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Abstract: Classical Lie algebras, like *sl*(2,**C**) can be represented using differential operators that act on polynomial space. These operators will take a different form when they are used on the space of polynomials of several variables and when the differentials are taken to be of higher order. We recall some known realizations and discuss possible deformations. In our two-parameter case we describe decomposition into indecomposable components.

Introduction: Representation theory is a branch of mathematics that looks at algebraic structures and represents them as linear transformations of a vector space. The goal of this paper is to study the representations of the Lie algebra *sl*(2,**C**) over the space of complex polynomials of one variables, **C**[x], and then again over the space of complex polynomials of two variables, **C**[x,y], in the general linear algebra associated with each space. We present two possible representations that will lead to the decomposition of the space. We find that when these representations take place over **C**[x,y], the operators of *e*, *f*, and *h* are actually constructed via the tensor product of **C**[x] and **C**[y]. It is also found that adding additional parameters to each representation over **C**[x], we will preserve the relations of the Lie bracket, an essential feature of the representation in this context. By adding these additional parameters we are able to see how this representation acts on the individual monomials of each space that form the space's basis. By doing this we are able to construct both finite and infinite dimensional modules which result in creating a decomposition of both spaces. We discuss under what circumstances these decompositions will result in finite or infinite dimensional modules. Once we have the decompositions for both spaces we are then able to prove the main result: $\boldsymbol{C}[x,y] \cong \oplus_{n=0}^{n=\infty} \boldsymbol{C}[x]_{\lambda+\mu-2n}$, where λ+μ-2n is an eigenvector which defines what can be contained in each module. The proof is outlined in several steps; first we show that there is an isomorphism between indecomposable modules of each space, followed by showing how some module of **C**[x,y] can be written as a sum of elements from finite dimensional modules (we decompose **C**[x,y] into these modules in Theorem 2) which will be isomorphic to modules of **C**[x]. The final step is then to show that these components must be uniquely determined, thereby making the sum direct.

We begin the paper by examining certain pieces of background information to help the unfamiliar reader get caught up to speed. This background will consist mostly of definitions and will serve to introduce vocabulary. It should be noted that these definitions are found in most texts on the subject but these are from the text, *An Introduction to Lie Algebras* by Karin Erdmann and Mark J. Wildon.

Background

We begin by defining fields, vector spaces, and Lie algebras and provide examples and counter examples. Certain vocabulary is introduced and the reader is reminded that only a prerequisite of abstract algebra is required to follow the material.

1. Definition: A field is a set $\boldsymbol{F}$ that is a commutative group with respect to two compatible operations, namely addition(+) and multiplication(·) that satisfy the following properties:

    a. $\forall\ a, b \in \boldsymbol{F}$ both $a + b$ and $a \cdot b$ are in $\boldsymbol{F}$. This is the property of additive and multiplicative closure.

    b. $\forall\ a, b$, and $c \in \boldsymbol{F}$, we have: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. This is the property of additive and multiplicative associativity

    c. $\forall\ a$ and b in $\boldsymbol{F}$, the following equalities hold: $a + b = b + a$ and $a \cdot b = b \cdot a$. This is multiplicative and additive commutivity.

    d. $\exists$ an element of $\boldsymbol{F}$, called the *additive identity* element and denoted by 0, such that $\forall\ a$ in F, $a + 0 = a$. Likewise, there is an element, called the *multiplicative identity* element and denoted by 1, such that $\forall\ a$ in $\boldsymbol{F}$, $a \cdot 1 = a$.

    e. $\forall\ a$ in $\boldsymbol{F}$, there exists an element $-a$ in $\boldsymbol{F}$, such that $a + (-a) = 0$. Similarly, for any a in $\boldsymbol{F}$ other than 0, there exists an element a^{-1} in F, such that $a \cdot a^{-1} = 1$. These are known as the additive and multiplicative inverses.

    f. $\forall\ a, b$ and c in $\boldsymbol{F}$, the following equality holds: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. This is known as the distributive property.

Examples of fields include the rational numbers, real numbers, and the complex numbers denoted $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$. The integers, for instance would not be considered a field since they lack multiplicative inverses. Now we define a vector space, which one will note has very similar properties to that of a field, but with some subtle, yet important differences.

2. Definition: A vector space V over a field $\boldsymbol{F}$, known as an $\boldsymbol{F}$-vector space, is a set, whose elements are vectors, that satisfy these axioms:

    a. $\forall\ \mathbf{u}, \mathbf{v}$, and $\mathbf{w} \in \mathrm{V}$, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

    b. $\forall\ \mathbf{v}$ and $\mathbf{w} \in \mathrm{V}$, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

c. ∀ **v** and **w** in V and ∀ a in ***F***, a(**v** + **w**) = a**v** + a**w**

d. ∀ **v** ∈ V, there exists an element **w** ∈ *V*, called the additive inverse of **v**, such that **v** + **w** = **0**. The additive inverse is denoted –**v**

e. V must contain an additive identity element know as the zero vector, such that ∀ **v** ∈ V, **v** + 0 = **v**

f. ∀ **v** in V and ∀ a, b in ***F***, (a + b)**v** = a**v** + b**v**

g. ∀ **v** in V and ∀ a, b in ***F***, a(b**v**) = (ab)**v**

h. 1**v** = **v**, where 1 denotes the multiplicative identity in ***F***\

i. Associated with vector addition and scalar multiplication we have closure.

Certain examples of Vector Spaces include the space of polynomial functions and any Euclidean Space. Associated with each vector space is a basis which is a set of vectors which, when put in any linear combination, can express any vector in the vector space. So in polynomial space of one variable, any vector can be written in the form $\boldsymbol{v} = \sum_{n=0}^{\boldsymbol{n}=\infty} a_n x^n$, because any polynomials is expressed as the sum of monomials with coefficients from whatever field the space is over. This point will be important when proving our final result.

3. Definition: Let ***F*** be a field. A Lie algebra over ***F*** is an ***F***-vector space L, together with a bilinear map, called the Lie bracket:

$$L \times L \longrightarrow L, \qquad (x, y) \rightarrow [x, y],$$

that satisfies the following properties:

a. [x,x] = 0 for all x ∈ L

b. [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, this is known as the Jacobi Identity.

Since the Lie bracket is bilinear, we have the following relation:

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$$

This implies that [x,y]=-[y,x] for all x, y in L. Now that we have our definition of a Lie algebra we can describe a few examples, the first one being the set of linear transformations from V→V. This Lie algebra is known as the general linear algebra and is denoted *gl*(V). The Lie bracket in this case is defined via a composition of maps:

$[x, y] = x \circ y - y \circ x,$ for all x,y in *gl*(V)

We know that the composition of linear maps will again be linear and that the difference of two linear maps will also yield another linear map so we can say that $x \circ y - y \circ x$ will again be an element of *gl*(V). Next we proceed to define a Lie algebra homomorphism:

4. Definition: Let L_1 and L_2 be Lie algebras. We say that the map $\Omega : L_1 \rightarrow L_2$ is a Lie algebra homomorphism if Ω is linear and

$$\Omega([x, y]) = [\Omega(x), \Omega(y)] \text{ for all x,y in } L_1.$$

Notice that the first Lie bracket is taken on elements from L_1, while the second bracket it taken on element from L_2. Of course if this map is also bijective then we can call it an isomorphism.

The last two things that we need to define are representations and modules. Representations and modules allow us to view abstract Lie algebras in very concrete ways to help to try to understand their structure. One of the most interesting things about Representations is there applications in other areas of mathematics and physics as we will illustrate with an example following the definition:

5. Definition: Let L be a Lie algebra over a field ***F*** and let V be some vector space over the same field ***F*** . A representation of L is a Lie algebra homomorphism

$$\varphi: L \rightarrow gl(V)$$

Now we can mention an example of a representation that is commonly seen in quantum physics. If we look at the angular momentum operators L_x, L_y, and L_z (these are most Lie algebras but angular momentum operators) we can describe their commutator relations as they are demonstrated in the context of quantum physics:

$$[L_x, L_y] = ihL_z, [L_y, L_z] = ihL_x, [L_z, L_x] = ihL_y.$$

One can see a direct analogue from the commutator relations of the angular momentum operators to Lie bracket operations associated with the space of rotations in $\boldsymbol{R}^3$, the Lie algebra *so*(3):

$$[x. y] = z, [y, z] = x, [z, x] = y,$$

The commutator relations of the x, y, and z components of the angular momentum operator in quantum physics form a representation of some 3-dimensional complex Lie algebra, but this is nothing other than the complexification of *so*(3). This example is merely to illustrate that if the operator, in this case angular momentum, is linear, than the only thing that needs to be checked is the Lie bracket relations are preserved, which they are up to an isomorphism.

We now begin our look at the alternative approach of representing Lie algebras, this time as modules. We start with a definition:

6. Definition: Let L be a Lie algebra over the field ***F***. A Lie module, or L-module, is a finite-dimensional ***F***-vector space V with a map defined as follows:

$$L \times V \to V, \qquad (x, v) \mapsto x \cdot v$$

This map must satisfy the following conditions:

a. $(\lambda x + \mu y) \cdot v = \lambda(x \cdot v) + \mu(y \cdot v)$,

b. $x \cdot (\lambda v + \mu w) = \lambda(x \cdot v) + \mu(x \cdot w)$,

c. $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$, for all x, y ∈ L, v, w ∈ V, and λ, μ ∈ ***F***.

If we look at parts a and b of the definition we can say that this mapping, $(x, v) \mapsto x \cdot v$, is bilinear. The sort of elementary example of this is to look at a vector space V and some Lie subalgebra of gl(v). It is easy to verify that V is an L-module when $x \cdot v$ is the image of v under the linear map x. One of the perfect synchronicities of mathematics is that we are able to use both Lie modules and representations to describe the same structures. If we let $\varphi : L \to gl(\mathrm{V})$ be a representation, we can construct an L-module out of V by the following mapping:

$$x \cdot v \coloneqq \varphi(x)(v) \quad \text{for some x} \in \text{L, v} \in \text{V}$$

To show that with this mapping we can go from a representation to a Lie module we just need to check that axioms a, b, and c for Lie modules are satisfied.

Proof: (a) Since φ is linear, we have:

$$(\lambda x + \mu y) \cdot v = \varphi(\lambda x + \mu y) \cdot v = \big(\lambda\varphi(x) + \mu\varphi(y)\big)(v)$$

$$= \lambda\varphi(x)(v) + \mu\varphi(y)(v) = \lambda(x \cdot v) + \mu(y \cdot v).$$

Axiom b is verified in the same fashion:

$$x \cdot (\lambda v + \mu w) = \varphi(x)(\lambda v + \mu w) = \lambda\varphi(x)(v) + \mu\varphi(x)(w) = \lambda(x \cdot v) + \mu(x \cdot w)$$

For axiom c we employ the definition of the mapping and the fact that φ is a Lie homomorphism.

$$[x, y] \cdot v = \varphi([x, y])(v) = [\varphi(x), \varphi(y)](v)$$

We know that the Lie bracket in gl(V) is the commutator of linear maps, so we have:

$$\varphi(x)\big(\varphi(y)(v)\big) - \varphi(y)\big(\varphi(x)(v)\big) = x \cdot (y \cdot v) - y \cdot (x \cdot v) \blacksquare$$

We can also talk about the converse process. Let V be an L-module, then we can say that V as a representation of L. We define a new map for φ:

$$\varphi : L \to gl(\mathrm{V}), \ \varphi(x)(v) \mapsto x \cdot v, \text{ for all x} \in \text{L, v} \in \text{V}$$

We will show that this is also a Lie algebra homomorphism:

Proof: The action of φ is clearly linear, so we only need to show that

$$\varphi([x,y])(v) = [\varphi(x), \varphi(y)] \cdot v \text{ for all x and y in L:}$$

$$\varphi([x,y])(v) = [x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

$$= x \cdot \big(\varphi(y)(v)\big) - y \cdot \big(\varphi(x)(v)\big) = \varphi(x)\big(\varphi(y)(v)\big) - \varphi(y)\big(\varphi(x)(v)\big)$$

$$= [\varphi(x), \varphi(y)] \cdot v \quad \blacksquare$$

It is an important feature of representations and modules that we are able to go back and forth between these two ways of expression. One thing that we notice is that it is sometimes advantageous to utilizes the framework of modules since it allows for certain concepts to appear more natural due to simpler notations, whereas we will find, at times, that it is useful to have an explicit homomorphism to work with. We will see that this will be necessary for the proof of the final result. We conclude the background with three definitions and a proof.

7. Definition: Let V be an L-module for some Lie algebra L. A submodule of V is a subspace W of V that is invariant under the action of L. This implies that for each $x \in L$ and for each $w \in W$, we have $x \cdot w \in W$. The analog for representations is called a subrepresentation.

For an example we can show that under the adjoint representation we turn a Lie algebra L into an L-module where the submodules of L are exactly the ideals of L.

Proof: Let L be a Lie algebra. We define the adjoint representation is this way:

$$\text{ad} : L \to gl(\text{L}), \qquad \text{ad}(x)y = [x,y]$$

Next we describe what it means to be an ideal:

8. Definition: Let L be a Lie algebra. An ideal I of L is a subspace of L such that

$$[x,y] \in I \text{ for all } x \in L \text{ and for all } y \in I$$

Now we say, let V be L with the L-module structure on V given by the adjoint representation of L and let W be some submodule of V. For some x in V and some w in W we have:

$$\text{ad}(x)w = [x,w]$$

Because W is a submodule under this operation [*x*,*w*], we know that [*x*,*w*] is contained in W. But that precisely what it means for W to be an ideal. ■

Now that we have the idea of what submodules are, we write our last definition:

9. Definition: the Lie module (or representation) V is said to be irreducible, or simple, if it is non-zero and the only submodules (or subrepresentations) it contains are {0} and V.

Thesis Problem: Now that we have a basic understanding of the necessary terminology, we are ready to look at the irreducible modules of *sl*(2,***C***). It is probably ideal to start by saying exactly what *sl*(2,***C***) is. The Lie algebra *sl*(2,***C***) is the space of 2 × 2 matrices with trace 0. It is easily checked that product of two trace-zero matrices will be again trace-zero, as will the difference, so that we have closure under the Lie bracket operation. We will begin by constructing a family of irreducible representation of *sl*(2,***C***) in the space of polynomials with complex coefficient, ***C***[x]. It should be noted that the basis of this space will be the infinite set of monomials {1, x, x^2, x^3…}. We begin by describing the basis of *sl*(2,***C***):

Proposition 1: The Lie algebra, *sl*(2,***C***), has a basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Proof: an arbitrary element A of *sl*(2,***C***) is of the form $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $a, b, c \in \boldsymbol{C}$. But we can rewrite A as a sum of e, f , and h in this way:

$$A = b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = b(e) + c(f) + a(h)$$

It should also be clear that we cannot write either e, f, or h as a linear combination of the other two, therefore e, f, and h form a basis of *sl*(2,***C***).■

It is easily seen that the commutator relations between these basis elements are as follows:

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f$$

We these commutator relations, we would first like to describe the decomposition of ***C***[x,y] into finite-dimensional irreducible submodules. This example is found in the text, *Introduction to Lie Algebras* by Karen Erdmann and Mark J. Wildon and provides a good framework for what we will do later when we attempt to describe the decomposition into both finite and infinite irreducible submodules. Let us begin.

We should start by having a more concrete definition of ***C***[x,y]. We can define ***C***[x,y] as the tensor product of ***C***[x] and ***C***[y]. We of course describe this as the tensor product over a field ***F***, which in this case will be the complex number.:

$$\boldsymbol{C}[x] \otimes \boldsymbol{C}[y] = \boldsymbol{C}[x, y], \qquad (cx^a, dy^b) \to cd(x^a \cdot y^b), \qquad x^a \in \boldsymbol{C}[x], y^b \in \boldsymbol{C}[y], \qquad c, d \in \boldsymbol{C}$$

Multiplication to obtain cross-term basis elements would be carried out in this fashion:

$$(a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n)(b_0 + b_1y + b_2y^2 + \cdots + b_{n-1}y^{n-1} + b_ny^n) =$$

$$= \sum_{m=0}^{m=m} a_m x^m \left(\sum_{n=0}^{n=n} b_n\, y^n \right)$$

This method of polynomial multiplication is something that we all learned in algebra, but what we do not learn until later is that this method of cross-term multiplication does in fact define a tensor product on these two one-variable spaces. We of course take all cross-terms to be linearly independent of each other and as such form the basis of the space. We now form a grade on this space by looking at the space of all basis elements of degree d.

We define the space $V_d = \text{Span}\{x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}$. This is the space of two-variable polynomials where each monomial is of degree d. We should also note that each V_d is of dimension d-1. We can take each V_d and make it an $sl(2,\boldsymbol{C})$-module by defining a Lie algebra homomorphism:

$$\gamma : sl(2, \boldsymbol{C}) \to gl(V_d)$$

We know that since γ is linear and that $sl(2,\boldsymbol{C})$ is spanned by *e*, *f*, and *h*, the γ will be defined on the space once it is defined on *e*, *f*, and *h*. We define in this way:

$$\gamma(e) \equiv x\frac{\partial}{\partial y}, \qquad \gamma(f) \equiv y\frac{\partial}{\partial x}, \qquad \gamma(h) \equiv x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

Since we will not be mentioning the matrices e, f, and h again, we shall neglect to write γ when referring to these operators and shall hereby refer to these operators as *e*, *f*, and *h*. We first prove that with these definitions, γ is a representation.

Proposition 2: The map γ is a representation of $sl(2,\boldsymbol{C})$.

Proof: These operators are the compositions of linear maps so they themselves are linear. Because of this we only have to check that the Lie bracket relations are maintained. This will be done by demonstrating how the commutator will act on some basis element $x^a y^b$:

$$[e,f](x^a y^b) = e \cdot f \cdot x^a y^b - f \cdot e \cdot x^a y^b = e \cdot a x^{a-1} y^{b+1} - f \cdot b x^{a+1} y^{b-1}$$

$$= a(b+1)x^a y^b - b(a+1)x^a y^b = (a-b)x^a y^b = h \cdot x^a y^b$$

We should note that the action of *h* on some basis element of V_d will just produce a scalar so we say that *h* acts diagonally on V_d. Now we check *h* and *f*:

$$[h,f] = h \cdot f \cdot x^a y^b - f \cdot h \cdot x^a y^b = h \cdot a x^{a-1} y^{b+1} - f \cdot (a-b) x^a y^b$$

$$= \big((a-1)-(b+1)\big) a x^{a-1} y^{b-1} - a(a-b) x^{a-1} y^{b+1}$$

$$= (a^2 - a - ba - a - a^2 + ab) x^{a-1} y^{b+1} = -2a x^{a-1} y^{b+1} = -2f \cdot x^a y^b$$

And finally we check *h* and *e*:

$$[h, e] = h \cdot e \cdot x^a y^b - e \cdot h \cdot x^a y^b = h \cdot b x^{a+1} y^{b-1} - e \cdot (a-b) x^a y^b$$

$$= b((a+1) - (b-1) x^{a+1} y^{b-1} - b(a-b) x^{a+1} y^{b-1}$$

$$= (ba + b - b^2 + b - ba + b^2) x^{a+1} y^{b-1} =$$

$$2b x^{a+1} y^{b-1} = 2e \cdot x^a y^b$$

We note that if either a or b is zero, the relations still hold true. At this point we can show what the matrices for the actions of e, f, and h look like. This is done by examining how these operators act on each basis vector. By doing this we will be able to verify again that h acts diagonally on elements of V_d:

$$e = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} & 0 & 0 & \cdots & 0 & 0 \\ & d & 0 & \cdots & 0 & 0 \\ 0 & & d-1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} d & 0 & \cdots & 0 & 0 \\ 0 & d-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d+2 & 0 \\ 0 & 0 & \cdots & 0 & -d \end{pmatrix}$$

We note that these matrices are of dimension d+1, as expected, and the *h* is a diagonal matrix with entries $d - 2k$ where $k = 0, 1, \ldots, d$. We can also give a diagram to show how *e* and *f* move us from one basis element to another:

$$y^d \xrightarrow{e} xy^{d-1} \xrightarrow{e} \ldots \xrightarrow{e} x^{d-2}y^2 \xrightarrow{e} x^{d-1}y \xrightarrow{e} x^d \xrightarrow{e} 0$$

$$0 \xleftarrow{f} y^d \xleftarrow{f} xy^{d-1} \xleftarrow{f} \ldots \xleftarrow{f} x^{d-2}y^2 \xleftarrow{f} x^{d-1}y \xleftarrow{f} x^d$$

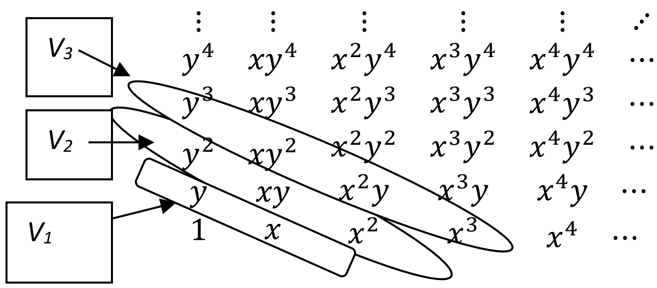
Of course these actions will introduce a scalar for each term, which we neglect to show in the diagram. We have shown that the action of *h* will *only* introduce a scalar and not change the basis element upon which it acts. We say that these vectors will be *h*-eigenvectors. Another fact that we can draw from this diagram is that by acting with *e* and *f* on some basis vector $x^a y^b$, we can generate the whole basis of V_d. This can be proved by showing that these modules are irreducible.

Theorem 1: The space V_d is an irreducible *sl*(2,***C***)-module.

Proof: This will be a proof by contradiction.

We assume that U is some nonzero $sl(2,\boldsymbol{C})$-submodule of V_d. U must contain some nonzero element u, which is some polynomial of degree d. We know that since u is in U, $h \cdot u \in U$ for all u. We know that since any element in U is also in V_d, h will act diagonally on U. We know that h will create an eigenspace that will be spanned by a single monomial of degree d, so U must contain this monomial. Then by virtue of our diagram above, U must also contain the basis of V_d. So $U = V_d$. We also point out that if we look at the matrices which represent the action of e and f, we can see that each nonzero entry corresponds to the scalar that is obtain by acting with either the e or f operator, depending upon which matrix we look at. For example, we can see that $f \cdot x^d = d \cdot x^{d-1}$. Based on these matrices, we can see that there will be no instances where these scalars will be zero, unless of course d is zero. With this last fact we can say that these modules will be irreducible. ■

We can actually show visually how $\boldsymbol{C}$[x,y] will decompose into these V_d via another diagram:



Having shown this diagram we can prove that $\boldsymbol{C}[x,y] \cong \oplus_{d=1}^{d=\infty} V_d$

Theorem 2: We have $\boldsymbol{C}[x,y] \cong \oplus_{d=0}^{d=\infty} V_d$.

Proof: The space $\boldsymbol{C}[x,y]$ contains some vector called $v_n = \sum_{a=0,b=0}^{a=n,b=m} cx^a y^b$. The actions of these operators will preserve the degree of where, $\deg(x^a y^b) = a + b$. So the sum $\sum_{d=0}^{d=\infty} V_d$ living inside $\boldsymbol{C}$[x,y] must be direct. Also, monomials of the form $x^a y^b$ form a basis of C[x,y]. Thus $\boldsymbol{C}[x,y] = \oplus_{d=0}^{d=\infty} V_d$. ■

We will later do a more involved proof that gets us our final result, but it will rely on many of the same ideas. For now we will change our representation to have it act on $\boldsymbol{C}$[x] and we will then show that we can create both finite and infinite dimensional modules

Since we have shown that e, f, and h form a basis we are in a position to make $\boldsymbol{C}$[x] into a $sl(2,\boldsymbol{C})$-module. We do this by specifying a Lie algebra homomorphism $\theta : sl(2,\boldsymbol{C}) \to gl(\boldsymbol{C}[x])$. Since $sl(2,\boldsymbol{C})$ is linearly spanned by the matrices e, f, and h, we will only have to describe how θ acts on the basis, so we define $\theta(e), \theta(f), \theta(h)$ in this way:

$$\theta(e) \equiv -\frac{d}{dx}, \qquad \theta(f) \equiv x^2 \frac{d}{dx}, \qquad \theta(h) \equiv -2x \frac{d}{dx}$$

If we look at h we can tell that if h acts on some basis element of $\boldsymbol{C}$[x], say x^a, we can tell that $h \cdot x^a = -2ax^a$, so since this is just a constant, we say h acts diagonally on $\boldsymbol{C}$[x]. We can also note that the way that we have defined e, the overall degree of the polynomial goes down by one, where as the overall degree of the polynomial will go up by one if we apply f. Now that we have the definition of our representation we must check that it is in fact a representation and we mention our first theorem.

Proposition 3: The map θ is a representation of $sl(2,\boldsymbol{C})$

Proof: With these definitions θ is a linear map. Both the actions of x^a and some differential will each act linearly so taking a product should also act linearly. This means that the only thing that we need to check is to make sure that θ preserves the Lie brackets:

1. $[e,f] = h$: So we need to check that $[\theta(e),\theta(f)] = \theta(h)$:

$$[\theta(e),\theta(f)] = -\frac{d}{dx}x^2\frac{d}{dx} + x^2\frac{d}{dx}\frac{d}{dx} = -2x\frac{d}{dx} - x^2\frac{d^2}{dx^2} + x^2\frac{d^2}{dx^2}$$

$$= -2x\frac{d}{dx} = \theta(h)$$

2. Similarly we check that $[\theta(h),\theta(e)] = 2\theta(e)$:

$$[\theta(h),\theta(e)] = -2x\frac{d}{dx}\cdot\left(-\frac{d}{dx}\right) + \frac{d}{dx}\cdot\left(-2x\ \frac{d}{dx}\right) = 2x\left(\frac{d^2}{dx^2}\right) + (-2)\left(\frac{d}{dx} + x\frac{d^2}{dx^2}\right)$$

$$= 2x\frac{d^2}{dx^2} - 2\frac{d}{dx} - 2x\frac{d^2}{dx^2} = -2\frac{d}{dx} = 2\theta(e)$$

3. For the last relation we check $[\theta(h),\theta(f)] = -2\theta(f)$:

$$[\theta(h),\theta(f)] = -2x\frac{d}{dx}x^2\frac{d}{dx} + x^2\frac{d}{dx}2x\frac{d}{dx} = -2x\left(2x\frac{d}{dx} + x^2\frac{d^2}{dx^2}\right) + 2x^2\left(\frac{d}{dx} + x\frac{d^2}{dx^2}\right)$$

$$-4x^2\frac{d}{dx} - 2x^3\frac{d^2}{dx^2} + 2x^2\frac{d}{dx} + 2x^3\frac{d^2}{dx^2} = -2x^2\frac{d}{dx} = -2\theta(f)$$

Now that we have verified that the commutator relations remain intact after the map, we know that θ must be a representation. ■

One thing that we would like to look at would be how the matrices that correspond to the actions of e, f, and h would look. The way that we do this is by examining the action of basis elements of $\boldsymbol{C}$[x]:

$$\theta(e) = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 2 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \theta(f) = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \theta(h) = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & -2 & 0 & \cdots \\ 0 & 0 & -4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

From now on I will refrain from using $\theta(e), \theta(f),$ and $\theta(h)$ and refer to them by their original names, *e*, *f* , and *h*, since we will not be referring to the 2 x 2 matrices for which these names were originally used. Let me make clear what these matrices are. We are trying to find out how *e*, *f*, and *h* act on the basis of ***C***[x]. So in the matrix of e we note that when it acts on the basis vector of 1, we get zero, whereas if it acts on x, we get the value of one, and if we use *e* on x^2 we get $2 \cdot x$. This pattern continues in each operator, where the values that you see in the entries merely refer to the constant in front of each term. Of course since the basis of ***C***[x] is infinite dimensional, so to should be these matrices. And again we have a physical example of *h* being diagonal. We can now construct our first module. We should note that this is not a very interesting example but it will be a good chance to provide an example of reducibility. Of course in a few moments we will generalize this, and this example will be handled within that context:

Theorem 3: With our definitions of *e*, *f*, and *h*, the module ***C***[x] will be reducible.

Proof: We know that ***C***[x] has a constant basis element. If we look at just the constant term, some complex number, a_0, we can show that scalars will form a submodule. If we apply f, then we get 0, as with *e*. This implies that we can't get any of the higher basis elements if all we are working with is the constant polynomial. This gives us the module with only the constant basis element contributing to the span so we have the simplest nontrivial submodule, which makes ***C***[x] is reducible. ■

At this point we would like change our definitions of *e*, *f*, and *h* in such a way that we will be able to construct bigger submodules of ***C***[x]. We can show that by the addition of a parameter term, we will be able to show that ***C***[x] will decompose into finite dimensional submodules. We define *e*, *f*, and *h* in the following manner:

$$e = -\frac{d}{dx}, \qquad f = x^2\frac{d}{dx} - mx, \qquad h = -2x\frac{d}{dx} + m, \qquad \text{for some m in } \boldsymbol{C}$$

Proposition 4: The new definitions of *e*, *f*, and *h* will form a representation of *sl*(2,***C***).

Proof: This will be perfectly analogous to the proof for the first definition of θ. It is clear that each component is linear, so we only check that the Lie brackets for *e*, *f*, and *h* are preserved:

1. First we check that $[e, f] = h$:

$$[e, f] = -\frac{d}{dx}\left(x^2\frac{d}{dx} - mx\right) + \left(x^2\frac{d}{dx} - mx\right)\frac{d}{dx} =$$

$$-2x\frac{d}{dx}-x^2\frac{d^2}{dx^2}+m+mx\frac{d}{dx}+x^2\frac{d^2}{dx^2}-mx\frac{d}{dx}=-2x\frac{d}{dx}+m=h$$

2. Next we show that $[h,e]=2e$:

$$[h,e]=\left(-2x\frac{d}{dx}+m\right)\left(-\frac{d}{dx}\right)+\left(\frac{d}{dx}\right)\left(-2x\frac{d}{dx}+m\right)=$$

$$2x\frac{d^2}{dx^2}-m\frac{d}{dx}-2\left(\frac{d}{dx}+x\frac{d^2}{dx^2}\right)+m\frac{d}{dx}+0=-2\frac{d}{dx}=2e$$

3. Finally we show that $[h,f]=-2f$:

$$[h,f]=\left(-2x\frac{d}{dx}+m\right)\left(x^2\frac{d}{dx}-mx\right)-\left(x^2\frac{d}{dx}-mx\right)\left(-2x\frac{d}{dx}+m\right)=$$

$$[-2x\left(2x\frac{d}{dx}+x^2\frac{d^2}{dx^2}-m\right)+mx^2\frac{d}{dx}-m^2x+2x^2\frac{d}{dx}+2x^3\frac{d^2}{dx^2}-mx^2\frac{d}{dx}$$

$$-2mx^2\frac{d}{dx}+m^2x]=-2x^2\frac{d}{dx}+2mx=-2\left(x^2\frac{d}{dx}-mx\right)=-2f \quad \blacksquare$$

Now that we have arrived at the same conclusion that we did earlier with regards to maintaining the commutator relations, we can update what our matrices for the actions of *e, f,* and *h* should look like:

$$\theta(e)=\begin{pmatrix}0 & 1 & 0 & \cdots\\ 0 & 0 & 2 & \cdots\\ 0 & 0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix},$$

$$\theta(f)=\begin{pmatrix}0 & 0 & 0 & \cdots\\ -m & 0 & 0 & \cdots\\ 0 & -m+1 & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}, \quad \theta(h)=\begin{pmatrix}m & 0 & 0 & \cdots\\ 0 & -2+m & 0 & \cdots\\ 0 & 0 & -4+m & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}$$

At this point we present one of the small, yet important, results in the paper:

Theorem 4: Under the new definition of *e, f,* and *h,* the complex polynomial space ***C***[x], will be reducible, iff *m* is a nonnegative integer.

Proof: (⟵) Here we assume *m* is some nonnegative integer. ***C***[x] contains some vector *v* of the form $\sum_{i=0}^{i=n} b_i x^i \neq 0$, where b_i is in ***C*** by definition. This polynomial will have $n+1$ terms. We will show in a similar manner to the proof of theorem 3 that we must eventually be left with a monomial after at most n applications of e.

Let n = 0: If we let n = 0 then of course we have just the constant term, b_0. If we apply f, then we get 0 + mx. If we let m = 0, then of course we get zero. This gives us the module with only the constant basis element contributing to the span so we have the simplest nontrivial submodule, which makes ***C***[x] is reducible.

Now we let $n = n$. So in U we have a polynomial of n + 1 terms, one of which must be a constant in ***C***. One application of e will make the constant term equal 0 (if it doesn't already), so the overall number of terms drops by one, as does the overall degree. We notice that if we apply e at most n-times we will obtain a constant term, and again we note that each application of e will produce some nonzero scalar in from of each term, but the movement between basis elements looks like this:

$$0 \overset{e}{\leftarrow} 1 \overset{e}{\leftarrow} x \overset{e}{\leftarrow} x^2 \overset{e}{\leftarrow} \cdots \overset{e}{\leftarrow} x^a \overset{e}{\leftarrow} \cdots$$

Now that we have just our constant term, we can build our basis back up with applications of f:

$$1 \overset{f}{\rightarrow} x \overset{f}{\rightarrow} x^2 \overset{f}{\rightarrow} \cdots \overset{f}{\rightarrow} x^a \overset{f}{\rightarrow} \cdots$$

When we do this we find that our scalar that we obtain from the action of f determines whether or not ***C***[x] will be reducible. We demonstrate this on some general basis element, x^a, where a must be nonnegative. We know that we can get this term for some a with applications of f as described above:

$$f \cdot x^a = x^2 \frac{d}{dx} x^a - m x^{a+1} = (a-m)x^{a+1}$$

We know that f acts linearly so this shows that if our f operator has $m = a$ where a is the degree of our polynomial, we will be unable to get any term of degree a + 1 since the action of f will distribute over each one and produce a zero in front of these terms. So if m is some nonnegative integer a, we can generate a basis like this:

$$1 \overset{f}{\rightarrow} x \overset{f}{\rightarrow} x^2 \overset{f}{\rightarrow} \cdots \overset{f}{\rightarrow} x^a \overset{f}{\rightarrow} 0$$

Now that we have a basis for our submodule so we can define $\boldsymbol{C}[x]_m$ in this way:

$$\boldsymbol{C}[x]_m = Span\{1, x, x^2, \dots, x^m\}, \qquad \text{when } m \in \boldsymbol{N}^+$$

This implies that if $m = a$ ***C***[x] is reducible.

(→) For this direction we assume that ***C***[x] contains some nontrivial submodule U. U must contain either one or more than one of the nonzero vectors found in ***C***[x]. Since U ≠ ***C***[x], we know that U cannot contain the basis of ***C***[x]. Take the highest degree vector in U. We know that there must be a highest degree, because if we were missing say one basis element x^i, then we could just apply e to some higher degree basis element to get any basis element with lower degree. So we say this maximal degree is some degree a, where a is the highest power of any monomial found in U. We can act on this element with f. As described before, f will increase the degree of the polynomial. Since a is the maximal degree in U, we must have that $f \cdot x^a = 0$. Of course by the same reasoning that we showed above, this will only happen when $m = a$. Since a is the exponent of some polynomial, it must be a nonnegative integer.■

Of course, if we act with operators where m is not a nonnegative integer, then under those operations we will find that $\boldsymbol{C}$[x] is irreducible. Also we should quickly point out that some submodule $\boldsymbol{C}[x]_m$, as described above, will be of dimension $m + 1$.

At this point, we would like to turn our attention to complex polynomial space of two variables, x and y. This space will be denoted $\boldsymbol{C}$[x,y] and we will denote any submodule that we find (and we will find them) as $\boldsymbol{C}[x,y]_{parameter}$. We shall see that the parameter that we are looking for will in fact be an h-eigenvalue, but more on that later. First we must rewrite our e, f, and h operators in such a way that they preserve the Lie brackets in this two variable space. We write at such:

$$e \equiv -\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \qquad f \equiv x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - \lambda x - \mu y, \qquad h \equiv -2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} + \lambda + \mu,$$

for some λ, μ in $\boldsymbol{C}$

We can look at these operators and tell that the y-component is just repeating what the x-component does. This is because the module $\boldsymbol{C}$[x,y] is isomorphic as $sl(2,\boldsymbol{C})$-modules to the tensor product of $\boldsymbol{C}$[x] and $\boldsymbol{C}$[y], where the first tensor operator is given by is given by $e = -\frac{d}{dx}$, $f = x^2\frac{d}{dx} - mx$, $h = -2x\frac{d}{dx} + m$ with m = λ while the second replaces x with y and has m = μ. We show this to emphasize the fact that the action in $\boldsymbol{C}[x, y]$ is actually constructed via the tensor product of $\boldsymbol{C}[x]$ and $\boldsymbol{C}[y]$.

Proposition 5: Under this definition, e, f, and h [[define a representation of sl(2,C)]].

Proof: This proof will be identical to the first two propositions with regards to determining if the defined map is indeed a representation. Of course this time our map goes from $sl(2, \boldsymbol{C}) \rightarrow gl(\boldsymbol{C}[\mathrm{x}, \mathrm{y}])$. These maps, as we can see, are compositions of linear maps. Based on this we know that they are linear, so much like the first two propositions, we need to make sure that the Lie bracket relations are preserved. One difference in the structure of the proofs is that in this case we rely on the bilinearity of the Lie bracket. Also we should note that any $\left[-\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y}\right] = -y^2\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial x\partial y} = 0$ because the mixed partials commute:

1. $[e, f] = \left[-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} - \lambda x - \mu y\right] =$

$$\left[-\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right] + \left[-\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y}\right] + \left[\frac{\partial}{\partial x}, \lambda x\right] + \left[\frac{\partial}{\partial x}, \mu y\right] + \left[-\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x}\right] + \left[-\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y}\right]$$

$$+\left[\frac{\partial}{\partial y}, \lambda x\right] + \left[\frac{\partial}{\partial y}, \mu y\right] =$$

$$-2x\frac{\partial}{\partial x} - x^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial x^2} - 0 + \lambda + \lambda x\frac{\partial}{\partial x} - \lambda x\frac{\partial}{\partial x} + \mu y\frac{\partial}{\partial x} - \mu y\frac{\partial}{\partial x} + 0 - 2y\frac{\partial}{\partial y}$$

$$-y^2\frac{\partial^2}{\partial y^2} + y^2\frac{\partial^2}{\partial y^2} + \lambda x\frac{\partial}{\partial y} - \lambda x\frac{\partial}{\partial y} + \mu + \mu y\frac{\partial}{\partial y} - \mu y\frac{\partial}{\partial y}$$

After the dust settles and we cancel all the proper terms we are left with:

$$-2x\frac{\partial}{\partial x} + \lambda - 2y\frac{\partial}{\partial y} + \mu = -2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} + \lambda + \mu = h$$

One of the good features of using the bilinearity of the Lie bracket is that it allows us to see quickly which parts of the bracket will produce cancelations.

2. $[h, e] = \left[-2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} + \lambda + \mu, -\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right] = \left[-2x\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\right] + \left[-2x\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right] +$

$$\left[\lambda, -\frac{\partial}{\partial x}\right] + \left[\mu, -\frac{\partial}{\partial x}\right] + \left[-2y\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right] + \left[-2y\frac{\partial}{\partial y}, -\frac{\partial}{\partial y}\right] + \left[\lambda, -\frac{\partial}{\partial y}\right] + \left[\mu, -\frac{\partial}{\partial y}\right] =$$

$$2x\frac{\partial^2}{\partial x^2} - 2\frac{\partial}{\partial x} - 2x\frac{\partial^2}{\partial x^2} + 0 - \lambda\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial x} - \mu\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial x} + 0 + 2y\frac{\partial^2}{\partial y^2} - 2\frac{\partial}{\partial y}$$

$$-2y\frac{\partial^2}{\partial y^2} - \lambda\frac{\partial}{\partial y} + \lambda\frac{\partial}{\partial y} - \mu\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial y} =$$

$$-2\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} = 2e$$

So that I won't have to write out so much, I will indicate which components of the Lie bracket will equal zero by writing$(= 0)$ next to them:

3. $[h, f] = \left[-2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} + \lambda + \mu, x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} - \lambda x - \mu y\right] =$

$$= \left[-2x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}\right] + \left[-2x\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y}\right](=0) +$$

$$+\left[-2x\frac{\partial}{\partial x}, -\lambda x\right] + \left[-2x\frac{\partial}{\partial x}, -\mu y\right](=0) + \left[-2y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x}\right](=0)$$

$$+\left[-2y\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y}\right] + \left[-2y\frac{\partial}{\partial y}, -\lambda x\right](=0) +$$

$$\left[-2y\frac{\partial}{\partial y}, -\mu y\right] + \left[\lambda, x^2\frac{\partial}{\partial x}\right](=0) + \left[\lambda, y^2\frac{\partial}{\partial y}\right](=0) + [\lambda, -\lambda x](=0)$$

$$+[\lambda, -\mu y](=0) + \left[\mu, x^2\frac{\partial}{\partial x}\right](=0) + \left[\mu, y^2\frac{\partial}{\partial y}\right](=0) + [\mu, -\lambda x](=0) + [\mu, -\mu y](=0) =$$

$$-2x\left(2x\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial x^2}\right) + x^2\left(2\frac{\partial}{\partial x} + 2x\frac{\partial^2}{\partial x^2}\right) + 2x\lambda + 2x^2\lambda\frac{\partial}{\partial x} - 2\lambda x^2\frac{\partial}{\partial x}$$

$$-2y\left(2y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial y^2}\right) + y^2\left(2\frac{\partial}{\partial y} + 2y\frac{\partial^2}{\partial y^2}\right) + 2y\mu + 2y^2\mu\frac{\partial}{\partial y} - 2\mu y^2\frac{\partial}{\partial y} =$$

$$-4x^2\frac{\partial}{\partial x} - 2x^3\frac{\partial^2}{\partial x^2} + 2x^2\frac{\partial}{\partial x} + 2x^3\frac{\partial^2}{\partial x^2} - 4y^2\frac{\partial}{\partial y}$$

$$-2y^3\frac{\partial^2}{\partial y^2} + 2y^2\frac{\partial}{\partial y} + 2y^3\frac{\partial^2}{\partial y^2} + 2x\lambda + 2y\mu =$$

$$-2x^2\frac{\partial}{\partial x} - 2y^2\frac{\partial}{\partial y} + 2x\lambda + 2y\mu = -2f \quad \blacksquare$$

While this calculation seemed more involved, especially in part 3, we are able to employ the linearity property of the Lie bracket to show that we do indeed maintain the commutator relations. Now that we have the commutator relations, I would like introduce a specific family of vectors in $\boldsymbol{C}[x,y]$. Let $v_n = c \cdot (x-y)^n, c \in \boldsymbol{C}$. This family of binomials has the property that if e acts on them, we get zero, no matter what degree they are:

$$e \cdot v_n = -\frac{\partial}{\partial x} c \cdot (x-y)^n - \frac{\partial}{\partial y} c \cdot (x-y)^n = -c \cdot n(x-y)^{n-1} + c \cdot n(x-y)^{n-1} = 0$$

We also notice that this family of vector produces other interesting properties when acted on by h, which will exploit shortly. The action is as follows:

$$h \cdot v_n = c \cdot \left[-2x\frac{\partial}{\partial x}(x-y)^n - 2y\frac{\partial}{\partial y}(x-y)^n + \lambda(x-y)^n + \mu(x-y)^n\right]$$

$$= c \cdot [-2n(x-y)^n + (\lambda + \mu)(x-y)^n] = c \cdot (\lambda + \mu - 2n)(x-y)^n$$

This makes perfect sense that h should only produce a constant, given the other ways that we have seen h acting diagonally. Now we write a lemma that relies on the fact that $h \cdot v_n = bv_n$ for some *b* in ***C***. This lemma, as well at the proof, is found in Erdmann and Walker's *Introduction to Lie Algebras*.

Lemma 1: Suppose that *U* is a *sl*(2,***C***)-module and $v \in U$ is an eigenvector of *h* with eigenvalue *b*:

1. Either $e \cdot v = 0 \ or \ e \cdot v$ is an eigenvector of *h* with eigenvalue $b + 2$
2. Either $f \cdot v = 0 \ or \ f \cdot v$ is an eigenvector of *h* with eigenvalue $b - 2$

Proof: Since *U* is a representation of *sl*(2,***C***), we have

1. $h \cdot (e \cdot v) = e \cdot (h \cdot v) + [h, e] \cdot v = e \cdot (bv) + 2e \cdot v = (b + 2)e \cdot v$
2. $h \cdot (f \cdot v) = f \cdot (h \cdot v) + [h, f] \cdot v = f \cdot (bv) - 2f \cdot v = (b - 2)e \cdot v$ ■

We should note that when we apply *h*, we find that we obtain certain eigenspaces. These eigenspaces will be nothing other than our V_d which we mentioned earlier, where *d* is the degree of the monomial that *h* acts on. With this lemma we know that if we take some *v* in ***C***[x,y], we get can construct irreducible submodules. We also know this by the fact that $\boldsymbol{C}[x] \subset \boldsymbol{C}[x, y]$ and we already showed the irreducibility in the smaller space.

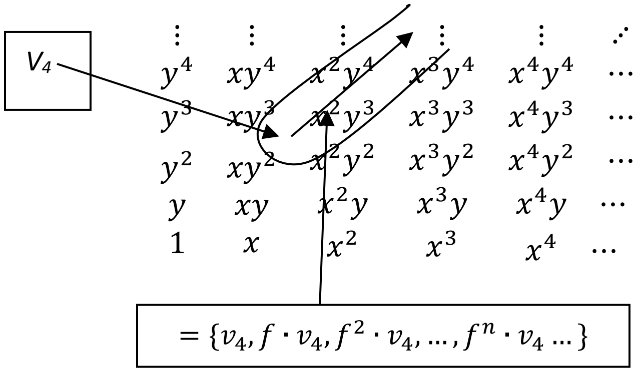
We are now ready to start the proof of the final result, but it will require several steps. So the way that the proof goes is this. We start with the most important step by showing that there are irreducible modules in $\boldsymbol{C}[x, y]$ that will be isomorphic to different irreducible modules in ***C***[x]. Once we have that we show that for some vector in ***C***[*x*,*y*], we can write is as the sum of uniquely determined components from different modules, which turns what was a summation into a direct sum. Let us begin:

Theorem 5: $\boldsymbol{C}[x, y]_{\lambda+\mu-2n} \cong \boldsymbol{C}[x]_{\lambda+\mu-2n}$for some λ, μ in ***C*** and *n* is some nonnegative integer.

Proof: We have a family of vectors that have the property $e \cdot v_n = 0$ and $h \cdot v_n = bv_n$. We can now consider this sequence of vectors:

$$v_n,\ f \cdot v_n,\ f^2 \cdot v_n, \ldots \in \boldsymbol{C}[x, y]$$

This sequence will either be infinite or finite depending on the choice of λ and μ in a manner similar to our value *m* in theorem 2. Now we claim that these vectors form a basis of $\boldsymbol{C}[x, y]_{\lambda+\mu-2n}$. We know from the lemma that was stated earlier that each of these vectors is an *h*-eigenvector that will produce a unique *h*-eigenvalue, so they are linearly independent. We can represent this with a diagram like we did before:



Of course our module would end if we found that we got the zero vector. This happens depending on our choice of lambda and mu. Next we must show that the span of these vectors is invariant. We know that if we act with *h* we only get a constant so we have invariance under *h* and we know that it will be invariant under *f* by construction, as it will just give you the next vector in the sequence. We now use an induction argument to show that we have invariance under *e*. The nice thing about this step is that it doesn't explicitly depend on the structure of the operators, but merely on the ways that *e*, *f*, and *h* relate to each other:

1. **Claim:** $e \cdot (f^j \cdot v_n) \in \text{Span}\{f^l \cdot v_n : 0 \leq l \leq j\}$

   Proof: If *j* = 0, we know that $e \cdot v_n = 0$. Now we assume this to be true for $l = j - 1$. We should also note that $e \cdot \left(f^j \cdot v_n\right) = (fe + h) \cdot (f^{j-1} \cdot v_n)$. This just comes from the fact that $h = [e, f] = ef - fe$. By our inductive hypothesis, $e \cdot (f^{j-1} \cdot v_n)$ is in $\text{Span}\{f^l \cdot v_n : 0 \leq l \leq j-1\}$. From this we know that $fe \cdot (f^{j-1} \cdot v_n)$ must also be in the Span of all $f^l \cdot v_n$ for all $l \leq j$. We also know that when *h* acts on $(f^{j-1} \cdot v_n)$ it will produce some eigenvalue so it too will be in the span. ■

Now we know that each of these vectors is linearly independent and that the applications of *e*, *f*, and *h* will allow us to stay within the space, we can create an explicit isomorphism between submodules of ***C***[x,y] and those of ***C***[x]. We define a map

$$\delta : \boldsymbol{C}[x, y]_{\lambda+\mu-2n} \to \boldsymbol{C}[x]_{\lambda+\mu-2n}$$

$$\delta(v_n) \equiv 1, \qquad \delta\left(f^j \cdot v_n\right) \equiv \tilde{f}^j \cdot 1, \text{where } \tilde{f} \text{ refers to the parametrized operator } f \text{ on } \boldsymbol{C}[x]$$

2. **Claim:** The map $\delta : \boldsymbol{C}[x, y]_{\lambda+\mu-2n} \to \boldsymbol{C}[x]_{\lambda+\mu-2n}$ is an isomorphism.

   Proof: We must show that the actions of *e*, *f*, and *h* commute with the mapping:

   For *h* we see: $\delta(h \cdot v_n) = \delta\left((\lambda + \mu - 2n)v_n\right) = (\lambda + \mu - 2n)\delta(v_n) = \tilde{h}\delta(v_n)$

   What we notice here is that the action of *h* must take the vector to eigenvectors of the same eigenvalue, but this is accomplished by letting *m* in $\tilde{h}$ equal λ + μ.

For *f* we see: $\delta\big(f^j \cdot v_n\big) = \delta\big(f \cdot f^{j-1} \cdot v_n\big) \equiv \tilde{f}^j \cdot 1 = \tilde{f} \cdot \tilde{f}^{j-1} \cdot 1 = \tilde{f} \cdot \delta(f^{j-1} \cdot v_n)$

For *e* we use an induction argument to show the commutation with the map. We start by letting *j* = 0, and then assume it to be true for *j* = *j* – 1. For *j* = 0 we have $\delta(e \cdot v_n) = 0 = e \cdot 1 = e \cdot \delta(v_n)$. When doing the inductive step we rely on the same trick that we did before and on the fact that δ commutes with *f* and *h*:

$$\delta\big(ef^j \cdot v_n\big) = \delta\Big((fe + h) \cdot \big(f^{j-1} \cdot v_n\big)\Big) = f \cdot \delta\big(ef^{j-1} \cdot v_n\big) + h \cdot \delta\big(f^{j-1} \cdot v_n\big)$$

Now, by using the inductive hypothesis we write:

$$(fe + h) \cdot \delta\big(f^{j-1} \cdot v_n\big) = ef \cdot \delta\big(f^{j-1} \cdot v_n\big) = e \cdot \delta\big(f^j \cdot v_n\big)$$

The final point that we need to make is that these submodules require a dimension count in order for them to be isomorphic. We have shown that within ***C***[x,y], we can create modules of dimension *a*, where *a* is some nonnegative integer. We also know that by using *f* we will either get 0 or a linearly independent eigenvector, so these submodules will either be finite or infinite. We can see that by sending the h-eigenvector to h-eigenvectors of corresponding eigenvalues, we are able to say that infinite dimensional submodules are isomorphic and finite submodules are isomorphic to those that have the same number of basis elements.■

We are now ready to show the final result:

Theorem 6: We have $\boldsymbol{C}[x, y] \;\cong \bigoplus_{d=0}^{d=\infty} \boldsymbol{C}[x]_{\lambda+\mu-2n}$:.

Proof: Suppose that there is some v_n in $\boldsymbol{C}[x, y]$. Say this v_n can be written in the form:

$$v_n = \sum_{i=0}^{n} b_i f^i v_{n-i}\,, \qquad b_i \in \boldsymbol{C}, \qquad v_{n-i} \in \boldsymbol{C}[x, y]$$

If we apply *e* to both sides of this equation, we can arrive at the fact that the coefficients must be zero. We can demonstrate this by an example:

$$v_2 = b_0 v_n + b_1 f v_{n-1} + b_2 f^2 v_{n-2},$$

$$e^2 \cdot v_2 = e^2 \cdot b_0 v_n + e^2 \cdot b_1 f v_{n-1} + e^2 \cdot b_2 f^2 v_{n-2}$$

$$0 = 0 + 0 + b_2 k v_{n-2}, \text{where } k \text{ is a nonzero scalar}$$

This implies that b_2 must be zero. We show this with induction: Let *i* = 1:

$$v_n = b_1 f \cdot v_{n-1}$$

$$e \cdot v_n = e \cdot b_1 f \cdot v_{n-1} = b_1 e \cdot f \cdot v_{n-1} = b_1(fe + h)v_{n-1} =$$

$$= b_1[fe \cdot v_{n-1} + h \cdot v_{n-1}] = b_1[0 + cv_{n-1}] = 0$$

The coefficient $c = \mu + \lambda - 2(n-1)$, will be nonzero in general, so we must say that $b_1 = 0$ since the expression must equal zero, regardless of c. We should also point out now that whatever c we get will determine what submodule we are working from and therefore what module we are going to. Now we assume this to be true $n = n - 1$, and show that it must be true for $n = n$:

$$v_n = b_0 v_n + b_1 f v_{n-1} + b_2 f^2 v_{n-2} + \cdots + b_n f^n v_0$$

With n applications of e, the LHS of the equation is 0 and the right hand side becomes:

$$b_n e^n \cdot f^n \cdot v_0 = b_n w_0 \cdot 1$$

Since we know that w_0 will be nonzero, we must conclude that $b_n = 0$. This implies that we cannot write these vectors in terms of vector from different modules, otherwise we would be able to show that one vector was a negative of another. Since we can vary the length of the polynomial, and the number of applications of e, we conclude that if this string of monomials is zero, then each coefficient must be zero. This implies that the sum is direct.

That last step that we need to discuss is that each monomial must be from this direct sum. We say that $\boldsymbol{C}[x, y]$ will decompose into $\oplus_{n=0}^{n=\infty} \boldsymbol{C}[x, y]^{\lambda+\mu-2n}$**.** We say that each of these submodules corresponds is an h-eigenspace where

$$\boldsymbol{C}[x, y]^{\lambda+\mu-2n} = \{v \in \boldsymbol{C}[x, y],\ h \cdot v = (\lambda + \mu - 2n) \cdot v\} = \text{Span}\{x^n, x^{n-1}y, \dots, y^n\}$$

These eigenspaces are created by the fact that h will preserve the degree, but could still yield zeroes with the proper number of applications and the proper choice of lambda and mu. These eigenspaces correspond to the finite dimensional modules where each basis element must be of the same degree, but these are none other than the V_d that we described earlier. We then say that each monomial must come from one of these h-eigenspaces, since they are certainly in one of the V_d. We can then close with an equality which allows which relies on linear independence of the basis elements and the fact that these spaces will have the same dimension:

$$\text{Span}\{f^n \cdot c, f^{n-1} \cdot (x - y), \dots, f \cdot (x - y)^{n-1}, (x - y)^n\} = \text{Span}\{x^n, x^{n-1}y, \dots, y^n\}$$

We can then say that $\boldsymbol{C}[x, y]_{\lambda+\mu-2n} = \boldsymbol{C}[x, y]^{\lambda+\mu-2n}$, so the direct sums must be equal as n goes to infinity. This shows that we can obtain the basis of $\boldsymbol{C}[x, y]$ by obtaining elements from these different V_d, where V_d will act as h-eigenspaces. With our previous demonstration of the isomorphism between modules of **C**[x] and **C**[x,y], we have our final isomorphism.■

Conclusion and Future Work: We are able to define a map that shows an equality between one set that is the result of a tensor product and another set that is the result of a direct sum of infinitely many of its finite dimensional submodules. We relied on known definitions, axioms, and the relations that apply to the Lie algebra we are working with, in this case $sl(2,\boldsymbol{C})$. By creating representations and modules we are able to see that there is an overall larger correspondence between these objects. Some of the future work that we hope to explore is to be able to describe explicitly the modules that we get for specific n applications of f since we get some rather nasty looking polynomials in two variables. That way we could actually show the elements of each module. We would also hope to work in higher dimensional Lie algebras to obtain some more general formulas and of course to characterize further n-parameter representations. We could also look at the structure of the differential operators to try to describe suitable representations for higher order operators. Again this would allow us to obtain general formulas to help further our understanding of the structure of these objects.

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