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Abstract. In this paper, we first recall Hermite polynomials, a particular family of orthogonal polynomials. We then evaluate their recurrence relations in terms of Sylvester type determinants of a certain tridiagonal matrix.

1. Introduction

Using orthogonal polynomials, R. Askey [3] evaluated the determinants

$$D_{N+1}(x) = \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ N & x & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & N-1 & x & 3 & \dots & 0 & 0 & 0 \\ \dots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 2 & x & N \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & x \end{vmatrix}, \tag{1.1}$$

These determinants were first considered by J. Sylvester in [6]. In addition, he obtained several generalizations of Sylvester type determinants and explored their connection to orthogonal polynomials [3].

Following the work of Askey [3], we will examine the Hermite polynomial in this matter. The Hermite polynomial of degree n is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}. \tag{1.2}$$

Hermite polynomials play an important role in probability theory and other areas of mathematics. For example, they arise in the Edgeworth and Gram-Charlier series: series that approximate a probability distribution using cumulants instead of moments. Gram-Charlier series are discussed at length by P. Hall [4].

The goal of the present paper is to evaluate the recurrence relation satisfied by the Hermite polynomial defined in (1.2) using Sylvester type determinants of certain tridiagonal matrices. More precisely, we prove the following

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Theorem 1.1. *One has*

$$\frac{H_{n+1}(x)}{2^{n+1}} = \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \dots & 0 & 0 & 0 \\ \dots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & \frac{(n-1)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{n}{2} & x \end{vmatrix}. \tag{1.3}$$

This result is not new; it is a special case of results in [3]. However, we believe that concentrating on this special case allows one to see the ideas of [3] more clearly.

We close the introduction with a brief synopsis of the paper. In §2, we will discuss some basic properties of orthogonal polynomials, Hermite polynomials, and Sylvester type determinants of tridiagonal matrices. In §3, we shall describe the three term recurrence relation of the Hermite polynomials. Finally, in §4, we prove Theorem 4.1.

2. Preliminaries

2.1. **Orthogonal polynomials.** Recall the definition of the orthogonal polynomials:

Definition (Definition 5.2.1[1]). *We say that a sequence of polynomials $p_n(x)_{n\in\mathbb{Z}_{>0}} \in \mathbb{R}[x]$, where $p_n(x)$ has exact degree n, is orthogonal with respect to the weight function $\omega(x)$ if there is a sequence of real numbers h_n such that*

$$\int_a^b p_n(x)p_m(x)\omega(x)dx = h_n\delta_{mn}. \tag{2.1}$$

Here as usual $\delta_{mn} = 1$ if $m = n$, and $\delta_{mm} = 0$ if $m \neq n$.

Remark. Note that the range of the integral may be infinite.

It is well–known that the sequence $\{p_n(x)\}$ satisfies a three-term recurrence relation.

Theorem 2.1 (Theorem 5.2.2 [1]). *Let $\{p_n(x)\}$ be a sequence of orthogonal polynomials. There are real numbers a_n, b_n, c_n indexed by $n \in \mathbb{Z}_{\geq 0}$ such that*

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x) \quad \text{for all} \quad n \geq 0, \tag{2.2}$$

where we set $p_{-1}(x) = 0$. Moreover,

$$a_{n-1}a_n c_n > 0,$$

for $n \geq 1$ and if the highest coefficients of $p_n(x)$ is $k_n > 0$, then

$$a_n = \frac{k_{n+1}}{k_n}, \quad c_{n+1} = \frac{a_{n+1}}{a_n}\frac{h_{n+1}}{h_n},$$

where h_n is given by (2.1).

2.2. **Hermite polynomials.** The normal integral $\int_{-\infty}^{\infty} e^{-x^2}dx$ plays a fundamental role in probability theory. Moreover the integrand e^{-x^2} has interesting properties. For example, it is basically its own Fourier transform [1, (6.1.1)]:

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{2ixt} dt. \tag{2.3}$$

The Hermite polynomials $H_n(x)$ defined in (1.2) are orthogonal polynomials with respect to the weight function $w(x) = e^{-x^2}$ [1]. It is easy to check that $H_n(x)$ is a polynomial in x of degree n. The first eleven Hermite polynomials are as follows:

$$\begin{aligned}
H_0(x) &= 1,\\
H_1(x) &= 2x,\\
H_2(x) &= 4x^2 - 2,\\
H_3(x) &= 8x^3 - 12x,\\
H_4(x) &= 16x^4 - 48x^2 + 12,\\
H_5(x) &= 32x^5 - 160x^3 + 120x,\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120,\\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x,\\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680,\\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x,\\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240.
\end{aligned}$$

Note that the Hermite polynomials have a simple generating function [1, (6.1.7)], namely

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} r^n = e^{2xr-r^2}. \tag{2.4}$$

By writing

$$e^{2xr-r^2} = \sum_{p=0}^{\infty} \frac{(2x)^p}{p!} r^p \sum_{q=0}^{\infty} \frac{(-1)^q r^{2q}}{(2q)!}$$

and equating the coefficient r^n on each side, we obtain

$$H_n(x) = \sum_{k=0}^{n/2} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}. \tag{2.5}$$

From (2.5), it is clear that $H_{2n}(x)$ is an even function and $H_{2n+1}(x)$ is an odd function.

2.3. **Sylvester type determinants.** The determinant $D_{N+1}(x)$ defined in (1.1) satisfies that $D_{N+1}(x) = (x-N)D_N(x+1)$. This gives

$$D_{N+1}(x) = \prod_{j=0}^{N}(x+N-2j) \tag{2.6}$$

(which is formulas (2.3), (2.4) from [3]). Another determinant of a tridiagonal matrix is

$$A_{N+1}(x) = \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -N & x-2 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & -N+1 & x-4 & 3 & \dots & 0 & 0 & 0 \\ \dots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & & x-2(N-1) & N \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & x-2N \end{vmatrix}$$

and the value of $A_{N+1}(x)$ is given as

$$A_{N+1}(x) = (x-N)^{N+1}.$$

An outline of how to evaluate $A_{N+1}(x)$ was given on page 229 in [6]. Askey [3] connected these Sylvester type determinants with orthogonal polynomials and obtained several generalizations of Sylvester type determinants.

3. The three term recurrence relation of $H_n(x)$

Recall the orthogonality property of $H_n(x)$:

Proposition 3.1 ((6.15) in [2])**.**

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = n!2^n\sqrt{\pi}\delta_{mn}. \tag{3.1}$$

Following the exposition of [2], an outline of the proof of this orthogonality relation is as follows:

If $m \neq n$, applying integration by parts, we can quickly check that

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 0.$$

If $m = n$, we have that

$$\begin{aligned}\int_{-\infty}^{\infty} H_n(x)H_n(x)e^{-x^2}dx =& (-1)^n\int_{-\infty}^{\infty}\frac{d^n}{dx^n}e^{-x^2}H_n(x)dx \\ =& \int_{-\infty}^{\infty}e^{-x^2}\frac{d^n}{dx^n}H_n(x)dx,\end{aligned}$$

where in the last equality we used integration by parts. From (2.5), it follows that

$$\frac{d^n}{dn} H_n(x) = 2^n n! \tag{3.2}$$

and therefore we obtain the result using the elementary fact that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Hermite polynomials satisfy the three term recurrence relation given by the following theorem:

Theorem 3.2 ((6.1.10) in [2]). *$H_n(x)$ satisfies that for $n \geq 1$,*

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0.$$

Proof. Consider the generating function of $H_n(x)$ as in (2.4). If we set

$$F(x, r) := e^{2xr - r^2}$$

then it clear that $F(x, r)$ satisfies the following differential equation:

$$\frac{\delta F}{\delta r} - (2x - 2r)F = 0.$$

Therefore we have that

$$\frac{d}{dr}\left(\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} r^n\right) - (2x - 2r) \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} r^n = 0.$$

A simple calculation shows that this is equal to

$$\sum_{n=0}^{\infty} \frac{H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x)}{n!} r^n = 0,$$

which gives the three term recurrence relation for $H_n(x)$. □

4. The connection between $H_n(x)$ and Sylvester type determinants

We restate the main theorem and prove it.

Theorem 4.1. *One has*

$$\frac{H_{n+1}(x)}{2^{n+1}} = \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & \frac{(n-1)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{n}{2} & x \end{vmatrix}. \tag{4.1}$$

Proof. We proceed by induction on n. Clearly it is true in the case $n = 0$, since we have that $\frac{H_1(x)}{2} = x$. Assume that

$$\frac{H_m(x)}{2^m} = \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & \frac{(m-2)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{m-1}{2} & x \end{vmatrix}.$$

for $m \leq n$. Now we want to evaluate the determinant

$$\begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & \frac{(n-1)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{n}{2} & x \end{vmatrix}. \tag{4.2}$$

We will use the cofactor expansion method along the last column of (4.2). Then we obtain that

$$\begin{aligned} \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & \frac{(n-1)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{n}{2} & x \end{vmatrix} =& x \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & \frac{(n-2)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{n-1}{2} & x \end{vmatrix} \\ &+ (-1)^{2n} \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & x & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{(n-2)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{n}{2} \end{vmatrix} \\ =& x\frac{H_n(x)}{2^n} + \begin{vmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & x & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{(n-2)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{n}{2} \end{vmatrix}, \end{aligned}$$

where in the last equality we used the induction hypothesis. In order to evaluate the last determinant, we again apply the cofactor expansion to the later determinant along the last row of it. Therefore we have that

$$\begin{vmatrix} x & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & ... & 0 & 0 & 0 \\ ... \\ 0 & 0 & 0 & 0 & ... & x & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & \frac{(n-2)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & \frac{n}{2} \end{vmatrix} = \frac{n}{2} \begin{vmatrix} x & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & ... & 0 & 0 & 0 \\ ... \\ 0 & 0 & 0 & 0 & ... & x & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & \frac{(n-3)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & \frac{n-2}{2} & x \end{vmatrix} = \frac{n}{2} \frac{H_{n-1}(x)}{2^{n-1}}.$$

All together we derive that

$$\begin{aligned} \begin{vmatrix} x & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & x & 1 & ... & 0 & 0 & 0 \\ ... \\ 0 & 0 & 0 & 0 & ... & \frac{(n-1)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & \frac{n}{2} & x \end{vmatrix} &= \frac{xH_n(x)}{2^n} + \frac{nH_{n-1}(x)}{2^n} \\ &= \frac{2xH_n(x) + 2nH_{n-1}(x)}{2^{n+1}} \\ &= \frac{H_{n+1}(x)}{2^{n+1}}, \end{aligned}$$

where in the last equality we use the three term recurrence relation for $H_n(x)$.

In all, we derive that

$$\frac{H_{n+1}(x)}{2^{n+1}} = \begin{vmatrix} x & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & x & 1 & ... & 0 & 0 & 0 \\ ... \\ 0 & 0 & 0 & 0 & ... & \frac{(n-1)}{2} & x & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & \frac{n}{2} & x \end{vmatrix}. \tag{4.3}$$

□

Here are some examples of expansions:

In case $n = 1$, we have that

$$\frac{H_2(x)}{2^2} = \begin{vmatrix} x & 1 \\ \frac{1}{2} & x \end{vmatrix} = x^2 - \frac{1}{2}.$$

Hence we have that $H_2(x) = 4x^2 - 2$.

If $n = 2$, we have that

$$\begin{aligned}
\frac{H_3(x)}{2^3} &= \begin{vmatrix} x & 1 & 0 \\ \frac{1}{2} & x & 1 \\ 0 & 1 & x \end{vmatrix} \\
&= 0 \cdot \begin{vmatrix} 1 & 0 \\ x & 1 \end{vmatrix} - \begin{vmatrix} x & 0 \\ \frac{1}{2} & 1 \end{vmatrix} + x \cdot \begin{vmatrix} x & 1 \\ \frac{1}{2} & x \end{vmatrix} \\
&= -x + x(x^2 - \frac{1}{2}) = x^3 - \frac{3x}{2}.
\end{aligned}$$

Therefore $H_3(x) = 8x^3 - 12x$ as desired.

For $n = 3$, we have that

$$\begin{aligned}
\frac{H_4(x)}{2^4} &= \begin{vmatrix} x & 1 & 0 & 0 \\ \frac{1}{2} & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & \frac{3}{2} & x \end{vmatrix} \\
&= x \begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & \frac{3}{2} & x \end{vmatrix} - \begin{vmatrix} \frac{1}{2} & 1 & 0 \\ 0 & x & 1 \\ 0 & \frac{3}{2} & x \end{vmatrix} \\
&= x^2 \begin{vmatrix} x & 1 \\ \frac{3}{2} & x \end{vmatrix} - x \begin{vmatrix} 1 & 1 \\ 0 & x \end{vmatrix} - \frac{1}{2} \begin{vmatrix} x & 1 \\ \frac{3}{2} & x \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & x \end{vmatrix} \\
&= x^2(x^2 - \frac{3}{2}) - x^2 - \frac{x^2}{2} + \frac{3}{4} \\
&= x^4 - 3x^2 + \frac{3}{4}.
\end{aligned}$$

Hence we obtain that $H_4(x) = 16x^4 - 48x^2 + 12$.

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