j(\alpha_i+\alpha_j)}\delta_{ji}\frac{h}{\alpha_j} + \frac{(\alpha_j+h)h}{\alpha_j(\alpha_i+\alpha_j)}\delta_j\frac{h}{\alpha_j} + \frac{h(\alpha_i+h)}{\alpha_j\alpha_i}\delta_i\frac{h}{\alpha_j} - \frac{h^2}{\alpha_j\alpha_i}\frac{h}{\alpha_j}$
$= \frac{(\alpha_j+h)(\alpha_i+\alpha_j+h)}{\alpha_j(\alpha_i+\alpha_j)}s_{ji}(\frac{\alpha_j+h}{\alpha_j})\delta_{jij} - \frac{(\alpha_j+h)h}{\alpha_j(\alpha_i+\alpha_j)}s_j(\frac{\alpha_j+h}{\alpha_j})\delta_j^2 - \frac{h(\alpha_i+h)}{\alpha_j\alpha_i}s_i(\frac{\alpha_j+h}{\alpha_j})\delta_{ij} +$
$\frac{h^2(\alpha_j+h)}{\alpha_j\alpha_i\alpha_j}\delta_j$
$- \frac{(\alpha_j+h)(\alpha_i+\alpha_j+h)}{\alpha_j(\alpha_i+\alpha_j)}s_{ji}(\frac{h}{\alpha_j})\delta_{ji} + \frac{(\alpha_j+h)h}{\alpha_j(\alpha_i+\alpha_j)}s_j(\frac{h}{\alpha_j})\delta_j + \frac{h(\alpha_i+h)}{\alpha_j\alpha_i}s_i(\frac{h}{\alpha_j})\delta_i - \frac{h^3}{\alpha_j\alpha_i\alpha_j}$
$= \frac{(\alpha_j+h)(\alpha_i+\alpha_j+h)}{\alpha_j(\alpha_i+\alpha_j)}(\frac{\alpha_i+h}{\alpha_i})\delta_{jij} - \frac{(\alpha_j+h)h}{\alpha_j(\alpha_i+\alpha_j)}(\frac{-\alpha_j+h}{-\alpha_j}) - \frac{h(\alpha_i+h)}{\alpha_j\alpha_i}(\frac{\alpha_i+\alpha_j+h}{\alpha_i+\alpha_j})\delta_{ij} +$
$\frac{h^2(\alpha_j+h)}{\alpha_j\alpha_i\alpha_j}\delta_j$
$- \frac{(\alpha_j+h)(\alpha_i+\alpha_j+h)}{\alpha_j(\alpha_i+\alpha_j)}(\frac{h}{\alpha_i})\delta_{ji} + \frac{(\alpha_j+h)h}{\alpha_j(\alpha_i+\alpha_j)}(\frac{h}{-\alpha_j})\delta_j + \frac{h(\alpha_i+h)}{\alpha_j\alpha_i}(\frac{h}{\alpha_i+\alpha_j})\delta_i - \frac{h^3}{\alpha_j\alpha_i\alpha_j}.$

$\square$

Chapter 3

Transition Matrices of the Demazure Elements and Demazure-Lusztig Elements

3.1 Simplification of Notation

In this section, we will introduce some shorthanded notation that will be utilized for the following computations. This notation makes computations simpler and easier to understand.

First, we redefine δ_i so that $\delta_w = \delta_{i_1}...\delta_{i_k} = \delta_{i_1...i_k}$ if $w = s_{i_1}...s_{1_k}$. For example, $\delta_{321} = \delta_3\delta_2\delta_1$. Similarly, we utilize this notation with s_w. For example, $s_{123} = s_1s_2s_3$. There are a couple of identities (which will be left to the reader to verify) that are used constantly in sections 3.2 and 3.3. They are as follows:

1. $s_i(\alpha_i) = -\alpha_i$.

2. $s_i(\alpha_j) = \alpha_i + \alpha_j$ if $|i - j| = 1$.

3. $s_i(\alpha_j) = \alpha_j$ if $|i - j| \geq 2$.

In addition, we also introduce some short hand for the α_i notation. Note that $\alpha_{-i} = -\alpha_i$ and that $\alpha_{i+j} = \alpha_i + \alpha_j$.

3.1.1 Demazure Element Operators for A_2

In this section, we compute the transition matrix between the δ basis with the Demazure elements in the case of S_3.

Consider $n = 3$, then $S_3 = \{e, s_1, s_2, s_{12}, s_{21}, s_{121} = s_{212}\}$. It follows:

$X_e = \frac{1}{\alpha_e} - \frac{1}{\alpha_e}\delta_e = 1.$

$X_1 = \frac{1}{\alpha_1} - \frac{1}{\alpha_1}\delta_1.$

$X_2 = \frac{1}{\alpha_2} - \frac{1}{\alpha_2}\delta_2.$

$X_{12} = (\frac{1}{\alpha_1} - \frac{1}{\alpha_1}\delta_1)(\frac{1}{\alpha_2} - \frac{1}{\alpha_2}\delta_2)$
$= \frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_1\alpha_2}\delta_2 - \frac{1}{\alpha_1}\delta_1\frac{1}{\alpha_2} + \frac{1}{\alpha_1}\delta_1\frac{1}{\alpha_2}\delta_2$
$= \frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_1\alpha_2}\delta_2 - \frac{1}{\alpha_1}s_1(\frac{1}{\alpha_2})\delta_1 + \frac{1}{\alpha_1}s_1(\frac{1}{\alpha_2})\delta_{12}$
$= \frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_1\alpha_2}\delta_2 - \frac{1}{\alpha_1\alpha_{1+2}}\delta_1 + \frac{1}{\alpha_1\alpha_{1+2}}\delta_{12}.$

$X_{21} = (\frac{1}{\alpha_2} - \frac{1}{\alpha_2}\delta_2)(\frac{1}{\alpha_1} - \frac{1}{\alpha_1}\delta_1)$
$= \frac{1}{\alpha_2\alpha_1} - \frac{1}{\alpha_2\alpha_1}\delta_1 - \frac{1}{\alpha_2}\delta_2\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\delta_2\frac{1}{\alpha_1}\delta_1$
$= \frac{1}{\alpha_2\alpha_1} - \frac{1}{\alpha_2\alpha_1}\delta_1 - \frac{1}{\alpha_2}s_2(\frac{1}{\alpha_1})\delta_2 + \frac{1}{\alpha_2}s_2(\frac{1}{\alpha_1})\delta_{21}$
$= \frac{1}{\alpha_2\alpha_1} - \frac{1}{\alpha_2\alpha_1}\delta_1 - \frac{1}{\alpha_2\alpha_{1+2}}\delta_2 + \frac{1}{\alpha_2\alpha_{1+2}}\delta_{21}.$

$X_{121} = X_{12}X_1 = (\frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_1\alpha_2}\delta_2 - \frac{1}{\alpha_1\alpha_{1+2}}\delta_1 + \frac{1}{\alpha_1\alpha_{1+2}}\delta_{12})(\frac{1}{\alpha_1} - \frac{1}{\alpha_1}\delta_1)$
$= \frac{1}{\alpha_1\alpha_2\alpha_1} - \frac{1}{\alpha_1\alpha_2}\delta_2\frac{1}{\alpha_1} - \frac{1}{\alpha_1\alpha_{1+2}}\delta_1\frac{1}{\alpha_1} + \frac{1}{\alpha_1\alpha_{1+2}}\delta_{12}\frac{1}{\alpha_1}$
$- \frac{1}{\alpha_1\alpha_2\alpha_1}\delta_1 + \frac{1}{\alpha_1\alpha_2}\delta_2\frac{1}{\alpha_1}\delta_1 + \frac{1}{\alpha_1\alpha_{1+2}}\delta_1\frac{1}{\alpha_1}\delta_1 - \frac{1}{\alpha_1\alpha_{1+2}}\delta_{12}\frac{1}{\alpha_1}\delta_1$
$= \frac{1}{\alpha_1\alpha_2\alpha_1} - \frac{1}{\alpha_1\alpha_2}s_2(\frac{1}{\alpha_1})\delta_2 - \frac{1}{\alpha_1\alpha_{1+2}}s_1(\frac{1}{\alpha_1})\delta_1 + \frac{1}{\alpha_1\alpha_{1+2}}s_{12}(\frac{1}{\alpha_1})\delta_{12}$
$- \frac{1}{\alpha_1\alpha_2\alpha_1}\delta_1 + \frac{1}{\alpha_1\alpha_2}s_2(\frac{1}{\alpha_1})\delta_{21} + \frac{1}{\alpha_1\alpha_{1+2}}s_1(\frac{1}{\alpha_1})\delta_1^2 - \frac{1}{\alpha_1\alpha_{1+2}}s_{12}(\frac{1}{\alpha_1})\delta_{121}$
$= \frac{1}{\alpha_1\alpha_2\alpha_1} - \frac{1}{\alpha_1\alpha_2\alpha_{1+2}}\delta_2 - \frac{1}{\alpha_1\alpha_{1+2}\alpha_{-1}}\delta_1 + \frac{1}{\alpha_1\alpha_{1+2}\alpha_{-1+2}}\delta_{12}$
$- \frac{1}{\alpha_1\alpha_2\alpha_1}\delta_1 + \frac{1}{\alpha_1\alpha_2\alpha_{1+2}}\delta_{21} + \frac{1}{\alpha_1\alpha_{1+2}\alpha_{-1}} - \frac{1}{\alpha_1\alpha_{1+2}\alpha_{-1+2}}\delta_{121}$
$= \frac{\alpha_{-1}\alpha_{1+2}+\alpha_1\alpha_2}{\alpha_{-1}\alpha_1^2\alpha_2\alpha_{1+2}} - \frac{1}{\alpha_1\alpha_2\alpha_{1+2}}\delta_2 - \frac{\alpha_{-1}\alpha_{1+2}+\alpha_1\alpha_2}{\alpha_{-1}\alpha_1^2\alpha_2\alpha_{1+2}}\delta_1 + \frac{1}{\alpha_1\alpha_{1+2}\alpha_{-1+2}}\delta_{12} + \frac{1}{\alpha_1\alpha_2\alpha_{1+2}}\delta_{21} -$
$\frac{1}{\alpha_1\alpha_{1+2}\alpha_{-1+2}}\delta_{121}.$

3.2 Demazure-Lusztig Operators for S_3:

In this section, we compute the transition matrix between the δ basis with the Demazure-Lusztig elements in the case of S_3.

$T_e = 1.$

$T_1 = \frac{\alpha_1+h}{\alpha_1}\delta_1 - \frac{h}{\alpha_1}.$

$T_2 = \frac{\alpha_2+h}{\alpha_2}\delta_2 - \frac{h}{\alpha_2}.$

$T_{12} = T_1T_2 = (\frac{\alpha_1+h}{\alpha_1}\delta_1 - \frac{h}{\alpha_1})(\frac{\alpha_2+h}{\alpha_2}\delta_2 - \frac{h}{\alpha_2})$
$= \frac{\alpha_1+h}{\alpha_1}\delta_1\frac{\alpha_2+h}{\alpha_2}\delta_2 - \frac{\alpha_1+h}{\alpha_1}\delta_1\frac{h}{\alpha_2} - \frac{h(\alpha_2+h)}{\alpha_1\alpha_2}\delta_2 + \frac{h^2}{\alpha_1\alpha_2}$
$= \frac{\alpha_1+h}{\alpha_1}s_1(\frac{\alpha_2+h}{\alpha_2})\delta_{12} - \frac{\alpha_1+h}{\alpha_1}s_1(\frac{h}{\alpha_2})\delta_1 - \frac{h(\alpha_2+h)}{\alpha_1\alpha_2}\delta_2 + \frac{h^2}{\alpha_1\alpha_2}$

$= \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)}{(\alpha_1)(\alpha_1+\alpha_2)}\delta_{12} - \frac{(\alpha_1+h)h}{\alpha_1(\alpha_1+\alpha_2)}\delta_1 - \frac{h(\alpha_2+h)}{\alpha_1\alpha_2}\delta_2 + \frac{h^2}{\alpha_1\alpha_2}.$

$T_{21} = T_2T_1 = (\frac{\alpha_2+h}{\alpha_2}\delta_2 - \frac{h}{\alpha_2})(\frac{\alpha_1+h}{\alpha_1}\delta_1 - \frac{h}{\alpha_1})$
$= \frac{\alpha_2+h}{\alpha_2}\delta_2\frac{\alpha_1+h}{\alpha_1}\delta_1 - \frac{\alpha_2+h}{\alpha_2}\delta_2\frac{h}{\alpha_1} - \frac{h(\alpha_1+h)}{\alpha_2\alpha_1}\delta_1 + \frac{h^2}{\alpha_2\alpha_1}$
$= \frac{\alpha_2+h}{\alpha_2}s_2(\frac{\alpha_1+h}{\alpha_1})\delta_{21} - \frac{\alpha_2+h}{\alpha_2}s_2(\frac{h}{\alpha_1})\delta_2 - \frac{h(\alpha_1+h)}{\alpha_2\alpha_1}\delta_1 + \frac{h^2}{\alpha_2\alpha_1}$
$= \frac{(\alpha_2+h)(\alpha_1+\alpha_2+h)}{\alpha_2(\alpha_1+\alpha_2)}\delta_{21} - \frac{\alpha_2+h}{\alpha_2}\delta_2 - \frac{h(\alpha_1+h)}{\alpha_2\alpha_1}\delta_1 + \frac{h^2}{\alpha_2\alpha_1}.$

$T_{121} = T_{12}T_1$
$= (\frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)}{(\alpha_1)(\alpha_1+\alpha_2)}\delta_{12} - \frac{(\alpha_1+h)h}{\alpha_1(\alpha_1+\alpha_2)}\delta_1 - \frac{h(\alpha_2+h)}{\alpha_1\alpha_2}\delta_2 + \frac{h^2}{\alpha_1\alpha_2})(\frac{\alpha_1+h}{\alpha_1}\delta_1 - \frac{h}{\alpha_1})$
$= \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)}{(\alpha_1)(\alpha_1+\alpha_2)}\delta_{12}\frac{\alpha_1+h}{\alpha_1}\delta_1 - \frac{(\alpha_1+h)h}{\alpha_1(\alpha_1+\alpha_2)}\delta_1\frac{\alpha_1+h}{\alpha_1}\delta_1 - \frac{h(\alpha_2+h)}{\alpha_1\alpha_2}\delta_2\frac{\alpha_1+h}{\alpha_1}\delta_1 + \frac{h^2(\alpha_1+h)}{\alpha_1\alpha_2\alpha_1}\delta_1$
$- \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)}{(\alpha_1)(\alpha_1+\alpha_2)}\delta_{12}\frac{h}{\alpha_1} + \frac{(\alpha_1+h)h}{\alpha_1(\alpha_1+\alpha_2)}\delta_1\frac{h}{\alpha_1} + \frac{h(\alpha_2+h)}{\alpha_1\alpha_2}\delta_2\frac{h}{\alpha_1} - \frac{h^3}{\alpha_1\alpha_2\alpha_1}$
$= \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)}{(\alpha_1)(\alpha_1+\alpha_2)}s_{12}(\frac{\alpha_1+h}{\alpha_1})\delta_{121} - \frac{(\alpha_1+h)h}{\alpha_1(\alpha_1+\alpha_2)}s_1(\frac{\alpha_1+h}{\alpha_1})\delta_1^2 - \frac{h(\alpha_2+h)}{\alpha_1\alpha_2}s_2(\frac{\alpha_1+h}{\alpha_1})\delta_{21} +$
$\frac{h^2(\alpha_1+h)}{\alpha_1\alpha_2\alpha_1}\delta_1$
$- \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)}{(\alpha_1)(\alpha_1+\alpha_2)}s_{12}(\frac{h}{\alpha_1})\delta_{12} + \frac{(\alpha_1+h)h}{\alpha_1(\alpha_1+\alpha_2)}s_1(\frac{h}{\alpha_1})\delta_1 + \frac{h(\alpha_2+h)}{\alpha_1\alpha_2}s_2(\frac{h}{\alpha_1})\delta_2 - \frac{h^3}{\alpha_1\alpha_2\alpha_1}$
$= \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)(\alpha_2+h)}{(\alpha_1)(\alpha_1+\alpha_2)(\alpha_2)}\delta_{121} - \frac{(\alpha_1+h)(h)(-\alpha_1+h)}{\alpha_1(\alpha_1+\alpha_2)(-\alpha_1)} - \frac{h(\alpha_2+h)(\alpha_1+\alpha_2+h)}{\alpha_1\alpha_2(\alpha_1+\alpha_2)}\delta_{21} + \frac{h^2(\alpha_1+h)}{\alpha_1\alpha_2\alpha_1}\delta_1$
$- \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)(h)}{(\alpha_1)(\alpha_1+\alpha_2)(\alpha_2)}\delta_{12} + \frac{(\alpha_1+h)(h^2)}{\alpha_1(\alpha_1+\alpha_2)(-\alpha_1)}\delta_1 + \frac{h(\alpha_2+h)h}{\alpha_1\alpha_2(\alpha_1+\alpha_2)}\delta_2 - \frac{h^3}{\alpha_1\alpha_2\alpha_1}$
$= \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)(\alpha_2+h)}{(\alpha_1)(\alpha_1+\alpha_2)(\alpha_2)}\delta_{121} - \frac{h(\alpha_2+h)(\alpha_1+\alpha_2+h)}{\alpha_1\alpha_2(\alpha_1+\alpha_2)}\delta_{21} - \frac{(\alpha_1+h)(\alpha_1+\alpha_2+h)(h)}{(\alpha_1)(\alpha_1+\alpha_2)(\alpha_2)}\delta_{12} + \frac{h^2(\alpha_1+h)[(\alpha_1+\alpha_2)(-\alpha_1)+(\alpha_2\alpha_1)]}{\alpha_1\alpha_2\alpha_1(\alpha_1+\alpha_2)(-\alpha_1)}\delta_1$
$+ \frac{h(\alpha_2+h)h}{\alpha_1\alpha_2(\alpha_1+\alpha_2)}\delta_2 - \frac{(\alpha_1+h)(h)(-\alpha_1+h)(\alpha_2\alpha_1) - h^3(\alpha_1+\alpha_2)(-\alpha_1)}{\alpha_1\alpha_2\alpha_1(\alpha_1+\alpha_2)(-\alpha_1)}.$

Bibliography

[1] Calmès, B., Zainoulline, K. and Zhong, C., A coproduct structure on the formal affine Demazure algebra, Math. Z. 282 (2016).

[2] Golubitsky, M. “Hyperplane Intuition.” Hyperplane Intuition, 26 Apr. 2018, golubitsky.github.io/ml-blog/2018/04/26/Hyperplane-Intuition.html.

[3] Humphreys, J. E. Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990. xii+204 pp.

[4] Kostant, B. and Kumar, S. The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Advances in Math., 62:187–237, 1986.

[5] Milne, J. S., Group theory. https://www.jmilne.org/math/CourseNotes/GT.pdf, 2013

[6] Rowland, T. ”Symmetric Bilinear Form.” From MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/SymmetricBilinearForm.html

[7] Zhao, G. and Zhong, C., Geometric representations of the formal affine Hecke algebra, M