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Gesture as a Window to Justification and Proof

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The role of the body, particularly gesture, in supporting mathematical reasoning is an emerging area of research in mathematics education. In the present study, we examine undergraduate students providing a justification for a task about a system of alternating gears, which involves concepts of number relating to even/odd patterns. Some participants were directed to perform gestures relevant to alternation and parity before attempting their justification, while others were not. Although these directed actions did not seem to influence the gestures participants used to solve the problem, we found an important relationship between gesture and mathematical reasoning. In particular, certain types of gestures during problem solving were associated with valid justifications. This research provides insight into the link between action and mathematical reasoning, and has implications for supporting students' proof activities.

Keywords: Cognition, Learning Theories, Reasoning and Proof

Learners display their mathematical thinking—and even engage in mathematical reasoning—with their bodies as well as their minds. Recent theoretical work in education and psychology has sought to broaden researchers' and educators' perspectives to address the role of the body in mathematical thinking and learning. In this research, we adopt this perspective to explore the role of learners' actions during proof and justification activities. Specifically, we investigate the connections between learners' *gestures* and their *justifications* as they talk aloud while attempting to solve a problem about an underlying numerical pattern presented in a gear-system task. By investigating how body movements relate to different forms of mathematical reasoning, we can better understand students' thinking and consider novel ways to support the construction of valid mathematical justifications. We contribute to *broadening perspectives on mathematics thinking and learning* by detailing a connection between specific gestures and types of proof that challenges the concept of mathematics as a disembodied system.

Theoretical Framework

Action and Gesture

An embodied cognition perspective highlights reciprocal connections between actions and cognition (e.g., Glenberg, 1997; Goldin-Meadow & Beilock, 2010). Specifically, these theories suggest that actions do not simply externalize the output of cognitive processes, but may also directly influence and cause changes in cognition and learning (e.g., Shapiro, 2011). Because this view postulates that actions and cognition are intrinsically linked, it stands to reason that there may be an association between an individual's actions and his or her performance on tasks requiring insight and problem solving (e.g., Thomas & Lleras, 2009). Research has shown that actions play an important, and potentially vital, role in the learning and using of mathematics concepts and procedures -- from the spatial bodily orientations that represent early concepts of number (Dehaene, Bossini, & Giraux, 1993) and the use of fingers in early counting (Alibali & DiRusso, 1999; Saxe & Kaplan, 1981), to the concrete and perceptual dimensions of mathematical symbols (Landy & Goldstone, 2007). Action can also be an important way to "ground" (Goldstone & Son, 2005) abstract mathematical ideas in students' experiences. Physically-grounded actions and manipulations of real and virtual objects can help students understand concepts like proportionality (Abrahamson & Howsin, 2010), fractions (Martin, 2009), and algebra (Nathan, Kintsch, & Young, 1992). It is clear that actions contribute to mathematical thinking.

One specific type of action that is of special interest to educators is *gesture* –the spontaneous hand movements that speakers produce as they talk. Recent theoretical work suggests that gestures manifest mental simulations of actions and perceptual states (Hostetter & Alibali, 2008). Even though gestures do not physically manipulate the environment, growing evidence suggests that the experience of producing gestures can directly influence cognition (Alibali & Kita, 2010; Beilock & Goldin-Meadow, 2010; Goldin-Meadow & Beilock, 2010). Gestures that represent mathematical objects may serve an important function of grounding mathematical ideas in bodily form (Alibali & Nathan, 2007; Nathan, 2008), and they may also communicate spatial and relational concepts (Alibali, Nathan, & Fujimori, 2011). Moreover, speakers' gestures may reveal unique aspects of thought that are based in perception and action, and that may be crucial to their reasoning about the ideas they communicate (e.g., Alibali & Nathan, 2012; Chu & Kita, 2011). Thus in mathematics, gestures not only can provide a window into students' thought processes; they can help students to represent and understand key ideas and relationships. This may be especially true in mathematical tasks that involve spatial reasoning.

Mathematical Proof and Justification

In the study presented here, we investigated how gesture is related to the type and quality of mathematical reasoning in the task of creating a justification for a mathematical task involving an underlying even/odd pattern. Our theoretical framework for conceptualizing mathematical proof and justification is based on Harel and Sowder's (1998) work, which states that producing a proof involves removing doubt about the truth of a conjecture, both from oneself and from others. Harel and Sowder further distinguish between three major subsets of proof: external conviction, empirical, and analytical. The first orients around the production of self-satisfying proofs that rely upon external resources, such as textbooks or teachers, which we do not focus on in this study. The second subset, empirical, involves validating conjectures using physical facts or perceptual experiences. Finally, the third subset, analytical, "is one that validates conjectures by means of logical deductions" (p. 258).

We rely upon a specific type of proof in this third subset, *transformational*, which Harel and Sowder define as proof activities that are characterized by generality and abstraction, deliberate mental operations, and image transformations. We also utilized a specific type of proof in the empirical subset, referred to as *perceptual* proofs. These proofs involve mental images similar to transformational proofs, but which Harel and Sowder contrast by noting that perceptual proofs "consist of perceptions and a coordination of perceptions, but lack the ability to transform or to anticipate the results of a transformation" (p. 255). Given that transformational and perceptual proofs both involve images, and given the close ties between gestures and mental images (e.g., McNeill, 1992), Harel and Sowder's proof scheme is particularly well suited to our approach.

Gear-Parity Problems

We examined the justifications participants provided as they attempted to generate and justify a conjecture about a system that follows the underlying numerical pattern of *parity*, which we here instantiate with a system of interlocking gears. Parity in this context refers to the idea that in systems with an odd number of gears, the final gear turns in the same direction as the initial gear, while in systems with even numbers of gears, the first and last gear turn in opposite directions.

When provided with a static display of a row of gears or asked to imagine such a system, participants typically begin solving the problem by simulating the turning of the gears with their hands (Schwartz & Black, 1996). After producing such rotating gestures, many participants shift to a new approach during which they note that the gears move in an alternating sequence. Past work with both undergraduates and young children has shown that concentrated, accurate simulation of the gear movements predicts the transition to recognizing such *alternation* (Boncoddo, Dixon & Kelley, 2010; Trudeau & Dixon, 2007). Thus, participants' ability to abstract the underlying mathematical relationship in the gear system seems to be related to their actions, and in particular, to the action of repeatedly tracing multiple circles as they think about and solve the problems. When a participant discovers parity, they display an understanding that the direction of movement of the final gear is related to the number of gears in the system.

Thus, reasoning about conjectures involving gear systems are of interest to mathematics education because they manifest the abstract parity rule that is a key component of understanding number systems. Moreover, gear systems are often utilized as a grounding context in mathematics lessons focusing on ratio, proportions, and linear functions (e.g., Lobato & Ellis, 2010; Ellis, 2007). For example, Ellis (2007) utilized gear systems in a teaching experiment on linear functions in order to give students a real-world situation within which to learn about linear relationships. Thus, gear systems embody important mathematical ideas, and examining how students solve such problems provides insight into how teachers can support their thinking about these ideas in the mathematics classroom.

Research Questions

In this study, we focus on the gestures that speakers produce when reasoning about a gear-parity conjecture. We first examine whether the types of gestures participants produced while thinking aloud about the conjecture varied depending on the specific actions they were directed to perform prior to being given the conjecture. Second, we examine whether the types of gestures participants produced during justification of the conjecture were related to the type and quality of the justifications they provided. Thus, our first research question was: Is the nature of students' directed actions *prior* to engaging in proof reasoning related to the nature of the justifications they provide? Our second yet primary research question was: Is the nature of students' gestures *during* proof reasoning related to the nature of the justifications they provide? This research provides insight into the link between action and mathematical reasoning, and has implications for supporting students' proof activities.

Methods

Participants

Participants were 120 undergraduate students enrolled in a psychology course at a large Midwestern university. Their average age was 19.6 years (*SD* = 1.08) and 51% of the participants were female.

Procedure

As part of a larger study, each participant was asked to solve a number of problems, including gear-parity, geometry, and transfer tasks; in this paper we focus only on the gear-parity problems. Participants were asked to think aloud (Ericsson & Simon, 1998) as they solved the problems. Instructions and interviewer prompts were standardized, and tasks were presented on an interactive white board. Participant data was coded separately for justifications and for gestures, as detailed later in this section.

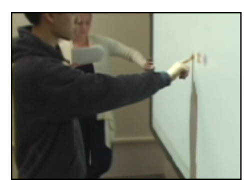
Before solving the gear-parity task, participants were randomly assigned to perform actions that were either relevant or irrelevant to providing a correct justification for the conjecture (see Table 1).

The relevant actions were based on previous studies that have shown that before participants discovered knowledge of the underlying numerical relationship of parity in the gear system, they typically use a strategy called “alternation,” in which they abstract the relation that adjacent gears move in alternating directions (Trudeau & Dixon, 2007). In particular, relevant actions involved students tapping back and forth on a screen, to model alternation. For participants who performed relevant actions, we also varied whether participants were told that the relevant actions they performed were related to the gear problem, which we call *projection*. Participants who received projection were explicitly informed that the tapping actions related to the solution to the problem, while other participants were not informed that their actions had any relevance. Finally, half of the participants performed small actions that were entirely within the periphery of their gesture space, while the other half performed larger actions that extended outside the periphery of their gesture space (McNeill, 1992). This factor did not affect participants’ success at abstracting the underlying mathematical relationship, so we do not discuss it further in this report.

Justification Coding

A coder blind to experimental condition used the proof categories from Harel and Sowder (1998) to code participants’ justifications on the gear task. For our purposes, participants’ justifications were coded as *analytical>transformational* if they acknowledged that connected gears form an alternating pattern of motion, and that the turning direction of the final gear is based on whether the total number of gears in the system is even or odd. We also extended the category of *empirical>perceptual* proof to account for a problem-specific nuance, distinguishing between *perceptual with alternation* and *perceptual without alternation*. Perceptual with alternation justifications included an understanding that the gears move in an alternating pattern, but the participant failed to acknowledge parity. Perceptual without alternation justifications included participants making comments that demonstrated a belief that the gears all turned in the same direction or that the last gear would turn in the same direction as the first gear, regardless of the number of gears present. Finally, the justifications of the participants who were unable to come up with a justification for how the gear system would turn were coded as *don't know*.

Table 1: Gear Conjecture and Directed Actions

Conjecture	An unknown number of gears are connected in a chain. You know what direction the first gear turns, how could you figure out what direction the last gear turns? Provide a justification as to why your answer is true.	
Relevant actions	<u>Large Actions</u>: Participant alternates between tapping a blue and yellow diamond placed an arm span apart on the Interactive White Board with their palm. 	<u>Small Actions</u>: Participant alternates between tapping a blue and yellow diamond placed a hands-length apart on the Interactive White Board with their index finger. 
Irrelevant actions	<u>Large Actions</u>: Participant taps only the blue diamond with their palm.	<u>Small Actions</u>: Participant taps only the blue diamond with their index finger.

Gesture Coding

A coder who was blind to experimental condition watched and coded each participant’s session without sound in order to classify gestures based on their form while uninfluenced by the verbalizations that accompanied those actions. Participants’ gestures were classified into three categories: *rotating* gestures, *ticking* gestures and *other* gestures. These categories were identified as important based on previous studies of undergraduates solving gear-system problems (e.g., Alibali, Spencer, Knox & Kita, 2011; Schwartz & Black, 1996; Trudeau & Dixon, 2007). *Rotating* gestures depicted one or more gears turning, typically using either one finger or the whole hand. Rotating gestures were further classified as depicting a single gear, multiple gears turning in the same direction, or multiple gears turning in alternate directions. *Ticking* gestures were defined as gestures that displayed a series of ticks, taps, or other discrete movements, typically produced using one finger or the whole hand and moving across space. Gestures that did not fall into either of these categorizes were labeled as "*other*" and descriptions of these gestures were recorded for further future analyses. Other gestures that we noted included beats, single points, and gestures that depicted movements other than rotating or ticking.

Results

Did participants' gestures vary by condition?

We first examined whether participants in the three conditions (relevant action with projection, relevant action without projection, and irrelevant action) varied in the types of gestures that they produced. First, participants were classified by whether they ever produced rotating gestures. Second, participants were then grouped according to whether they ever produced multiple rotating gestures with alternation, or only produced rotating gestures without alternation. As noted above, alternation is of particular importance to the gear-parity conjecture. We expected that more participants in the relevant-action conditions (regardless of whether they also received projection) than in the irrelevant-action condition would produce gestures that manifested alternation. Specifically, we hypothesized that tapping back and forth embodied the alternation relationship found in the gear system, which we expected to be subsequently manifested in the gestures that participants produced during justification. As seen in Table 2 (leftmost column), this was numerically the case. However, the difference across conditions was not significant.

Table 2: Percentage of Types of Gestures Produced by Participants in Each Condition

	Rotating Gestures		
Condition	Any multiple with alternation	Other (no multiple with alternation)	Ticking Gestures
Irrelevant	53	18	20
Relevant + Projection	55	30	8
Relevant	68	23	25

A third coding dimension classified participants in terms of whether they ever produced ticking gestures. We had anticipated that more participants in the relevant-action conditions than in the irrelevant-action condition would produce ticking gestures, since the relevant actions they had performed involved ticking back and forth. Instead, more participants in both the relevant and irrelevant conditions produced ticking gestures than did participants in the relevant-action-plus-projection condition. The difference across conditions in the proportion of participants who produced ticking gestures approached significance, $\chi^2(2, N = 120) = 4.50$, $p = .105$.

Thus, we did not find statistically significant relationships between action condition and patterns of gesture production. However, trends in the data suggest that the relevant actions may have encouraged participants to acknowledge alternation, and that projection may have inhibited ticking gestures during the justification.

Were variations in gesture associated with variations in justifications?

Given that variations in gesture production amongst conditions were not significant, we collapsed across conditions in order to examine whether participants' gestures during problem solving were associated with the quality of the justifications they provided. Figure 1 displays the percent of participants who provided each type of justification, as a function of the type of gestures participants produced during their explanation. Very few participants produced ticking gestures only ($N = 1$), or ticking gestures with non-alternating rotating gestures ($N = 2$), so these categories are not displayed in the figure. The number of participants in the other categories ranged from 18 to 52.

As seen in Figure 1, participants who produced rotating gestures with alternation (the right two bars) were much more likely to provide valid, transformational justifications than were participants who did not produce such gestures (the left two bars), $\chi^2(1, N = 117) = 18.71, p < .001$ (83% vs. 43%). Importantly, producing rotating gestures *with* alternation was more strongly associated with transformational justifications than was producing rotating gestures *without* alternation (i.e., single circles, or multiple circles that did not alternate direction), $\chi^2(1, N = 78) = 7.13, p = .007$ (comparing the middle two bars to one another, 46% vs. 15%). In addition, participants who produced ticking gestures *along with* rotating gestures with alternation (the rightmost bar) were slightly more likely to provide transformational justifications than were participants who produced rotating gestures with alternation without ticking gestures (the center right bar) (67% vs. 54%). However, this difference was not significant.

Finally, among participants who produced perceptually-based justifications (the two light gray categories), participants who produced rotating gestures with alternation were more likely to incorporate alternation into their justifications than were participants who did not produce such gestures (the left two bars), $\chi^2(1, N = 68) = 18.50, p < .001$.

In sum, producing rotation gestures with alternation was associated with expressing more sophisticated justifications. Participants who produced rotation gestures with alternation were more likely to express transformational justifications, and, if they expressed perceptual justifications, they were more likely to incorporate alternation.

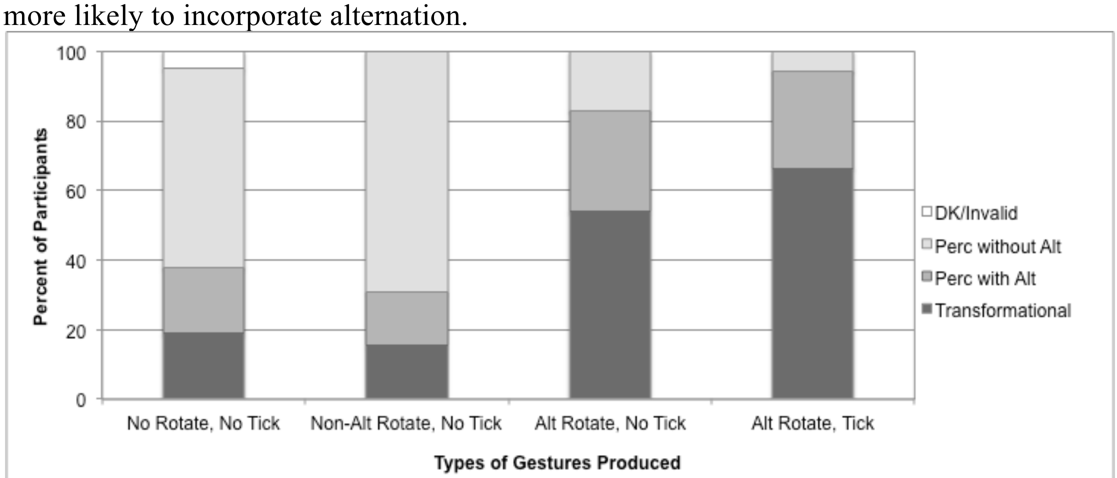


Figure 1: Percent of Participants who Provided Each Type of Justification as a Function of the Types of Gestures Produced

Conclusion and Implications

This study investigated the gestures participants produced when generating and justifying a conjecture about a gear system. Specifically, we found that participants who produced multiple, alternating rotating gestures were the most likely to solve the problem utilizing transformational justifications. This finding aligns with past research indicating that simulation of the gear movements was associated with participants' generating an alternation approach (Boncoddo et al., 2010). Additionally, this finding adds to the body of work that shows that gesture is closely related to, and can perhaps affect, learners' approaches to mathematical justification. The fact that certain, identifiable gestures were closely aligned with valid justifications highlights the importance of considering both speech and gesture when examining mathematical reasoning. Producing these gestures may be a key aspect of students' understanding and adopting the mathematical insights behind the gear problem.

Participants who produced ticking gestures along with rotating gestures with alternation were most likely to produce transformational justifications, raising the possibility that ticking motions are a potentially useful form of abstraction for the concept of parity. Rotating gestures and ticking gestures may each play different roles in supporting mathematical reasoning, and when used in combination they may be especially powerful. In this light, it is interesting that participants in the relevant-action-with-projection condition were especially *unlikely* to use ticking gestures. The high rate of multiple, alternating gestures, but low rate of ticking gestures in this condition suggests that projection may have encouraged participants to focus too strongly on the concrete movement of the gears, making it more difficult for them to make the mathematical abstraction. In other words, being directed to perform alternating ticking gestures and being explicitly told of their relevance to the gear problem may have made abstraction to the even/odd pattern more difficult, as it focused participants on concrete, perceptual aspects of the situation (e.g., Kaminski, Sloutsky, & Heckler, 2005).

This work has important implications for broadening perspectives on mathematical thinking and learning. While students in mathematics classrooms may or may not be involved in gear tasks specifically, we have shown that certain kinds of gestures can be strongly associated with producing transformational proofs. Thus, it may be beneficial for teachers to encourage students to gesture while reasoning, and to pay close attention to students' gestures to look for key aspects of their reasoning processes and current levels of understanding.

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